AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **69(2)** (2017), Pages 222–235

Orientable $\mathbb{Z}_n$-distance magic labeling of the Cartesian product of two cycles

Bryan Freyberg

Department of Mathematics and Computer Science
Southwest Minnesota State University
Marshall, MN 56258
U.S.A.
bryan.freyberg@smsu.edu

Melissa Keranen

Department of Mathematical Sciences
Michigan Technological University
Houghton, MI 49931
U.S.A.
msjukuri@mtu.edu

Abstract

A *directed $\mathbb{Z}_n$-distance magic labeling* of an oriented graph $\overrightarrow{G} = (V, A)$ of order n is a bijection $\overrightarrow{\ell} : V \to \mathbb{Z}_n$ with the property that there exists $\mu \in \mathbb{Z}_n$ (called the *magic constant*) such that

$$w(x) = \sum_{y \in N_G^+(x)} \overrightarrow{\ell}(y) - \sum_{y \in N_G^-(x)} \overrightarrow{\ell}(y) = \mu \text{ for every } x \in V(G).$$

If for a graph G there exists an orientation $\overrightarrow{G}$ such that there is a directed $\mathbb{Z}_n$-distance magic labeling $\overrightarrow{\ell}$ for $\overrightarrow{G}$, we say that G is *orientable $\mathbb{Z}_n$-distance magic.* In this paper, we prove that the Cartesian product of any two cycles is orientable $\mathbb{Z}_n$-distance magic.

1 Definitions and known results

A *distance magic labeling* of a graph $G = (V, E)$ of order n is a bijection $f : V \to \{1, 2, \ldots, n\}$ with the property that there is a positive integer k (called the *magic constant*) such that

$$w(x) = \sum_{y \in N(x)} f(y) = k \text{ for every } x \in V(G),$$

where $N(x) = \{y|xy \in E\}$ is the *open neighborhood* of vertex x. We call $w(x)$ the *weight* of vertex x. See [1] for a survey of results regarding distance magic graphs. Froncek adapted distance magic labeling by using the elements from an abelian group as labels rather than integers in [5]. Let $G = (V, E)$ be a graph of order n and let Γ be an abelian group of order n. If there exists a bijection $\ell : V \to \Gamma$ with the property that there is an element $\mu \in \Gamma$ such that

$$w(x) = \sum_{y \in N(x)} \ell(y) = \mu \text{ for every } x \in V(G),$$

we say the labeling ℓ is a Γ-*distance magic labeling* and we say the graph G is Γ-*distance magic*. If such a labeling exists for every abelian group of order n, then we say G is *group distance magic*.

For a given natural number p, let $[p]$ denote the set $\{0, 1, \dots, p-1\}$. For a set of integers S and a number c, let $S + c = \{x + c : x \in S\}$. For an element g of a group G, we use the notation $ord_G(g)$ to denote the order of g.

The Cartesian product $G \square H$ of two graphs G and H is a graph with vertex set $V(G) \times V(H)$ and two vertices (g, h) and (g', h') are adjacent in $G \square H$ if and only if $g = g'$ and h is adjacent to h' in H, or $h = h'$ and g is adjacent to g' in G. Let $C_n = \{x_0, x_1, \dots, x_{n-1}, x_0\}$ denote a cycle of length n.

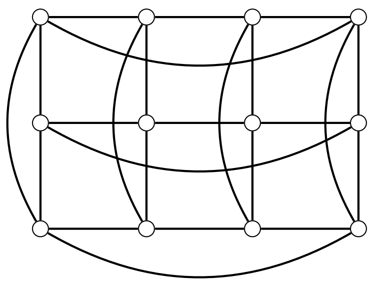

Figure 1: Cartesian product $C_3 \square C_4$

Froncek proved the following result in [5].

Theorem 1. [5] *The Cartesian product $C_m \square C_n$ is $\mathbb{Z}_{mn}$-distance magic if and only if mn is even.*

Cichacz made progress towards settling when $C_m \square C_n$ is group distance magic by proving the following in [2].

Theorem 2. [2] *Let $l = \operatorname{lcm}(m, n)$. If m or n is even, then $C_m \square C_n$ is $Z_\alpha \times \Gamma$-distance magic for any $\alpha \equiv 0 \pmod{l}$ and any abelian group Γ of order $\frac{mn}{\alpha}$.*

Cichacz and Froncek proved the following non-existence result in [4].

Theorem 3. *If G is an r-regular graph of order n and r is odd, then G is not $\mathbb{Z}_n$-distance magic.*

The following analog of group distance magic labeling for directed graphs was introduced in [3]. Let $G = (V, E)$ be an undirected graph on n vertices. Assigning a direction to the edges of G gives an *oriented graph* $\overrightarrow{G}(V, A)$. We will use the notation $\overrightarrow{xy}$ to denote an edge directed from vertex x to vertex y. Let $N^+(x) = \{y|\overrightarrow{yx} \in A\}$ and $N^-(x) = \{z|\overrightarrow{xz} \in A\}$. Let Γ be an abelian group of order n. A *directed* Γ-*distance magic labeling* of an oriented graph $\overrightarrow{G} = (V, A)$ of order n is a bijection $\overrightarrow{\ell} : V \to \Gamma$ with the property that there is a $\mu \in \Gamma$ (called the *magic constant*) such that

$$w(x) = \sum_{y \in N^+(x)} \overrightarrow{\ell}(y) - \sum_{y \in N^-(x)} \overrightarrow{\ell}(y) = \mu \text{ for every } x \in V(G).$$

If for a graph G there exists an orientation $\overrightarrow{G}$ such that there is a directed Γ-distance magic labeling $\overrightarrow{\ell}$ for $\overrightarrow{G}$, we say that G is *orientable* Γ-*distance magic.*

In this paper, we focus on orientable $\mathbb{Z}_n$-distance magic labeling, where $\mathbb{Z}_n$ is the cyclic group of order n. For the sake of orienting a cycle C_n, if the edges are oriented such that every arc has the form $\overrightarrow{x_i x_{i+1}}$ for all $i \in \{0, 1, \ldots, n-1\}$ (where the addition in the subscript is taken modulo n), then we say the cycle is oriented *clockwise.* On the other hand, if all the edges of the cycle are oriented such that every arc has the form $\overrightarrow{x_i x_{i-1}}$ for all $i \in \{0, 1, \ldots, n-1\}$, then we say the cycle is oriented *counter-clockwise.*

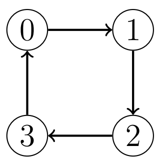

Figure 2: Orientable $\mathbb{Z}_4$-distance magic labeling of C_4 with clockwise orientation

It is an easy observation that C_n is orientable $\mathbb{Z}_n$-distance magic for all $n \geq 3$ (orient all the edges in the same direction around the cycle and label the vertices consecutively $0, 1, \ldots, n-1$).

The following theorem was proved by Cichacz et al. in [3].

Theorem 4. [3] *Let G be a graph of order n in which every vertex has odd degree. If $n \equiv 2 \pmod 4$, then G is not orientable $\mathbb{Z}_n$-distance magic.*

Regarding the Cartesian product of two cycles, they obtained the following partial result.

Theorem 5. [3] *If* $\gcd(m, n) = 1$, *then the Cartesian product $C_m \square C_n$ is orientable $\mathbb{Z}_{mn}$-distance magic.*

In Section 3 we prove the Cartesian product $C_m \square C_n$ is orientable $\mathbb{Z}_{mn}$-distance magic for all $m, n \geq 3$.

2 Lemmas

In this section, we prove a series of lemmas regarding the labelings used in the main theorem of Section 3. Let $m, n \geq 3$ be given and let $\gcd(m,n) = d$. Define $\lambda = \frac{m+n}{d}$ and let $\gcd(\frac{m}{d}, d) = \alpha$. For a given integer a, let $0 \leq \mathcal{R}(a) < d$ represent the remainder when a is divided by d. That is, $a = qd + \mathcal{R}(a)$ for some positive integer q. We begin by establishing some relationships between m, n, d, and α.

Observation 6. *If* $\alpha^2 \nmid d$, *then* $\gcd\left(\alpha\frac{m}{d}, d\right) = \alpha$ *and* $\gcd\left(\frac{m}{d}, \frac{n}{\alpha}\right) = 1$.

Proof. By elementary properties of the greatest common divisor, $\gcd\left(\frac{m}{d}, \frac{n}{d}\right) = 1$ implies $\gcd\left(\frac{m}{d} \cdot \frac{n}{d}, \frac{n}{\alpha}\right) = \gcd\left(\frac{m}{d}, \frac{n}{\alpha}\right) \gcd\left(\frac{n}{d}, \frac{n}{\alpha}\right)$. But $\gcd\left(\frac{m}{d} \cdot \frac{n}{d}, \frac{n}{\alpha}\right) = \frac{n}{d} \gcd\left(\frac{m}{d}, \frac{d}{\alpha}\right)$, and $\gcd\left(\frac{n}{d}, \frac{n}{\alpha}\right) = \frac{n}{d} \gcd\left(1, \frac{d}{\alpha}\right) = \frac{n}{d}$. Therefore, $\gcd\left(\frac{m}{d}, \frac{d}{\alpha}\right) = \gcd\left(\frac{m}{d}, \frac{n}{\alpha}\right)$. Multiplying both sides by α gives $\gcd\left(\alpha\frac{m}{d}, d\right) = \alpha \gcd\left(\frac{m}{d}, \frac{n}{\alpha}\right)$. But since $\alpha^2 \nmid d$, we have $\gcd\left(\alpha\frac{m}{d}, d\right) = \alpha$ and hence, $\gcd\left(\frac{m}{d}, \frac{n}{\alpha}\right) = 1$. □

Observation 7. *If* $\alpha^2 \mid d$, *then* $\gcd\left(\alpha^2\frac{m}{d}, d\right) = \alpha^2$ *and* $\gcd\left(\frac{m}{d}, \frac{n}{\alpha^2}\right) = 1$.

Proof. Essentially the same argument as in the proof of Observation 6 gives $\gcd\left(\alpha\frac{m}{d}, \frac{d}{\alpha}\right) = \alpha \gcd\left(\frac{m}{d}, \frac{n}{\alpha^2}\right)$. Since $\alpha^2 \mid d$, we have $\gcd\left(\alpha\frac{m}{d}, d\right) = \alpha^2$ and thus $\gcd\left(\alpha\frac{m}{d}, \frac{d}{\alpha}\right) = \alpha$. Hence, $\gcd\left(\frac{m}{d}, \frac{n}{\alpha^2}\right) = 1$. The fact that $\gcd\left(\alpha^2\frac{m}{d}, d\right) = \alpha^2$ follows from $\gcd\left(\alpha\frac{m}{d}, \frac{d}{\alpha}\right) = \alpha$. □

For the following lemmas, let $\mathbb{Z}_{mn}$ be the cyclic group of order mn, let $V = \{(i,j) : i \in [m], j \in [n]\}$, and for a given function $g : V \longmapsto \mathbb{Z}_{mn}$, define $g'(i,j) = g(i,j) - \mathcal{R}(j-i)$. For an element $g \in \mathbb{Z}_{mn}$, we denote by $< g >$, the subgroup generated by g. Assume $1 < d < \min\{m,n\}$ for all of the lemmas.

Lemma 8. *If* $\gcd(\lambda, d) = 1$, *then the mapping* $g : V \longmapsto \mathbb{Z}_{mn}$ *given by* $g(i,j) = jm + in + \mathcal{R}(j-i)$, *is a bijection.*

Proof. To show that g is injective suppose that $g'(i,j) = g'(a,b)$ for some (a,b), $(i,j) \in V$. Therefore, we have

$$jm + in \equiv bm + an \pmod{mn}. \tag{1}$$

Rearranging this equation gives $(j-b)m + (i-a)n \equiv 0 \pmod{mn}$. For ease of notation, let $x = j - b$ and $y = i - a$. Then since $|x| \leq n-1$, $|y| \leq m-1$, and $xm + yn \equiv 0 \pmod{mn}$, we have that $xm + yn = kmn$ for some $k \in \{-1, 0, 1\}$. Suppose $k = \pm 1$. Then $|y| = |i-a| = \frac{m(n-x)}{n} \in \mathbb{Z}$ if and only if $x = 0$ since $n \nmid m$ by assumption (recall $d < \min\{m,n\}$). But if $x = 0$, then $yn = \pm mn$, but this is impossible since $|y| < m$. Hence, $xm + yn = 0$. Then dividing by d, we have $x\frac{m}{d} + y\frac{n}{d} = 0$. Since $\frac{m}{d}$ and $\frac{n}{d}$ are relatively prime, the solutions have the form $(x,y) = \left(\frac{n}{d}r, -\frac{m}{d}r\right)$, $\forall r \in [d]$. We have now established that there are exactly d ordered pairs in V which have the same value under g'. This means that in order for g to be a bijection, we must show that $\{\mathcal{R}(y_r - x_r) : r \in [d]\} = [d]$. To this end, observe that $\mathcal{R}(y_r - x_r) \equiv (y_r - x_r) \equiv -\frac{m}{d}r - \frac{n}{d}r \equiv -r\lambda \pmod{d}$ for each

$r \in [d]$. Since $\gcd(\lambda, d) = 1$, we have $\langle \lambda \rangle \cong \mathbb{Z}_d$, hence $\langle -\lambda \rangle \cong \mathbb{Z}_d$. Therefore, $\{\mathcal{R}(y_r - x_r) : r \in [d]\} = [d]$, so g is an injection, hence bijection. $\square$

Lemma 9. *If* $\gcd(\lambda, d) > 1$, *let* $k = 1$ *if* $\alpha^2 \nmid d$ *and let* $k = 2$ *if* $\alpha^2 \mid d$. *Then the mapping* $g_{\alpha^k} : V \longmapsto \mathbb{Z}_{mn}$ *given by* $g_{\alpha^k}(i, j) = jm + in\frac{d}{\alpha^k} + \mathcal{R}(j - i)$, *is a bijection.*

Proof. Suppose that $g'_{\alpha^k}(i, j) = g'_{\alpha^k}(a, b)$ for some (i, j), $(a, b) \in V$. Then we have $jm + in\frac{d}{\alpha^k} \equiv bm + an\frac{d}{\alpha^k} \pmod{mn}$. Letting $t = j - b$, and $u = i - a$, dividing by d, and observing that $\alpha^k \mid n$ gives

$$t\frac{m}{d} + u\frac{n}{\alpha^k} \equiv 0 \pmod{\frac{m}{d}n}. \tag{2}$$

Now observe that $\gcd\left(\frac{m}{d}, d\right) = \alpha$ implies $\alpha^2 \mid m$. Therefore, $\alpha \nmid \frac{n}{\alpha^k}$ since otherwise, $\alpha \mid \frac{n}{\alpha^k}$ implies $\alpha^{k+1} \mid n$. Then if $k = 1$, we have $\alpha^2 \mid n$ and $\alpha^2 \nmid d$ implies that $\alpha \mid \frac{n}{d}$ which in turn implies $\gcd(\frac{m}{d}, \frac{n}{d}) > 1$, contradicting the assumption, $\gcd(m, n) = d$. While if $k = 2$, we have $\alpha^3 \mid n$ implies $\gcd(\frac{m}{d}, \frac{n}{\alpha^2}) > 1$, a contradiction of Observation 7. Then since $\alpha \mid \frac{m}{d}$ but $\alpha \nmid \frac{n}{\alpha^k}$, we have that $\alpha \mid u$ from (2). But also, $\frac{m}{d} \mid u\frac{n}{\alpha^k}$. By (2) and Observations 6 and 7, $\gcd\left(\frac{m}{d}, \frac{n}{\alpha^k}\right) = 1$ which implies $\frac{m}{d} \mid u$. Therefore, both α and $\frac{m}{d}$ must divide u. Similarly, $\frac{n}{\alpha^k}$ must divide $t\frac{m}{d}$, which implies that $\frac{n}{\alpha^k} \mid t$. This allows us to provide a full description of the pairs (u, t) satisfying (2). Let S be the set of all such pairs. Then for all $p \in \left[\frac{d}{\alpha^k}\right]$, we have

$$\begin{aligned} S &= \{(\tfrac{m}{d}\alpha^k p, 0), (\tfrac{m}{d}\alpha^k p - \tfrac{m}{d}, \tfrac{n}{\alpha^k}), (\tfrac{m}{d}\alpha^k p - 2\tfrac{m}{d}, \tfrac{2n}{\alpha^k}), \\ &\quad \ldots, (\tfrac{m}{d}\alpha^k p - (\alpha^k - 1)\tfrac{m}{d}, \tfrac{(\alpha^k - 1)n}{\alpha^k})\}. \end{aligned}$$

Note that there are exactly $\alpha^k \cdot \frac{d}{\alpha^k} = d$ pairs in S. Therefore, we have established that exactly d ordered pairs in V share the same value under g'_{α^k}. Now it remains to show that these ordered pairs have distinct values under $\mathcal{R}$. For ease of notation, let $x = \mathcal{R}(-\frac{m}{d}\alpha^k)$, $y = \mathcal{R}(\frac{n}{\alpha^k})$, and $z = \mathcal{R}(\frac{m}{d})$. Furthermore, let $H = \langle x \rangle \leqslant \mathbb{Z}_d$. Then, $|H| = ord_{\mathbb{Z}_d}(x) = \frac{d}{\gcd(x,d)} = \frac{d}{\alpha^k}$, by Observations 6 and 7. Applying $\mathcal{R}$ to each member of S defines the multiset,

$$\mathcal{R}(S) = \left\{H + 0, H + (y + z), H + 2(y + z), \ldots, H + (\alpha^k - 1)(y + z)\right\}.$$

It remains to show that the cosets of H in $\mathcal{R}(S)$ partition $[d]$. First observe that $y + z \not\equiv 0 \pmod{d}$ since otherwise we have $\alpha \mid \frac{n}{\alpha^k}$ which we have already established is a contradiction. Secondly, suppose $(y + z) \in H$. Then $\frac{n}{\alpha^k} + \frac{m}{d} \equiv -\frac{m}{d}\alpha^k q \pmod{d}$ for some $q \in \left[\frac{d}{\alpha^k}\right]$. But since $\alpha \mid \frac{m}{d}$, it must be the case that $\alpha \mid \frac{n}{\alpha^k}$, which leads to the same contradiction as before. Therefore, $(y + z) \notin H$. Hence $\mathcal{R}(S) = [d]$, and so g_{α^k} is an injection, hence bijection. $\square$

Lemma 10. *Let* m *be even and* n *be odd. If* $\gcd(\lambda, d) = 1$, *then the mapping* $g : V \longmapsto \mathbb{Z}_{mn}$ *given by* $g(i, j) = \begin{cases} jm + in + \mathcal{R}(j - i), & i \text{ even} \\ (j - 1)m + (i - 1)n + d + \mathcal{R}(j - i), & i \text{ odd} \end{cases}$ *is a bijection.*

Proof. Suppose that $g'(i,j) = g'(a,b)$ for some (i,j), $(a,b) \in V$. It cannot be the case that i and a have different parities. For the sake of contradiction, suppose i is even and a is odd. Then we have $jm + in \equiv d + (b-1)m + (a-1)n \pmod{mn}$. Therefore, $(j-b+1)m + (i-a+1)n \equiv d \pmod{mn}$. But this is a contradiction since $(j-b+1)m$ and $(i-a+1)n$ are both even and d is necessarily odd. So it cannot be the case that i is even and a is odd. Essentially the same argument shows it cannot be the case that i is odd and a is even. Therefore, i and a must be of the same parity. If i and a are both even, then $g'(i,j) = g'(a,b)$ implies equation (1) from Lemma 8, while if i and a are both odd, then we have $d + (j-1)m + (i-1)n \equiv d + (b-1)m + (a-1)n \pmod{mn}$, which also is equivalent with (1). Thus g is a bijection by the same argument as in Lemma 8. □

Lemma 11. *Let m be even and n be odd. If $\gcd(\lambda, d) > 1$, let $k = 1$ when $\alpha^2 \nmid d$, and let $k = 2$ when $\alpha^2 \mid d$. Then the mapping $g_{\alpha^k} : V \longmapsto \mathbb{Z}_{mn}$ given by* $g_{\alpha^k}(i,j) = \begin{cases} jm + in\frac{d}{\alpha^k} + \mathcal{R}(j-i), \ i \,\text{even} \\ (j-1)m + (i-1)n\frac{d}{\alpha^k} + d + \mathcal{R}(j-i), \ i \,\text{odd} \end{cases}$ *is a bijection.*

Proof. Suppose that $g'_{\alpha^k}(i,j) = g'_{\alpha^k}(a,b)$ for some (i,j), $(a,b) \in V$. As in Lemma 10, i and a must be of the same parity. If i and a are both even, then necessarily $jm + in\frac{d}{\alpha^k} \equiv bm + an\frac{d}{\alpha^k} \pmod{mn}$. Whereas, if i and a are both odd, then we have that $d + (j-1)m + (i-1)n\frac{d}{\alpha^k} \equiv d + (b-1)m + (a-1)n\frac{d}{\alpha^k} \pmod{mn}$. However, letting $t = j - b$, $u = i - a$, dividing by d, and observing that $\alpha^k \mid n$, we see that both equations are equivalent to (2) from Lemma 9. Hence in either case, g_{α^k} is a bijection by the same argument used in Lemma 9. □

In the next three lemmas, assume m and n are even. Then let $V_2 = \{(i,j) \in V : i \equiv j \pmod 2\} \subseteq V$. Let $2\mathbb{Z}_{mn} = \{2h : h \in \mathbb{Z}_{mn}\}$ denote the subgroup of $\mathbb{Z}_{mn}$ consisting of the even integers contained in $\mathbb{Z}_{mn}$. Similarly, let $2[d] = \{2h : h \in \mathbb{Z}_d\}$. Also note that since m and n are both even, then at most one of $\frac{m}{d}$ and $\frac{n}{d}$ may be even. So assume without loss of generality that $\frac{n}{d}$ is always odd.

Lemma 12. *Let m and n be even. If $\gcd(\lambda, d) = 1$, then the mapping $g : V_2 \longmapsto 2\mathbb{Z}_{mn}$ given by*

$$g(i,j) = \begin{cases} jm + in + \mathcal{R}(j-i), \text{ for } i \equiv j \equiv 0 \pmod 2 \\ (j-1)m + (i-1)n + d + \mathcal{R}(j-i), \text{ for } i \equiv j \equiv 1 \pmod 2 \end{cases}$$

is a bijection.

Proof. Suppose $g'(i,j) = g'(a,b)$ for some (i,j), $(a,b) \in V$. Observe that it cannot be the case that i, j are both even and a, b are both odd, since otherwise $(j - b + 1)m + (i - a + 1)n = kmn + d$ for some integer k would imply $(j-b+1)\frac{m}{d} + (i-a+1)\frac{n}{d} = \frac{kmn}{d} + 1$, a contradiction since the left hand side of the equation is even and the right hand side is odd (recall that $\frac{n}{d}$ is odd). For the same reason, it cannot be the case that i, j are both odd while a, b are both even. Therefore, i, j, a, and b are all of the same parity. Consequently, $g'(i,j) = g'(a,b)$ implies equation (1) from Lemma 8. With no restriction on the parities of $x = j - b$ and $y = i - a$, this equation was found to have the d solutions $(x_r, y_r) = \left(\frac{n}{d}r, -\frac{m}{d}r\right)$ for each $r \in [d]$ in the proof of Lemma 8. However, in the present case we require that x and y both be even. Recall that $\frac{n}{d}$ is odd. Therefore, the $\frac{d}{2}$ solutions to (1) are $(x_r, y_r) = \left(\frac{n}{d}2r, -\frac{m}{d}2r\right)$ for each $r \in \left[\frac{d}{2}\right]$. We have now established that there are exactly $\frac{d}{2}$ ordered pairs in V_2 having the same value under g'. This means that in order for g to be a bijection, we must show that the set $\left\{\mathcal{R}(y_r - x_r) : r \in \left[\frac{d}{2}\right]\right\} = 2[d]$. To this end, observe $\mathcal{R}(y_r - x_r) \equiv (y_r - x_r) \equiv -\frac{m}{d}2r - \frac{n}{d}2r \equiv -2r\lambda \pmod{d}$ for each $r \in \left[\frac{d}{2}\right]$. Since $\gcd(\lambda, d) = 1$, we have $\langle\lambda\rangle \cong \mathbb{Z}_d$, hence $\langle-\lambda\rangle \cong \mathbb{Z}_d$. Therefore, $\left\{\mathcal{R}(x_r, y_r) : r \in \left[\frac{d}{2}\right]\right\} = 2[d]$. Therefore, the $\frac{d}{2}$ ordered pairs of V_2 having the same value under g' have distinct and even values under $\mathcal{R}$. Hence, $g : V_2 \longmapsto 2\mathbb{Z}_{mn}$ is an injection, hence bijection. □

Lemma 13. *Let m and n both be even. If $\gcd(\lambda, d) > 1$, let $k = 1$ when $\alpha^2 \nmid d$, and let $k = 2$ when $\alpha^2 \mid d$. Then the mapping $g_{\alpha^k} : V_2 \longmapsto 2\mathbb{Z}_{mn}$ given by*

$$g_{\alpha^k}(i,j) = \begin{cases} jm + in\frac{d}{\alpha^k} + \mathcal{R}(j-i), & \text{for } i \equiv j \equiv 0 \pmod 2 \\ (j-1)m + (i-1)n\frac{d}{\alpha^k} + d + \mathcal{R}(j-i), & \text{for } i \equiv j \equiv 1 \pmod 2 \end{cases}$$

is a bijection.

Proof. Suppose $g'_{\alpha^k}(i,j) = g'_{\alpha^k}(a,b)$ for some $(i,j), (a,b) \in V$. As in Lemma 12, it must be the case that i, j, a, and b are all of the same parity. Then letting $u = i - a$, $t = j - b$, dividing by d, and observing that $\alpha^k \mid n$ we have that $g'_{\alpha^k}(i,j) = g'_{\alpha^k}(a,b)$ implies equation (2) from Lemma 9. With no restriction on the parities of u and t, we observed in the proof of Lemma 9 that a full description of the d pairs (u,t) satisfying (2) is given by

$$\begin{aligned} S \;=\; & \{(\tfrac{m}{d}\alpha^k p, 0), (\tfrac{m}{d}\alpha^k p - \tfrac{m}{d}, \tfrac{n}{\alpha^k}), (\tfrac{m}{d}\alpha^k p - 2\tfrac{m}{d}, \tfrac{2n}{\alpha^k}), \\ & \ldots, (\tfrac{m}{d}\alpha^k p - (\alpha^k - 1)\tfrac{m}{d}, \tfrac{(\alpha^k-1)n}{\alpha^k})\}, \end{aligned}$$

for all $p \in \left[\frac{d}{\alpha^k}\right]$. However, in this case we are restricted to the pairs in S such that u and t are both even.

If $\frac{m}{d}$ is odd, then $gcd(\frac{m}{d}, d) = \alpha$ is odd, since d is even. So α^k is also odd, and hence $\frac{n}{\alpha^k}$ is even, since n is even. Then for all $p \in \{0, 2, \ldots, \frac{d}{\alpha^k}\}$ and all $l \in \{1, 3, \ldots, \frac{d}{\alpha^k} - 1\}$, $S_1 \subset S$ where

$$\begin{aligned} S_1 \;=\; & \{(\tfrac{m}{d}\alpha^k p, 0), (\tfrac{m}{d}\alpha^k l - \tfrac{m}{d}, \tfrac{n}{\alpha^k}), (\tfrac{m}{d}\alpha^k p - 2\tfrac{m}{d}, \tfrac{2n}{\alpha^k}), \\ & \ldots., (\tfrac{m}{d}\alpha^k l - (\alpha^k - 2)\tfrac{m}{d}, \tfrac{(\alpha^k-2)n}{\alpha^k}), (\tfrac{m}{d}\alpha^k p - (\alpha^k - 1)\tfrac{m}{d}, \tfrac{(\alpha^k-1)n}{\alpha^k})\}, \end{aligned}$$

is the full set of $\frac{d}{2}$ solutions to (2) in this case.

On the other hand, if $\frac{m}{d}$ is even we have $\gcd(\frac{m}{d}, d) = \alpha$ is even, so α^k is also even. Then since $\gcd\left(\frac{m}{d}, \frac{d}{\alpha^k}\right) = 1$ by Observations 6 and 7, we have that $\frac{d}{\alpha^k}$ is odd and hence $\frac{n}{\alpha^k} = \frac{d}{\alpha^k} \cdot \frac{n}{d}$ is odd, since $\frac{n}{d}$ is odd. Then for all $p \in \left[\frac{d}{\alpha^k}\right]$, $S_2 \subset S$ where

$$\begin{aligned} S_2 \;=\; & \{(\tfrac{m}{d}\alpha^k p, 0), (\tfrac{m}{d}\alpha^k p - 2\tfrac{m}{d}, \tfrac{2n}{\alpha^k}), (\tfrac{m}{d}\alpha^k p - 4\tfrac{m}{d}, \tfrac{4n}{\alpha^k}), \\ & \ldots, (\tfrac{m}{d}\alpha^k p - (\alpha^k - 2)\tfrac{m}{d}, \tfrac{(\alpha^k-2)n}{\alpha^k})\}, \end{aligned}$$

is the full set of $\frac{d}{2}$ solutions to (2) in this case. Therefore, in either case we have established that exactly $\frac{d}{2}$ ordered pairs in V_2 share the same value under g'_{α^k}. Now we will show that these ordered pairs have distinct values under $\mathcal{R}$. We have already observed that $\mathcal{R}(S) = [d]$ in the proof of Lemma 9. Then since $S_1, S_2 \subseteq S$ and $\frac{|S_1|}{|S|} = \frac{|S_2|}{|S|} = \frac{1}{2}$ and both S_1 and S_2 contain ordered pairs of the form (u, t) where u and t are both even, we conclude that $\mathcal{R}(S_1) = \mathcal{R}(S_2) = 2[d]$. Hence, the mapping $g_{\alpha^k} : V_2 \longmapsto 2\mathbb{Z}_{mn}$ is an injection, hence bijection. $\square$

3 Cartesian product of two cycles

We begin with a construction for the case when one cycle length is a multiple of the other.

Theorem 14. *The Cartesian product $C_m \square C_{km}$ is orientable $\mathbb{Z}_{km^2}$-distance magic for all $m \geq 3$ and $k \geq 1$.*

Proof. Let $G = C_m = \{g_0, g_1, \ldots, g_{m-1}, g_0\}$ and $H = C_{km} = \{h_0, h_1, \ldots, h_{km-1}, h_0\}$. Then orient each copy of $G \square H$ as follows. Fix $j \in [km]$. Then for all $i \in [m]$, orient counter-clockwise each cycle of the form $\{(g_i, h_j), (g_{i+1}, h_j), \ldots, (g_{i-1}, h_j), (g_i, h_j)\}$, where the arithmetic in the subscript is performed modulo m. Similarly, fix $i \in [m]$. Then for all $j \in [km]$, orient counter-clockwise each cycle of the form

$$\{(g_i, h_j), (g_i, h_{j+1}), \ldots, (g_i, h_{j-1}), (g_i, h_j)\},$$

where the arithmetic in the subscript is performed modulo km. Since the graph $G \square H$ can be edge-decomposed into cycles of those two forms, we have oriented every edge in $G \square H$. Let x_i^j denote the vertex $(g_i, h_j) \in V(G \square H)$ for $i \in [m]$, $j \in [km]$. Define $\overrightarrow{l} : V \to \mathbb{Z}_{km^2}$ by

$$\vec{l}(x_i^j) = mj + \mathcal{R}(i - j).$$

Expressing $\vec{l}(x_i^j)$ in the following alternative way,

$$\vec{l}(x_i^j) = \begin{cases} mj, & \text{for } i \equiv j \pmod{m} \\ mj + 1, & \text{for } i \equiv j + 1 \pmod{m} \\ mj + 2, & \text{for } i \equiv j + 2 \pmod{m} \\ \vdots & \vdots \\ mj + (m-1), & \text{for } i \equiv j - 1 \pmod{m} \end{cases},$$

we see that $\vec{l}$ is clearly bijective.

Then for all x_i^j we have $N^+\left(x_i^j\right) = \{x_i^{j+1}, x_{i+1}^j\}$ and $N^-\left(x_i^j\right) = \{x_i^{j-1}, x_{i-1}^j\}$ where the arithmetic is performed modulo km in the superscript and modulo m in the subscript. Therefore,

$$\begin{aligned} w(x_i^j) &= \vec{l}(x_i^{j+1}) + \vec{l}(x_{i+1}^j) - [\vec{l}(x_{i-1}^j) + \vec{l}(x_i^{j-1})] \\ &= m(j+1) + mj - mj - m(j-1) \\ &\quad + [\mathcal{R}(i-j-1) + \mathcal{R}(i-j+1) - \mathcal{R}(i-j-1) - \mathcal{R}(i-j+1)] \\ &= \begin{cases} m(2 - km), \ j \in \{0, km-1\} \\ m \cdot 2, \ j \in \{1, \ldots, km-2\} \end{cases} \\ &= 2m, \end{aligned}$$

since $m(2-km) \equiv 2m \pmod{km^2}$.

Thus, $\vec{l}$ is an orientable $\mathbb{Z}_{km^2}$-distance magic labeling. □

Each case in the proof of the next theorem uses a directed labeling which was shown to be a bijection from the vertex set of the graph to the appropriate group in Section 2.

Theorem 15. *The Cartesian product $C_m \square C_n$ is orientable $\mathbb{Z}_{mn}$-distance magic for all $m, n \geq 3$.*

Proof. Let $m, n \geq 3$ be given and let $\gcd(m, n) = d$. Then define $\lambda = \frac{m+n}{d}$ and let $\gcd(\frac{m}{d}, d) = \alpha$. If $d = 1$, we are done by Theorem 5. If $d = \min\{m, n\}$ (i.e. one cycle length is a multiple of the other), we are done by Theorem 14. So assume $1 < d < min\{m, n\}$. Notice this is the same assumption used to prove the lemmas in Section 2. Let $G = C_m = \{g_0, g_1, \ldots, g_{m-1}, g_0\}$ and $H = C_n = \{h_0, h_1, \ldots, h_{n-1}, h_0\}$. Then orient each copy of $G \square H$ as in Theorem 14. Now let x_i^j denote the vertex $(g_i, h_j) \in V(G \square H)$ for all $i \in [m]$ and $j \in [n]$. We proceed in three cases based on the parity of m and n. Since we will define a different directed labeling for each case, we first pause to make the following observation. For any directed labeling $\vec{l}: V(G \square H) \to \mathbb{Z}_{mn}$ of $\overrightarrow{G \square H}$, we have

$$\begin{aligned} w(x_i^j) &= \sum_{y \in N^+(x_i^j)} \vec{l}(y) - \sum_{y \in N^-(x_i^j)} \vec{l}(y) \\ &= \vec{l}(x_i^{j+1}) + \vec{l}(x_{i+1}^j) - [\vec{l}(x_{i-1}^j) + \vec{l}(x_i^{j-1})], \end{aligned}$$

for every vertex $x_i^j \in V(G \square H)$, where the arithmetic is performed modulo n in the superscript and modulo m in the subscript. However, it should be emphasized that the weight calculation is performed in the group $\mathbb{Z}_{mn}$.

Case 1.1. m and n both odd and $\gcd(\lambda, d) = 1$.

For all $x_i^j \in V(G \square H)$, define $\vec{l}: V(G \square H) \to \mathbb{Z}_{mn}$ where

$$\vec{l}\left(x_i^j\right) = jm + in + \mathcal{R}\left(j - i\right).$$

Then $\vec{l}$ is a bijection by Lemma 8 and

$$\begin{aligned} w(x_i^j) &= (j+1)m + in + jm + (i+1)n \\ &- [jm + (i-1)n + (j-1)m + in] \\ &+ \mathcal{R}(j-i+1) - \mathcal{R}(j-i+1) + \mathcal{R}(j-i-1) - \mathcal{R}(j-i-1) \\ &= sm + rn, \end{aligned}$$

where $s \in \{2, 2-n\}$ and $r \in \{2, 2-m\}$. But, $sm + rn \equiv 2m + 2n \pmod{mn}$, so

$$w(x_i^j) = 2m + 2n.$$

In the remaining cases we will omit the equality involving s and r above and only show the final congruence modulo mn.

Case 1.2. m and n both odd and $\gcd(\lambda, d) > 1$.

Let $k = 1$ when $\alpha^2 \nmid d$ and let $k = 2$ when $\alpha^2 \mid d$. Then for all $x_i^j \in V(G\Box H)$, define $\vec{l_{\alpha^k}} : V(G\Box H) \to \mathbb{Z}_{mn}$ where

$$\vec{l_{\alpha^k}}\left(x_i^j\right) = jm + in\frac{d}{\alpha^k} + \mathcal{R}\left(j - i\right).$$

Then $\vec{l_{\alpha^k}}$ is a bijection by Lemma 9 and

$$\begin{array}{rcl} w(x_i^j) & = & (j+1)m + in\frac{d}{\alpha^k} + jm + (i+1)n\frac{d}{\alpha^k} \\ & - & \left[jm + (i-1)n\frac{d}{\alpha^k} + (j-1)m + in\frac{d}{\alpha^k}\right] \\ & + & \mathcal{R}\left(j-i+1\right) - \mathcal{R}\left(j-i+1\right) + \mathcal{R}\left(j-i-1\right) - \mathcal{R}\left(j-i-1\right) \\ & = & 2m + 2n\frac{d}{\alpha^k}\,. \end{array}$$

Suppose exactly one of m and n is odd. Since the Cartesian product is commutative, we may assume without loss of generality that m is even and n is odd. Then as in the previous case, $\gcd(\lambda, d)$ establishes two subcases.

Case 2.1. m even, n odd, and $\gcd(\lambda, d) = 1$.

For all $x_i^j \in V(G\Box H)$, define $\vec{l} : V(G\Box H) \to \mathbb{Z}_{mn}$ where

$$\vec{l}(x_i^j) = \begin{cases} jm + in + \mathcal{R}(j-i), & i \text{ even} \\ (j-1)m + (i-1)n + d + \mathcal{R}(j-i), & i \text{ odd} \end{cases}.$$

Then $\vec{l}$ is a bijection by Lemma 10, and if i is even we have

$$\begin{array}{rcl} w(x_i^j) & = & (j+1)m + in + (j-1)m + in + d \\ & - & [(j-1)m + (i-2)n + d + (j-1)m + in] \\ & + & \mathcal{R}\left(j-i+1\right) - \mathcal{R}\left(j-i+1\right) + \mathcal{R}\left(j-i-1\right) - \mathcal{R}\left(j-i-1\right) \\ & = & 2m + 2n. \end{array}$$

While if i is odd we have

$$\begin{array}{rcl} w(x_i^j) & = & jm + (i-1)n + d + jm + (i+1)n \\ & - & [jm + (i-1)n + (j-2)m + (i-1)n + d] \\ & + & \mathcal{R}\left(j-i+1\right) - \mathcal{R}\left(j-i+1\right) + \mathcal{R}\left(j-i-1\right) - \mathcal{R}\left(j-i-1\right) \\ & = & 2m + 2n. \end{array}$$

Case 2.2. m even, n odd, and $\gcd(\lambda, d) > 1$.

As in Case 1.2, let $k = 1$ when $\alpha^2 \nmid d$ and let $k = 2$ when $\alpha^2 \mid d$. For all $x_i^j \in V(G\Box H)$, define $\vec{l_{\alpha^k}} : V(G\Box H) \to \mathbb{Z}_{mn}$ where

$$\vec{l_{\alpha^k}}(x_i^j) = \begin{cases} jm + in\frac{d}{\alpha^k} + \mathcal{R}(j-i), & i \text{ even} \\ (j-1)m + (i-1)n\frac{d}{\alpha^k} + d + \mathcal{R}(j-i), & i \text{ odd} \end{cases}.$$

Then $\vec{l_{\alpha^k}}$ is a bijection by Lemma 11, and if i is even we have

$$\begin{array}{rcl} w(x_i^j) & = & (j+1)m + in\frac{d}{\alpha^k} + (j-1)m + in\frac{d}{\alpha^k} \\ & - & \left[(j-1)m + (i-2)n\frac{d}{\alpha^k} + d + (j-1)m + in\frac{d}{\alpha^k}\right] \\ & + & \mathcal{R}\left(j-i+1\right) - \mathcal{R}\left(j-i+1\right) + \mathcal{R}\left(j-i-1\right) - \mathcal{R}\left(j-i-1\right) \\ & = & 2m + 2n\frac{d}{\alpha^k}\,. \end{array}$$

While if i is odd we have

$$\begin{aligned} w(x_i^j) &= jm + (i-1)n\tfrac{d}{\alpha^k} + d + jm + (i+1)n\tfrac{d}{\alpha^k} \\ &- \left[jm + (i-1)n\tfrac{d}{\alpha^k} + (j-2)m + (i-1)n\tfrac{d}{\alpha^k} + d\right] \\ &+ \mathcal{R}(j-i+1) - \mathcal{R}(j-i+1) + \mathcal{R}(j-i-1) - \mathcal{R}(j-i-1) \\ &= 2m + 2n\tfrac{d}{\alpha^k}\,. \end{aligned}$$

Now suppose that both m and n are even. Since $\gcd(m,n) = d$, at most one of $\frac{m}{d}$ and $\frac{n}{d}$ is even. Since the Cartesian product is commutative, if one of $\frac{m}{d}$ and $\frac{n}{d}$ is even, we may assume without loss of generality that $\frac{n}{d}$ is odd. Then as in the previous cases, $\gcd(\lambda, d)$ establishes two subcases.

Case 3.1. $\frac{m}{d}$ even, $\frac{n}{d}$ odd, and $\gcd(\lambda, d) = 1$.

For all $x_i^j \in V(G\Box H)$, define $\vec{f} : V(G\Box H) \to \mathbb{Z}_{mn}$ and $\vec{l} : V(G\Box H) \to \mathbb{Z}_{mn}$ where

$$\vec{f}(x_i^j) = \begin{cases} jm + in + \mathcal{R}(j-i), \text{ for } i \equiv j \equiv 0 \pmod 2 \\ (j-1)m + (i-1)n + d + \mathcal{R}(j-i), \text{ for } i \equiv j \equiv 1 \pmod 2 \end{cases},$$

and

$$\vec{l}(x_i^j) = \begin{cases} \vec{f}(x_i^j), \text{ for } i \equiv j \pmod 2 \\ \vec{f}(x_i^{j-1}) + 1, \text{ for } i \not\equiv j \pmod 2 \end{cases}.$$

By Lemma 12, $\vec{f}$ maps the vertices $\{x_i^j : i \equiv j \pmod 2\}$ bijectively to $2\mathbb{Z}_{mn}$. Then clearly $\vec{f} + 1$ maps the vertices $\{x_i^{j-1} : i \not\equiv j \pmod 2\}$ bijectively to $2\mathbb{Z}_{mn} + 1$. Therefore, $\vec{l} : V(G\Box H) \to \mathbb{Z}_{mn}$ is a bijection since $\mathbb{Z}_{mn} \cong 2\mathbb{Z}_{mn} \cup 2\mathbb{Z}_{mn} + 1$. If $i \equiv j \pmod 2$ we have

$$\begin{aligned} w(x_i^j) &= \vec{l}(x_i^{j+1}) + \vec{l}(x_{i+1}^j) - [\vec{l}(x_{i-1}^j) + \vec{l}(x_i^{j-1})] \\ &= \vec{f}(x_i^j) + 1 + \vec{f}(x_{i+1}^{j-1}) + 1 - \left[\vec{f}(x_{i-1}^{j-1}) + 1 + \vec{f}(x_i^{j-2}) + 1\right] \\ &= \vec{f}'(x_i^j) + \vec{f}'(x_{i+1}^{j-1}) - \left[\vec{f}'(x_{i-1}^{j-1}) + \vec{f}'(x_i^{j-2})\right] \\ &+ \mathcal{R}(j-i) + \mathcal{R}(j-i-2) - \mathcal{R}(j-i) - \mathcal{R}(j-i-2) \\ &= 2m + 2n. \end{aligned}$$

While if $i \not\equiv j \pmod 2$ we have

$$\begin{aligned} w(x_i^j) &= \vec{l}(x_i^{j+1}) + \vec{l}(x_{i+1}^j) - [\vec{l}(x_{i-1}^j) + \vec{l}(x_i^{j-1})] \\ &= \vec{f}(x_i^{j+1}) + \vec{f}(x_{i+1}^j) - \left[\vec{f}(x_{i-1}^j) + \vec{f}(x_i^{j-1})\right] \\ &= \vec{f}'(x_i^{j+1}) + \vec{f}'(x_{i+1}^j) - \left[\vec{f}'(x_{i-1}^j) + \vec{f}'(x_i^{j-1})\right] \\ &+ \mathcal{R}(j-i+1) + \mathcal{R}(j-i-1) - \mathcal{R}(j-i+1) - \mathcal{R}(j-i-1) \\ &= 2m + 2n. \end{aligned}$$

Figure 3 provides an example of this case.

Case 3.2. $\frac{m}{d}$ even, $\frac{n}{d}$ odd, and $\gcd(\lambda, d) > 1$.

As in the previous cases, let $k = 1$ when $\alpha^2 \nmid d$, let $k = 2$ when $\alpha^2 \mid d$, and for all $x_i^j \in V(G\Box H)$, define $\vec{f}_{\alpha^k} : V(G\Box H) \to \mathbb{Z}_{mn}$ and $\vec{l}_{\alpha^k} : V(G\Box H) \to \mathbb{Z}_{mn}$ where

$$\vec{f}_{\alpha^k}(x_i^j) = \begin{cases} jm + in\frac{d}{\alpha^k} + \mathcal{R}(j-i), \text{ for } i \equiv j \equiv 0 \pmod 2 \\ (j-1)m + (i-1)n\frac{d}{\alpha^k} + d + \mathcal{R}(j-i), \text{ for } i \equiv j \equiv 1 \pmod 2 \end{cases},$$

and

$$\vec{l}_{\alpha^k}(x_i^j) = \begin{cases} \vec{f}_{\alpha^k}(x_i^j), \text{ for } i \equiv j \pmod 2 \\ \vec{f}_{\alpha^k}(x_i^{j-1}) + 1, \text{ for } i \not\equiv j \pmod 2 \end{cases}.$$

By Lemma 13 and essentially the same argument used in Case 3.1, we conclude that $\vec{l}_{\alpha^k} : V(G\Box H) \to \mathbb{Z}_{mn}$ is a bijection. Finally, if $i \equiv j \pmod 2$ we have

$$\begin{aligned} w(x_i^j) &= \vec{l}_{\alpha^k}(x_i^{j+1}) + \vec{l}_{\alpha^k}(x_{i+1}^j) - [\vec{l}_{\alpha^k}(x_{i-1}^j) + \vec{l}_{\alpha^k}(x_i^{j-1})] \\ &= \vec{f}_{\alpha^k}(x_i^j) + 1 + \vec{f}_{\alpha^k}(x_{i+1}^{j-1}) + 1 - \left[\vec{f}_{\alpha^k}(x_{i-1}^{j-1}) + 1 + \vec{f}_{\alpha^k}(x_i^{j-2}) + 1\right] \\ &= \vec{f'}_{\alpha^k}(x_i^j) + \vec{f'}_{\alpha^k}(x_{i+1}^{j-1}) - \left[\vec{f'}_{\alpha^k}(x_{i-1}^{j-1}) + \vec{f'}_{\alpha^k}(x_i^{j-2})\right] \\ &+ \mathcal{R}(j-i) + \mathcal{R}(j-i-2) - \mathcal{R}(j-i) - \mathcal{R}(j-i-2) \\ &= 2m + 2n\frac{d}{\alpha^k}. \end{aligned}$$

While if $i \not\equiv j \pmod 2$ we have

$$\begin{aligned} w(x_i^j) &= \vec{l}_{\alpha^k}(x_i^{j+1}) + \vec{l}_{\alpha^k}(x_{i+1}^j) - [\vec{l}_{\alpha^k}(x_{i-1}^j) + \vec{l}_{\alpha^k}(x_i^{j-1})] \\ &= \vec{f}_{\alpha^k}(x_i^{j+1}) + \vec{f}_{\alpha^k}(x_{i+1}^j) - \left[\vec{f}_{\alpha^k}(x_{i-1}^j) + \vec{f}_{\alpha^k}(x_i^{j-1})\right] \\ &= \vec{f'}_{\alpha^k}(x_i^{j+1}) + \vec{f'}_{\alpha^k}(x_{i+1}^j) - \left[\vec{f'}_{\alpha^k}(x_{i-1}^j) + \vec{f'}_{\alpha^k}(x_i^{j-1})\right] \\ &+ \mathcal{R}(j-i+1) + \mathcal{R}(j-i-1) - \mathcal{R}(j-i+1) - \mathcal{R}(j-i-1) \\ &= 2m + 2n\frac{d}{\alpha^k}. \end{aligned}$$

In every case, $w(x_i^j)$ is constant for all $x_i^j \in V(G\Box H)$. Hence, $C_m\Box C_n$ is orientable $\mathbb{Z}_{mn}$-distance magic. □

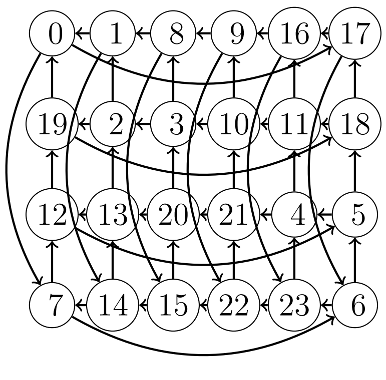

Figure 3: Orientable $\mathbb{Z}_{24}$-distance magic labeling of $C_4\Box C_6$

4 Future work

We have shown that the Cartesian product of any two cycles is orientable $\mathbb{Z}_n$-distance magic. In this paper, we fix the orientation of our cycles. However, because any cycle can be oriented in a non-unique way, it would be interesting to see if there are other orientations for which an accompanying orientable Z_n-distance magic labeling could be found.

Hypercubes are an important class of graphs which can be constructed using the Cartesian product of cycles. In [6] we showed the repeated Cartesian product of a cycle is orientable $\mathbb{Z}_n$-distance magic, proving that even-ordered hypercubes are orientable $\mathbb{Z}_n$-distance magic. A natural direction forward is to generalize to the Cartesian product of many cycles of various lengths. We pose the following problem.

Problem 16. For what numbers $n_1, n_2, \ldots, n_k$ is the Cartesian product $C_{n_1} \square C_{n_2} \square \ldots \square C_{n_k}$ orientable $\mathbb{Z}_{n_1,n_2,\ldots,n_k}$-distance magic?

Another possibility for future work is to consider abelian groups other than the cyclic group.

Problem 17. Determine all abelian groups Γ such that $C_m \square C_n$ is orientable Γ-distance magic.

One may wonder whether an orientable $\mathbb{Z}_n$-distance magic graph G of order n is also $\mathbb{Z}_n$-distance magic, or vice versa. Since $\mathbb{Z}_n$-distance magic labeling is more restrictive than orientable $\mathbb{Z}_n$-distance magic labeling, intuitively it should not be the case that orientable $\mathbb{Z}_n$-distance magic implies $\mathbb{Z}_n$-distance magic. Indeed, Theorems 1 and 15 show that this is not the case.

But perhaps $\mathbb{Z}_n$-distance magic implies orientable $\mathbb{Z}_n$-distance magic. Theorems 3 and 4 indicate that the contrapositive checks for the case of odd regular graphs on $n \equiv 2 \pmod 4$ vertices. We pose the following conjecture.

Conjecture 18. *If a graph G of order n is $\mathbb{Z}_n$-distance magic, then it is orientable $\mathbb{Z}_n$-distance magic.*

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(Received 14 Jan 2017; revised 9 Sep 2017)