Maximum Packings of K_n with Hexagons

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Abstract

A *complete solution* of the maximum packing problem of K_n with hexagons is given.

1 Introduction

A *hexagon system* is a pair (S, H) where H is a collection of edge-disjoint hexagons which partition the edge set of the complete undirected graph K_n with vertex set S. The number $|S| = n$ is called the *order* of the hexagon system (S, H) and $|H| = n(n-1)/12$. In what follows we will denote the hexagon

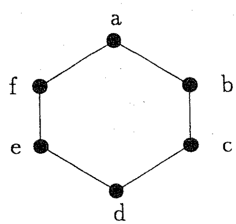

Figure 1:

by any cyclic shift of (a, b, c, d, e, f) or (a, f, e, d, c, b).

Example 1.1 (Hexagon systems of orders 9 and 13):

(1) $S = \{1,2,3,4,5,6,7,8,9\}$; $H_1 = \{(1,2,3,6,7,8), (3,4,5,6,8,9), (1,3,7,4,6,9), (2,4,1,5,3,8), (2,9,4,8,5,7), (1,6,2,5,9,7)\}$

(2) $S = \{1,2,3,4,5,6,7,8,9,10,11,12,13\}$; $H_2 = \{(1,2,4,7,3,8), (13,1,3,6,2,7), (12,13,2,5,1,6), (11,12,1,4,13,5), (10,11,13,3,12,4), (9,10,12,2,11,3), (8,9,11,1,10,2), (7,8,10,13,9,1), (6,7,9,12,8,13), (5,6,8,11,7,12), (4,5,7,10,6,11), (3,4,6,9,5,10), (2,3,5,8,4,9)\}$

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It is well-known that the *spectrum* (that is, set of all n such that hexagon system of order n exists) is *precisely* the set of all $n \equiv 1$ or 9 (mod 12). (See, for example [2, 3].)

If $n \not\equiv 1$ or 9 (mod 12), we cannot construct a hexagon system of order n. However, it is of interest to see just how "close" we can come to a hexagon system. A *packing* of K_n with hexagons is a pair (S, P) where P is an edge-disjoint collection of hexagons. The difference between a hexagon system (S, H) and a packing (S, P) is that the hexagons in H partition the edge set of K_n whereas the only requirement on the hexagons in P is that they are edge-disjoint. (They may or may not partition the edge set of K_n.) If (S, P) is a packing of order n, then the set of uncovered edges L is called the *leave*. Hence $E(K_n) = E(P) \bigcup E(L)$ and $E(P) \bigcap E(L) = \emptyset$. If (S, P) is a packing and $|P|$ is as large as possible (so that $|L|$ is as small as possible), then P is called a *maximum packing*. Of course, a hexagon system is just a maximum packing with leave the empty set.

The object of this paper is to give a *complete answer* to each of the following questions. For a given n:

(1) What is the number of hexagons in a maximum packing? For example, when $n \equiv 1$ or 9 (mod 12), the number of hexagons is $n(n-1)/12$.

(2) How is a maximum packing achieved?

(3) What does the leave of a maximum packing look like?

We will divide our work into six parts: (i) $n \equiv 1$ or 9 (mod 12) (hexagon systems), (ii) $n \equiv 0, 2, 6,$ or 8 (mod 12) (leave a 1-factor), (iii) $n \equiv 3$ or 7 (mod 12) (leave a 3-cycle), (iv) $n \equiv 5$ (mod 12) (leave a 4-cycle), (v) $n \equiv 11$ (mod 12) (leave a 7-cycle or a not necessarily disjoint 3-cycle and 4-cycle), and (vi) $n \equiv 4$ or 10 (mod 12) (leave a spanning subgraph with $(n+8)/2$ edges, with all vertices of odd degree).

Not too surprisingly, we will begin with $n \equiv 1$ or 9 (mod 12); i.e., with the construction of hexagon systems.

2 Hexagon Systems

Before plunging into the construction of hexagon systems we will need a theorem due to D. Sotteau, as well as the following definitions. A *bipartite 2k-cycle system* (X, Y, C) is a collection C of edge-disjoint $2k$-cycles, which partition the edges of the complete undirected bipartite graph $K_{x,y}$ with vertex set $X \bigcup Y$ ($X \bigcap Y = \emptyset$). If $x = |X|$ and $y = |Y|$, then (X, Y, C) is said to have order (x, y). As one might expect, a *bipartite hexagon system* (BHS) is a triple (X, Y, B) where B is a collection of edge-disjoint hexagons which partition the edge set of $K_{x,y}$.

Theorem 2.1 *(D. Sotteau [4]) A bipartite 2k-cycle system of order (x, y) exists if and only if*

(1) x and y are both even,

(2) $x \geq k$ and $y \geq k$, and

(3) $2k|xy$. □

The $n+12$ Construction. [1] Let (K_n, H_1) be a hexagon system of order n based on $X \bigcup \{\infty\}$ and (K_{13}, H_2) a hexagon system of order 13 based on $Y \bigcup \{\infty\}$. Since $|Y| = 12$ and $n \equiv 1$ or 9 (mod 12) implies $|X|$ is even, Sotteau's Theorem guarantees that a BHS (X, Y, B) of order $(|X|, |Y|)$ exists. Define a collection of hexagons H on $X \bigcup Y \bigcup \{\infty\}$ by $H = H_1 \bigcup H_2 \bigcup B$. It is easily seen that (K_{n+12}, H) is a hexagon system. □

Theorem 2.2 *(Folk Theorem) The spectrum for hexagon systems is precisely the set of all $n \equiv 1$ or 9 (mod 12).*

Proof: Beginning with the hexagon systems (K_9, H_1) and (K_{13}, H_2) in Example 1.1, the $n + 12$ Construction yields a hexagon system of every order $n \equiv 1$ or 9 (mod 12). □

□

3 Necessary Conditions for Maximum Packings

If n is odd, every vertex of K_n has even degree, and since each vertex in a hexagon is incident with 2 edges in that hexagon, we know the leave of a maximum packing, if any, must have each of its vertices incident with an even number of edges. As we have stated, if $n \equiv 1$ or 9 (mod 12) a hexagon system exists and the leave is the empty set. If $n \equiv 3$ or 7 (mod 12) ≥ 7, $6|[\binom{n}{2} - 3]$, hence the smallest possible leave is a 3-cycle. If $n \equiv 5$ (mod 12) ≥ 17, $6|[\binom{n}{2} - 4]$, hence the smallest possible leave is a 4-cycle. If $n \equiv 11$ (mod 12), $6|[\binom{n}{2} - 1]$, but, as we have noted, each vertex in the leave must be incident with an even number of edges in the leave, so the smallest possible leave has 7 edges: a 7-cycle, or a not necessarily disjoint 3-cycle and 4-cycle.

If n is even, since each vertex of K_n has odd degree, it is easily seen that the leave must be a spanning subgraph with each vertex having odd degree. The smallest such graph is a 1-factor and is the smallest possible leave for $n \equiv 0, 2, 6,$ or 8 (mod 12) ≥ 6, since $6|[\binom{n}{2} - \frac{n}{2}]$ for such n. However, if $n \equiv 4$ or 10 (mod 12), $6|[\binom{n}{2} - \frac{n}{2} - 4]$, hence the smallest possible leave has $(n + 8)/2$ edges. The only possible degree sequences for such a leave are: $(9, 1, \ldots, 1), (7, 3, 1, \ldots, 1), (5, 5, 1, \ldots, 1), (5, 3, 3, 1, \ldots, 1)$, and $(3, 3, 3, 3, 1, \ldots, 1)$.

With this information, we can proceed with the examples necessary for our construction.

4 Small Cases of Maximum Packings

In this section, we give a collection of the necessary small examples of maximum packings for the general construction to follow.

Example 4.1 (K_6, P): $P = \{(1,3,2,5,4,6), (1,2,4,3,6,5)\}$; $L = \{(1,4), (2,6), (3,5)\}$.

Example 4.2 (K_8, P): $P = \{(1,5,2,8,3,7), (1,8,4,7,6,2), (1,4,2,3,5,6), (3,4,5,7,8,6)\}$; $L = \{(1,3), (2,7), (4,6), (5,8)\}$.

Example 4.3 (K_7, P): $P = \{(1,2,3,4,6,7), (1,4,2,5,6,3), (1,6,2,7,3,5)\}$; $L = \{(4,5,7)\}$.

Example 4.4 (K_{15}, P): $P = \{(1,2,3,4,6,15), (1,4,2,5,6,3), (1,6,2,15,3,5), (15,7,8,11,12,13), (8,9,10,11,13,14), (15,8,12,9,11,14), (7,9,15,10,8,13), (7,14,9,13,10,12), (15,11,7,10,14,12), (1,7,2,8,3,9), (4,9,5,10,6,8), (1,8,5,7,3,10), (2,9,6,7,4,10), (1,11,2,12,3,13), (4,13,5,14,6,12), (1,12,5,11,3,14), (2,13,6,11,4,14)\}$; $L = \{(4,5,15)\}$.

Example 4.5 (K_{17}, P): $P = \{(1,3,5,7,9,17), (1,5,6,7,8,16), (1,6,2,7,3,8), (1,7,4,6,8,9), (2,17,16,15,14,13), (1,15,17,14,12,11), (1,14,16,13,15,12), (4,5,8,10,11,9), (4,8,11,13,12,17), (2,4,10,6,12,5), (1,10,2,11,3,13), (3,6,9,10,12,16), (2,8,12,7,10,14), (2,12,9,5,10,15), (2,9,14,11,7,16), (3,17,5,16,9,15), (3,14,5,15,4,12), (4,14,8,15,6,16), (5,13,7,14,6,11), (6,17,11,16,10,13), (4,11,15,7,17,13), (3,9,13,8,17,10)\}$; $L = \{(1,2,3,4)\}$.

Example 4.6 (K_{11}, P): $P = \{(1,11,2,10,3,9), (1,10,9,11,7,8), (1,7,9,8,10,6), (1,4,2,6,11,5), (2,5,3,6,4,9), (2,7,3,11,4,8), (3,4,10,7,5,8), (8,6,9,5,10,11)\}$; $L = \{(1,2,3), (4,5,6,7)\}$.

Example 4.7 (K_{11}, P): $P = \{(1,11,2,10,3,9), (2,9,10,11,8,7), (1,8,2,6,10,5), (1,10,8,9,11,6), (1,4,11,5,7,3), (2,4,6,8,3,5), (4,7,11,3,6,9), (4,8,5,9,7,10)\}$; $L = \{(1,2,3,4,5,6,7)\}$.

Example 4.8 (K_{11}, P): $P = \{(1,4,6,7,10,11), (1,5,11,9,10,8), (1,6,2,10,5,9), (1,7,2,8,6,10), (2,4,8,5,7,9), (2,5,3,10,4,11), (3,7,4,9,8,11), (3,8,7,11,6,9)\}$; $L = \{(1,2,3), (3,4,5,6)\}$.

Example 4.9 (K_{11}, P): $P = \{(1,6,2,7,9,10), (1,7,3,8,9,11), (1,8,10,11,6,9), (2,4,5,7,11,8), (2,5,8,7,6,10), (2,9,4,10,5,11), (3,10,7,4,8,6), (3,9,5,6,4,11)\}$; $L = \{(1,3,5), (1,2,3,4)\}$.

Example 4.10 (K_{10}, P): $P = \{(1,3,2,5,4,6), (1,2,4,3,6,5), (1,7,2,8,3,9), (4,9,5,10,6,8), (1,8,5,7,3,10), (2,9,6,7,4,10)\}$; $L = \{(1,4), (2,6), (3,5), (7,9), (8,9), (9,10), (7,8,10)\}$.

Example 4.11 (K_{10}, P): $P = \{(1,4,2,3,6,8,), (1,5,2,6,7,9), (1,6,9,8,5,10), (2,7,3,8,4,10), (2,9,4,6,10,8), (3,9,5,4,7,10)\}$; $L = \{(1,2), (7,8), (5,6), (3,4), (9,10), (1,3,5,7)\}$.

Example 4.12 (K_{10}, P): $P = \{(1,2,3,6,7,8),(3,4,5,6,8,9),(1,3,7,4,6,9),$ $(2,4,1,5,3,8),(2,9,4,8,5,7),(1,6,2,5,9,7)\}; L = \{(1,10),(2,10),(3,10),(4,10),$ $(5,10),(6,10),(7,10),(8,10),(9,10)\}$.

Example 4.13 (K_{10}, P): $P = \{(1,3,6,4,5,7),(1,4,2,6,9,8),(1,5,9,7,8,10),$ $(1,6,10,7,3,9),(2,10,5,8,4,7),(2,8,3,10,4,9)\}; L = \{(1,2),(3,4),(5,6),(6,7),$ $(6,8),(9,10),(2,3,5)\}$.

Example 4.14 (K_{10}, P): $P = \{(1,3,6,4,8,9),(1,4,7,10,8,6),(1,5,7,9,6,10),$ $(1,7,2,9,5,8),(2,6,7,3,10,5),(2,8,3,9,4,10)\}; L = \{(1,2),(3,4),(5,6),(7,8),$ $(9,10),(3,5,4,2)\}$.

Example 4.15 (K_{10}, P): $P = \{(1,2,9,10,8,6),(1,3,2,8,9,5),(1,4,2,7,5,8),$ $(1,7,4,6,3,9),(2,5,3,4,9,6),(4,5,6,7,3,8)\}; L = \{(1,10),(2,10),(3,10),(4,10),$ $(5,10),(6,10),(7,10),(7,8),(7,9)\}$.

Example 4.16 (K_{10}, P): $P = \{(1,2,10,9,8,4),(1,3,8,5,7,9),(1,6,2,3,4,7),$ $(1,8,7,2,4,10),(2,8,10,7,3,9),(3,6,4,9,5,10)\}; L = \{(1,5),(2,5),(3,5),(4,5),$ $(5,6),(6,7),(6,8),(6,9),(6,10)\}$.

Example 4.17 (K_{10}, P): $P = \{(1,2,9,8,10,6),(1,3,2,8,6,9),(1,4,6,3,9,5),$ $(1,7,2,6,5,8),(2,4,7,3,10,5),(3,5,7,9,4,8)\}; L = \{(1,10),(2,10),(9,10),$ $(4,10),(4,5),(3,4),(7,10),(7,8),(6,7)\}$.

Example 4.18 (K_{10}, P): $P = \{(1,2,3,10,4,9),(1,3,9,6,7,8),(1,4,8,5,3,7),$ $(1,5,2,8,3,6),(4,6,10,2,9,7),(2,6,8,9,5,7)\}; L = \{(1,10),(7,10),(8,10),(9,10),$ $(5,10),(5,6),(4,5),(2,4),(3,4)\}$.

Example 4.19 (K_{10}, P): $P = \{(1,5,6,7,10,9),(1,7,4,8,5,10),(2,8,1,6,9,7),$ $(2,3,4,6,8,9),(8,10,3,9,5,7),(5,3,6,10,2,4)\}; L = \{(1,2),(1,3),(1,4),(2,5),(2,6),$ $(3,7),(3,8),(4,9),(4,10)\}$.

Example 4.20 (K_{10}, P): $P = \{(1,3,2,10,8,9),(1,4,5,10,9,7),(1,5,2,7,8,6),$ $(2,8,1,10,6,4),(5,6,2,9,4,8),(3,9,5,7,4,10)\}; L = \{(1,2),(3,4),(3,5),(3,6),(3,7),$ $(3,8),(6,7),(6,9),(7,10)\}$.

Example 4.21 (K_{10}, P): $P = \{(1,4,6,10,9,7),(1,5,7,10,4,9),(1,6,9,5,10,2),$ $(1,8,2,6,3,10),(2,4,7,3,8,5),(2,9,3,4,8,7)\}; L = \{(1,3),(2,3),(3,5),(4,5),(5,6),$ $(6,7),(6,8),(8,9),(8,10)\}$.

Example 4.22 (K_{10}, P): $P = \{(1,3,6,7,8,10),(1,4,6,10,7,5),(1,6,8,5,4,9),$ $(1,7,2,9,3,8),(2,8,4,7,3,10),(2,6,9,5,10,4)\}; L = \{(1,2),(5,6),(3,4)(7,9),(8,9),$ $(9,10),(2,3,5)\}$.

Example 4.23 (K_{16}, P): $P = \{(1,3,5,10,16,15),(1,4,6,13,14,11),$ $(1,5,7,12,15,10),(1,16,12,10,9,14),(2,3,6,9,11,13),(2,4,7,10,13,9),$ $(2,5,13,7,11,10),(1,6,2,7,3,8),(1,7,8,11,6,12),(3,13,1,9,7,14),(2,11,3,10,6,14),$

$(3,15,2,16,8,12),(4,9,16,7,15,11),(4,10,14,8,13,15),(4,13,16,11,5,14),$ $(8,2,12,5,9,15),(16,3,9,12,4,5),(5,15,6,16,4,8)\}$; $L=\{(1,2),(3,4),(5,6),(6,7),$ $(6,8),(8,9),(8,10),(11,12),(12,13),(12,14),(14,15),(14,16)\}$.

Example 4.24 (K_{16},P): $P=\{(1,3,5,8,9,16),(1,4,5,7,8,10),(2,3,14,4,9,15),$ $(1,5,16,15,6,11),(1,6,10,11,12,13),(1,7,12,14,11,8),(1,15,12,16,10,14),$ $(2,9,1,12,10,13),(2,4,6,12,8,14),(3,7,9,12,4,13),(3,8,13,11,15,10),$ $(2,7,16,8,15,5),(2,6,9,13,7,11),(3,16,2,10,4,11),(3,12,2,8,4,15),$ $(5,10,7,14,16,11),(6,13,5,14,9,3),(4,7,15,14,6,16)\}$; $L=\{(1,2),(3,4),(5,6),$ $(5,9),(5,12),(6,7),(6,8),(9,10),(9,11),(13,14),(13,15),(13,16)\}$.

Example 4.25 (K_{16},P) $P=\{(1,3,5,9,16,15),(1,4,2,16,12,14),$ $(1,5,10,11,15,13),(1,6,2,15,4,7),(3,6,7,8,11,12),(3,7,9,15,12,10),$ $(4,5,11,7,12,6),(2,7,16,13,14,5),(2,9,1,16,6,11),(2,10,1,11,14,8),$ $(2,3,8,10,13,12),(3,13,2,14,6,15),(3,14,7,13,8,16),(4,11,3,9,6,13),$ $(10,16,11,13,5,15),(8,1,12,5,16,4),(8,15,7,10,4,12),(4,14,10,6,8,9)\}$; $L=\{(1,2),(3,4),(5,6),(5,7),(5,8),(9,10),(9,11),(9,12),(9,13),(9,14),(14,15),$ $(14,16)\}$.

Example 4.26 (K_{16},P): $P=\{(1,3,7,8,11,14),(1,4,2,5,16,13),(1,5,11,4,14,12),$ $(1,6,9,10,11,16),(1,7,2,16,14,10),(2,8,1,9,15,6),(3,5,12,15,11,9),$ $(2,3,6,4,10,13),(2,11,1,15,3,12),(2,9,8,12,4,15),(10,2,14,5,13,3),$ $(7,11,3,14,9,12),(16,3,8,13,14,7),(4,5,15,10,6,8),(6,7,4,9,16,12),$ $(7,15,8,14,6,13),(16,8,10,12,11,6),(10,7,9,13,4,16)\}$; $L=\{(1,2),(3,4),(5,6),$ $(5,7),(5,8),(5,9),(5,10),(11,13),(12,13),(13,15),(14,15),(15,16)\}$.

Example 4.27 (K_{16},P): $P=\{(1,9,2,3,12,16),(1,10,16,14,6,15),$ $(1,11,14,15,12,13),(1,12,11,9,8,14),(2,4,3,16,5,8),(2,5,4,16,6,10),$ $(2,6,9,16,8,15),(2,7,9,15,10,13),(2,11,6,13,8,12),(3,14,2,16,7,13),$ $(3,11,16,13,15,5),(3,6,7,8,4,10),(4,7,3,8,6,12),(4,15,3,9,12,14),$ $(4,6,5,14,7,11),(9,4,13,5,10,14),(5,12,7,10,8,11),(5,7,15,11,13,9)\}$; $L=\{(1,2),$ $(1,3),(1,4),(1,5),(1,6),(1,7),(1,8),(9,10),(10,11),(10,12),(13,14),(15,16)\}$.

Example 4.28 (K_{16},P) : $P=\{(1,7,2,3,4,8),(2,4,5,6,7,16),(1,9,10,12,16,13),$ $(1,10,2,5,13,15),(1,16,3,5,8,14),(1,11,3,15,14,12),(3,6,16,14,11,8),$ $(2,8,16,9,11,15),(2,6,11,10,14,9),(2,11,12,15,9,13),(4,6,8,12,2,14),$ $(4,9,6,14,7,13),(5,10,6,13,8,15),(5,12,6,15,4,16),(3,7,4,11,5,14),$ $(3,9,5,7,15,10),(8,9,12,3,13,10),(4,10,16,11,13,12)\}$; $L=\{(1,3),(1,4),(1,5),$ $(1,6),(1,2),(12,7),(7,8),(7,9),(7,10),(7,11),(13,14),(15,16)\}$.

Example 4.29 (K_{16},P): $P=\{(1,2,4,7,9,10),(1,4,5,6,8,9),(1,5,12,16,9,15),$ $(1,6,2,5,15,12),(1,7,2,15,11,16),(1,8,2,12,14,13),(2,11,1,14,9,13),$ $(3,5,9,12,13,8),(3,6,4,11,13,10),(3,9,2,10,8,12),(3,16,2,14,8,11),$ $(6,15,3,7,10,14),(4,14,3,13,16,10),(4,9,6,10,15,8),(4,12,7,14,5,13),$ $(5,16,4,15,7,11),(6,13,7,16,14,11),(5,10,12,6,16,8)\}$; $L=\{(1,3),(2,3),(3,4),$ $(5,7),(6,7),(7,8),(9,11),(10,11),(11,12),(13,15),(14,15),(15,16)\}$.

Example 4.30 (K_{16}, P): $P = \{(1,11,3,15,7,14),(1,7,8,10,11,12),$ $(1,8,2,9,11,16),(1,9,3,4,5,15),(1,10,12,14,15,13),(2,3,10,7,11,14),$ $(2,5,3,6,4,10),(2,4,7,6,8,12),(2,6,9,12,7,13),(2,7,16,14,8,15),$ $(3,7,5,16,10,13),(3,8,16,9,15,12),(5,8,11,15,10,14),(4,11,6,13,5,12),$ $(4,13,9,14,6,15),(4,16,12,6,5,9),(6,10,5,11,2,16),$ $(4,14,3,16,13,8)\}$; $L = \{(1,2),(1,3),(1,4),(1,5),(1,6),(7,9),(8,9),(9,10),(11,13),$ $(12,13),(13,14),(15,16)\}$.

5 Maximum Packings

We will construct maximum packings according to the leave.

$n \equiv 0, 2, 6,$ or 8 (mod 12). In this case the leave is a 1-factor. The cases $n = 6$ and $n = 8$ are handled in Examples 4.1 and 4.2. So we can assume $n \geq 12$. The following construction will allow us to take care of the remaining cases.

The $n+6$ Construction. Let (K_n, P_1) be a maximum packing of even order n based on X with leave L_1 and (K_6, P_2) the maximum packing of order 6 in Example 4.1 based on Y with leave L_2. Let (X, Y, B) be a BHS of order $(|X|, |Y|)$. (See [4].) Then $(K_{n+6}, P_1 \bigcup P_2 \bigcup B)$ is a maximum packing of order $n+6$ based on $X \bigcup Y$ with leave $L_1 \bigcup L_2$. □

Theorem 5.1 *If $n \equiv 0, 2, 6,$ or 8 (mod 12) the leave of a maximum packing is a 1-factor and such a maximum packing exists for all admissible $n \geq 6$.*

Proof: Starting with the examples of orders 6 and 8, the $n+6$ Construction produces a maximum packing of every order $n \equiv 0, 2, 6,$ or 8 (mod 12) ≥ 12. □

□

$n \equiv 3$ or 7 (mod 12). In this case the leave is a 3-cycle. The cases for $n = 7$ and 15 are handled in Examples 4.3 and 4.4, respectively. We use the following obvious modification of the $n+12$ Construction.

The $n+12$ *MP* Construction. Let (K_n, P) be a maximum packing of odd order n based on $X \bigcup \{\infty\}$ with leave L and (K_{13}, H) the hexagon system of order 13 in Example 1.1 based on $Y \bigcup \{\infty\}$. Let (X, Y, B) be a BHS of order $(|X|, |Y|)$. Then $(K_{n+12}, P \cup H \cup B)$ is a maximum packing of order $n+12$ based on $X \cup Y \cup \{\infty\}$ with leave L. □

Theorem 5.2 *If $n \equiv 3$ or 7 (mod 12) the leave of a maximum packing is a 3-cycle and such a maximum packing exists for admissible $n \geq 7$.*

Proof: Beginning with the examples of orders 7 and 15, the $n+12$ MP Construction yields a maximum packing of every order $n \equiv 3$ or 7 (mod 12) ≥ 7. □

□

$n \equiv 5$ (mod 12). For this case the leave is a 4-cycle. The case for $n = 17$ is given in Example 4.5.

Theorem 5.3 *If $n \equiv 5 \pmod{12} \geq 17$ the leave of a maximum packing is a 4-cycle and such a maximum packing exists for admissible $n \geq 17$.*

Proof: Beginning with the example of order 17, the $n+12$ MP Construction yields a maximum packing of every order $n \equiv 5 \pmod{12} \geq 17$. □

□

$n \equiv 11$ **(mod 12).** In this case the leave is a 7-cycle or a not necessarily disjoint 3-cycle and 4-cycle. The 4 possible leaves are given in Examples 4.6, 4.7, 4.8, and 4.9.

Theorem 5.4 *If $n \equiv 11 \pmod{12}$ a maximum packing has leave a 7-cycle or a not necessarily disjoint 3-cycle and 4-cycle.*

Proof: Starting with any one of the maximum packings in Examples 4.6, 4.7, 4.8, and 4.9 the $n+12$ MP Construction yields a maximum packing of every order $n \equiv 11 \pmod{12}$. □

□

$n \equiv 4$ **or** 10 **(mod 12).** In this case the leave is a spanning subgraph of odd degree with $(n+8)/2$ edges. If $n = 10$ the only leaves are those in Examples 4.10 – 4.22. If $n = 16$ the leave is either one of the leaves from Examples 4.23 – 4.30 or one of the leaves from Examples 4.10 – 4.22 plus a disjoint 1-factor (the leave from (K_6, P)). For $n \geq 22$ the leave is one of those in Examples 4.10 – 4.30 plus a disjoint 1-factor.

Theorem 5.5 *If $n \equiv 4$ or 10 $\pmod{12}$ a maximum packing has one of the leaves in Examples 4.10 – 4.30 plus a disjoint 1-factor, and all 21 l eaves are possible for all $n \equiv 4$ or 10 $\pmod{12} \geq 16$. For $n = 10$, the only possible leaves are those in Examples 4.10 – 4.22.*

Proof: Beginning with the packings in Examples 4.10 – 4.30, the $n+6$ Construction yields all maximum packings of every order $n \equiv 4$ or 10 $\pmod{12} \geq 22$. □

□

6 Summary

We summarize the results in the following easy-to-read table.

<table>
<tr><th>K_n</th><th>Number of Hexagons in a Maximum Packing</th><th>Leave</th></tr>
<tr><td>all $n \equiv 1$ or $9 \pmod{12}$</td><td>$n(n-1)/12$</td><td>$\emptyset$</td></tr>
<tr><td>all $n \equiv 0, 2, 6,$ or $8 \pmod{12}$ ≥ 6</td><td>$n(n-2)/12$</td><td>1-factor</td></tr>
<tr><td>all $n \equiv 3$ or $7 \pmod{12}$</td><td>$(n^2-n-6)/12$</td><td>3-cycle</td></tr>
<tr><td>all $n \equiv 5 \pmod{12} \geq 17$</td><td>$(n^2-n-8)/12$</td><td>4-cycle</td></tr>
<tr><td>all $n \equiv 11 \pmod{12}$</td><td>$(n^2-n-14)/12$</td><td>4 leaves are possible: a 7-cycle or the union of a (not necessarily disjoint) 3-cycle and 4-cycle</td></tr>
<tr><td>all $n \equiv 4$ or $10 \pmod{12}$ ≥ 10</td><td rowspan="3">$(n^2-2n-8)/12$</td><td>spanning subgraph of odd degree with $(n+8)/2$ edges:</td></tr>
<tr><td>$n = 10$</td><td>leaves in Examples 4.10 – 4.21</td></tr>
<tr><td>$n \equiv 4$ or $10 \pmod{12}$ ≥ 16</td><td>the 13 leaves for $n = 10$ plus a disjoint 1-factor and the leaves in Examples 4.23 – 4.30 plus a disjoint 1-factor when $n \geq 22$</td></tr>
</table>

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