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Equitable block colourings for 8-cycle systems

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Abstract

Let $\Sigma = (X, \mathcal{B})$ be an 8-cycle system of order $v = 1+16k$. A c-colouring of type s is a map $\phi\colon \mathcal{B} \to \mathcal{C}$, with $\mathcal{C}$ set of colours, so that exactly c colours are used and for every vertex x all the blocks containing x are coloured with exactly s colours. Let $8k = qs + r$, with $q, r \geq 0$. The colouring ϕ is called *equitable* if for every vertex x the set of the $8k$ blocks containing x is partitioned into r colour classes of cardinality $q+1$ and $s-r$ colour classes of cardinality q. This paper deals with a study of bicolourings, tricolourings and quadricolourings with $s = 2, 3, 4$.

1 Introduction

Block colourings of 4-cycle systems have been introduced and studied in [3, 4, 7, 8], and in [1, 2] block colourings were also studied for 6-cycle systems and systems of 4-kites. The purpose of this paper is to study block colourings of 8-cycle systems.

Let K_v be the complete simple graph on v vertices. The graph on vertex set $\{a_1, a_2, \ldots, a_k\}$ with edge set $\{\{a_1, a_k\}, \{a_i, a_{i+1}\} \mid 1 \leq i \leq k\}$ is called a k-cycle, and it is denoted by $(a_1, a_2, \ldots, a_k)$. An n-cycle system of order v, briefly $nCS(v)$, is a pair $\Sigma = (X, \mathcal{B})$, where X is the set of vertices of K_v and $\mathcal{B}$ is a set of n-cycles, called *blocks*, that partitions the edges of K_v.

A colouring of an $nCS(v)$ $\Sigma = (X, \mathcal{B})$ is a mapping $\phi\colon \mathcal{B} \to \mathcal{C}$, where $\mathcal{C}$ is a set of colours. A c-colouring is a colouring where c colours are used. The set of blocks coloured with a colour of $\mathcal{C}$ is a *colour class*. A c-colouring of type s is a colouring in which, for every vertex x, all of the blocks containing x are coloured with s colours.

Let $\Sigma = (X, \mathcal{B})$ be an $nCS(v)$, let $\phi\colon \mathcal{B} \to \mathcal{C}$ be a c-colouring of type s, and let $\frac{v-1}{2} = qs + r$ with $q \geq 0$ and $0 \leq r < s$. Each vertex of an $nCS(v)$ is contained in $\frac{v-1}{2}$ blocks. The mapping ϕ is *equitable* if for every vertex x the set of the $\frac{v-1}{2}$ blocks containing x is partitioned into r colour classes of cardinality $q+1$ and $s-r$ colour classes of cardinality q. A bicolouring, tricolouring or quadricolouring is an equitable colouring of type 2, 3 or 4, respectively.

The colour spectrum of $\Sigma = (X, \mathcal{B})$ is the set:

$$\Omega_s^{(n)}(\Sigma) = \{c \mid \text{there exists a } c\text{-block-colouring of type } s \text{ of } \Sigma\}.$$

The focus of our study is the set:

$$\begin{aligned}\Omega_s^{(n)}(v) &= \bigcup \Omega_s^{(n)}(\Sigma) \\ &= \{c \mid \text{there exists a } c\text{-block-colouring of type } s \text{ of some } nCS(v)\ \Sigma\},\end{aligned}$$

where Σ varies in the set of all the $nCS(v)$.

The *lower s-chromatic index* is defined as:

$$\chi_s^{(n)}(\Sigma) = \min \Omega_s^{(n)}(\Sigma)$$

and the *upper s-chromatic index* is

$$\overline{\chi}_s^{(n)}(\Sigma) = \max \Omega_s^{(n)}(\Sigma).$$

If $\Omega_s^{(n)}(\Sigma) = \emptyset$, then we say that Σ is uncolourable.

In the same way we define

$$\chi_s^{(n)}(v) = \min \Omega_s^{(n)}(v) \quad \text{and} \quad \overline{\chi}_s^{(n)}(v) = \max \Omega_s^{(n)}(v).$$

Block colourings for $s = 2$, $s = 3$ and $s = 4$ of $4CS$ have been studied in [3, 7, 8]. The problem arose as a consequence of colourings of Steiner systems studied in [6, 9, 10, 15].

This paper deals with nCS of odd order v, with n even. In Section 2 we will look more closely at bicolourings with $v = 2kn + 1$, and we completely give the spectrum in such a case. In particular, the complete spectrum of bicolourings for $8CS$ is shown. The following result is known (see [11] and [5, p. 374]):

Theorem 1.1. *There exists an* $8CS(v)$ *if and only if* $v = 1 + 16k$ *for some* $k \in \mathbb{N}$.

In Sections 3, 4 and 5 the block colourings for $8CS$ with $s = 3$ and $s = 4$ are studied.

From now on, we construct 8-cycle systems from difference methods. This means that we fix the vertex set $\mathbb{Z}_v$, and define a base block $B = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$; its translates will be all the blocks of type $B + i = (a_1 + i, a_2 + i, a_3 + i, a_4 + i, a_5 + i, a_6 + i, a_7 + i, a_8 + i)$, for every $i \in \mathbb{Z}_v$. Then, given $x, y \in X$, $x \neq y$, the edge $\{x, y\}$ will belong to one of the blocks $B + i$ for some i if and only if $|x - y| \in \{|a_i - a_{i+1}| : i = 1, \ldots, 8\}$, where the indices are taken modulo 8.

2 Bicolourings

This section deals with the study of block colourings of type 2 for n-cycle systems, where n is even. It begins by determining an upper bound on the number of colours used in such colourings.

Theorem 2.1. *Let $\Sigma = (V, \mathcal{B})$ be an $nCS(2kn+1)$, with $n \in \mathbb{N}$, n even, and $k \in \mathbb{N}$, and let $\phi\colon \mathcal{B} \to C$ be a c-bicolouring of Σ. Then $c \leq 3$.*

Proof. Let $|C| = c$ and let $\gamma \in C$. Any element $v \in V$ is incident with kn blocks, and if it is incident with blocks colored γ, then it must be incident with precisely $\frac{kn}{2}$ blocks colored γ. This implies that there are at least $kn + 1$ vertices incident with blocks colored γ. Thus

$$c(1 + kn) \leq 2(1 + 2kn),$$

so that $c \leq 3$. □

In this section we completely determine the colour spectrum of bicolourings for $nCS(v)$, with $v = 2kn + 1$. In order to do this, the following lemma must first be proven. Given a graph $G = (V, E)$ and given two disjoint sets $X, Y \subset V$, let $e_G(X, Y)$ denote the number of edges in G incident to one vertex in X and one in Y.

Lemma 2.2. *Let C_m be a cycle of length m whose vertices belong to two disjoint sets X and Y. Then $e_{C_m}(X, Y)$ is even.*

Proof. The statement is proven by induction on m. If $m = 3$, it is trivial.

So let $m \geq 4$. If $e_{C_m}(X, Y) = 0$, then the statement is proved. Suppose that $\{x_1, x_2\}$ is an edge of C_m so that $x_1 \in X$ and $x_2 \in Y$. If $C_m = (x_1, x_2, \ldots, x_m)$, let $C = (x_1, x_3 \ldots, x_m)$. So C has length $m - 1$ and

$$E(C) \;=\; E(C_m) \cup \{\{x_1, x_3\}\} \setminus \{\{x_1, x_2\}, \{x_2, x_3\}\}.$$

By induction on m we can say that $e_C(X, Y)$ is even. At this point there are two possibilities. If $x_3 \in X$, then $e_{C_m}(X, Y) = e_C(X, Y)+2$. If $x_3 \in Y$, then $e_{C_m}(X, Y) = e_C(X, Y)$. Since $e_C(X, Y)$ is even, the statement is proven. □

We can now formulate our main results of this section.

Theorem 2.3. *If k is odd, then $\Omega_2^{(n)}(2kn + 1) = \emptyset$.*

Proof. Let $\Sigma = (V, \mathcal{B})$ be an $nCS(v)$, where $v = 2kn + 1$, and let $\phi\colon \mathcal{B} \to C$ be a 2-bicolouring of Σ. Let $\gamma \in C$ and let $\mathcal{B}_\gamma$ be the set of blocks of $\mathcal{B}$ colored γ. Then any vertex of V belongs to $\frac{kn}{2}$ blocks of $\mathcal{B}_\gamma$. Thus

$$|\mathcal{B}_\gamma| \;=\; \frac{v \cdot \frac{kn}{2}}{n} = \frac{v \cdot k}{2}.$$

Since k is odd, we get a contradiction.

Now suppose that $\Sigma = (V, \mathcal{B})$ is an $nCS(v)$, where $v = 2kn+1$, and let $\phi\colon \mathcal{B} \to C$ be a 3-bicolouring of Σ. In this case we proceed as in [7, Lemma 2.1]. We can assume that $C = \{1, 2, 3\}$ and let X denote the set of vertices incident with blocks of colour 1 and 2, and Y denote the set of vertices incident with blocks of colour 1 and 3, and Z denote the set of vertices incident with blocks of colour 2 and 3. Let $x = |X|$, $y = |Y|$ and $z = |Z|$.

We note that these sets are pairwise disjoint and that in each block there are vertices belonging to at most two of the sets X, Y and Z. Moreover, by Lemma 2.2 a block cannot contain an odd number of edges having vertices incident to two different sets. This implies that the products xy, xz and yz are even. It follows that among x, y and z at most one is odd. However, since $x+y+z=v$, one of them is odd, while the others are even. Since

$$|B_1| = \frac{\frac{kn}{2}\cdot(x+y)}{n} = \frac{k(x+y)}{2},$$
$$|B_2| = \frac{\frac{kn}{2}\cdot(x+z)}{n} = \frac{k(x+z)}{2},$$
$$|B_3| = \frac{\frac{kn}{2}\cdot(y+z)}{8} = \frac{k(y+z)}{2},$$

we obtain a contradiction, because k is odd. This shows that $3 \notin \Omega_2^{(n)}(2kn+1)$ and so $\Omega_2^{(n)}(2kn+1)=\emptyset$ by Theorem 2.1. □

Now let us recall two results:

Theorem 2.4 ([11, 13],[5, p. 382])**.** *For any $n \in \mathbb{N}$, n even, and $k \in \mathbb{N}$, there exists a cyclic decomposition of K_{2kn+1} into n-cycles.*

Theorem 2.5 ([14, Theorem B])**.** *The complete bipartite graph $K_{m,n}$ can be decomposed into $2k$-cycles if and only if m and n are even, $m \geq k$, $n \geq k$ and $2k$ divides mn.*

Theorem 2.4 and Theorem 2.5 are used to prove the following:

Theorem 2.6. *If k and n are even, then $\Omega_2^{(n)}(2kn+1)=\{2,3\}$.*

Proof. Let $V=\mathbb{Z}_{2kn+1}$. From Theorem 2.4, let us consider a cyclic decomposition of the complete graph over $\mathbb{Z}_{2kn+1}$ with base blocks A_i for $i \in \{1,\dots,k\}$. If $k=2h$, assign colour 1 to the blocks A_i and all of their translated forms for $i \in \{1,\dots,h\}$. Also assign colour 2 to the blocks A_i and all their translated forms for $i \in \{h+1,\dots,2h\}$. Let $\mathcal{B}$ be the set of all these blocks; then $\Sigma=(\mathbb{Z}_{2kn+1},\mathcal{B})$ is an $nCS(2kn+1)$ and the previous assignment determines a 2-bicolouring of Σ. In particular, any vertex is contained in $2hn$ blocks, hn of them colored 1 and hn colored 2.

We now prove that $3 \in \Omega_2^{(n)}(2kn+1)$. Let $k=2h$ and consider two disjoint sets A and B, with $|A|=|B|=2hn$, and a vertex $\infty \notin A \cup B$. By Theorem 2.4 let us consider two $nCS(2hn+1)$, $\Sigma_1=(A\cup\{\infty\},\mathcal{B}_1)$ and $\Sigma_2=(B\cup\{\infty\},\mathcal{B}_2)$. According to Theorem 2.5 it is possible to take an nCS $\Sigma_3=(K_{A,B},\mathcal{B}_3)$ on the bipartite graph $K_{A,B}$. Then $\Sigma=(A\cup B\cup\{\infty\},\mathcal{B}_1\cup\mathcal{B}_2\cup\mathcal{B}_3)$ is an $nCS(2kn+1)$. By assigning colour i to the blocks of $\mathcal{B}_i$, for $i=1,2,3$, we get a 3-bicolouring of Σ.

This implies that $3 \in \Omega_2^{(n)}(2kn+1)$ and by Theorem 2.1 the statement is proved. □

Note that if $n=2^r$ for some $r \geq 2$ then an $nCS(v)$ exists if and only if $v=2kn+1$ for some $k \geq 1$ (see [5, p. 374]). Thus the previous results provide the complete spectrum of nCS in this particular case.

3 Lower 3-chromatic index for an $8CS$

In this section we treat an $8CS$, and only in the case $s = 3$ since the case $s = 2$ has been covered completely in Section 2.

Theorem 3.1. $\chi_3^{(8)}(16k+1) = 3$ *for any* $k \geq 1$.

In the proof of Theorem 3.1 we need to distinguish between the cases $k \equiv 0, 1, 2 \mod 3$. Theorem 3.1 will be proven for $k = 1$ and $k = 2$.

Theorem 3.2. $\chi_3^{(8)}(17) = 3$.

Proof. Let us consider the following blocks on $\mathbb{Z}_{17}$:

$$\begin{aligned}
A_1 &= (0, 1, 3, 5, 8, 6, 4, 2)\\
A_2 &= (0, 3, 6, 10, 9, 5, 1, 4)\\
A_3 &= (0, 5, 7, 4, 3, 2, 1, 6)\\
A_4 &= (11, 14, 13, 16, 12, 9, 15, 8)\\
A_5 &= (9, 13, 12, 7, 14, 15, 11, 16)\\
A_6 &= (7, 8, 16, 14, 10, 15, 12, 11)\\
A_7 &= (0, 7, 1, 12, 2, 11, 3, 8)\\
A_8 &= (0, 9, 1, 8, 2, 7, 3, 10)\\
A_9 &= (0, 11, 1, 10, 2, 9, 3, 12)\\
A_{10} &= (4, 13, 5, 16, 7, 15, 6, 14)\\
A_{11} &= (4, 15, 5, 14, 8, 13, 6, 16)\\
A_{12} &= (9, 11, 13, 15, 16, 10, 12, 14)\\
A_{13} &= (0, 13, 1, 16, 3, 15, 2, 14)\\
A_{14} &= (0, 15, 1, 14, 3, 13, 2, 16)\\
A_{15} &= (2, 5, 4, 8, 10, 13, 7, 6)\\
A_{16} &= (4, 9, 8, 12, 6, 11, 5, 10)\\
A_{17} &= (4, 11, 10, 7, 9, 6, 5, 12).
\end{aligned}$$

The system $\Sigma = (\mathbb{Z}_{17}, \bigcup_{i=1}^{17} A_i)$ is an $8CS$ of order 17. Let $\phi\colon \bigcup A_i \to \{1, 2, 3\}$ be the colouring assigning the colour 1 to the blocks A_i, for $i = 1, \ldots, 6$, the colour 2 to the blocks A_i, for $i = 7, \ldots, 12$ and the colour 3 to the blocks A_i for $i = 13, \ldots, 17$. Then ϕ is a 3-tricolouring of Σ. In particular, all vertices occur in exactly three blocks coloured 1, except for vertices 2, 10 and 13, which belong to two blocks colored 1; all vertices occur in exactly three blocks coloured 2, except for vertices 4, 5 and 6, which belong to two blocks colored 2; the vertices $0, 1, 3, 7, 8, 9, 11, 12, 14, 15, 16$ occur in exactly two blocks colored 3 while the remaining ones belong to three blocks colored 3. This proves the statement. □

Let us now consider the case $k = 2$.

Theorem 3.3. $\chi_3^{(8)}(33) = 3$.

Proof. On the set $X = \{x_i \colon x \in \mathbb{Z}_{11},\ i = 1, 2, 3\}$, consider the following blocks:

$$A_i = (0_i, 1_i, 3_i, 6_i, 2_i, 7_i, 1_{i+1}, 1_{i+2})$$
$$B_i = (0_{i+1}, 1_{i+2}, 10_{i+1}, 2_{i+2}, 9_{i+1}, 7_{i+2}, 1_{i+1}, 8_{i+2}),$$

where we take indices modulo 3, so that $x_4 := x_1$ and $x_5 := x_2$ for any $x \in \mathbb{Z}_{11}$. Let $\mathcal{B}_i$ be the set of blocks A_i and B_i and their translated forms, for any $i = 1, 2, 3$, where i is kept fixed. So this means that

$$A_i + j = (j_i, (j+1)_i, (j+3)_i, (j+6)_i, (j+2)_i, (j+7)_i, (j+1)_{i+1}, (j+1)_{i+2})$$

and

$$B_i + j =$$
$$(j_{i+1}, (j+1)_{i+2}, (j+10)_{i+1}, (j+2)_{i+2}, (j+9)_{i+1}, (j+7)_{i+2}, (j+1)_{i+1}, (j+8)_{i+2}),$$

for any $j \in \mathbb{Z}_{11}$. Then $\Sigma = (X, \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3)$ is an $8CS$ of order 33. Moreover, the colouring $\phi\colon \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \to \{1, 2, 3\}$ which assigns colour i to the blocks of $\mathcal{B}_i$ is a 3-tricolouring of Σ. The statement is proven because for a fixed $i = 1, 2, 3$ the vertices $0_i, \ldots, 10_i$ belong to six blocks colored i, while the other vertices $0_j, \ldots, 10_j$, with $j \neq i$, belong to five blocks colored i. □

Let us now proceed to the proof of Theorem 3.1.

Proof of Theorem 3.1. We distinguish three cases.

(1) Let $k \equiv 0 \mod 3$, so that $k = 3h$ for some $h \geq 1$. If $v = 16k + 1 = 48h + 1$ we need to consider three pairwise disjoints sets A_1, A_2, A_3 so that $|A_i| = 16h$ for any i, and take $\infty \notin A_i$ for any i. According to Theorem 1.1 it is possible to consider three $8CS$ $\Sigma_i = (A_i \cup \{\infty\}, \mathcal{B}_i)$ for $i = 1, 2, 3$. By Theorem 2.5 we can decompose the complete bipartite graph K_{A_1,A_2} into 8-cycles C_i, $i = 1, \ldots, 32h^2$, the complete bipartite graph K_{A_1,A_3} into 8-cycles D_i, $i = 1, \ldots, 32h^2$, and the complete bipartite graph K_{A_2,A_3} into 8-cycles E_i, $i = 1, \ldots, 32h^2$. If

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \bigcup_{i=1}^{32h^2} (C_i \cup D_i \cup E_i),$$

then the system $\Sigma = (A_1 \cup A_2 \cup A_3 \cup \{\infty\}, \mathcal{B})$ is an $8CS$ of order v. Let us define a colouring assigning the colour 1 to the blocks of $\mathcal{B}_1$ and to the blocks E_i, the colour 2 to the blocks of $\mathcal{B}_2$ and to the blocks of D_i, and the colour 3 to the blocks of $\mathcal{B}_3$ and to the blocks C_i. Thus this is a 3-tricolouring of Σ, because any element of $A_1 \cup A_2 \cup A_3 \cup \{\infty\}$ belongs to precisely $8h$ blocks colored i, for $i = 1, 2, 3$.

(2) Let $k \equiv 1 \mod 3$, so that $k = 3h + 1$ for some $h \geq 0$, and let $v = 48h + 17$. According to Theorem 3.2 it is possible to suppose that $h \geq 1$. Let us consider pairwise disjoint sets X_1, X_2, X_3, Y_1, Y_2, Y_3, i with $|X_1| = 4$, $|X_2| = |X_3| = 6$, $|Y_1| = |Y_2| = |Y_3| = 16h$, and consider an element $\infty \notin \bigcup X_i \cup \bigcup Y_j$.

By Theorem 3.2 we can consider an $8CS$ $\Sigma_1 = (X_1 \cup X_2 \cup X_3 \cup \{\infty\}, \mathcal{B}_1)$ with a 3-tricolouring. The blocks of $\mathcal{B}_1$ are divided into three subsets $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, where the blocks of $\mathcal{C}_i$ are colored i.

Similarly, as seen in the case $k \equiv 0 \mod 3$, it is possible to consider an $8CS$ $\Sigma_2 = (Y_1 \cup Y_2 \cup Y_3 \cup \{\infty\}, \mathcal{B}_2)$ with a 3-tricolouring. The blocks of $\mathcal{B}$ are divided inito three subsets $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, where the blocks of $\mathcal{D}_i$ are colored i. Moreover, by Theorem 2.5 the bipartite graphs K_{X_i,Y_j}, for any $i, j = 1, 2, 3$, can be decomposed into a family $\mathcal{E}_{ij}$ of 8-cycles.

Now let us consider the system

$$\Sigma = (\bigcup_{i=1}^{3} X_i \cup \bigcup_{j=1}^{3} Y_j \cup \{\infty\}, \bigcup_{i=1}^{3} \mathcal{C}_i \cup \bigcup_{j=1}^{3} \mathcal{D}_j \cup \bigcup_{i,j=1,2,3} \mathcal{E}_{ij}).$$

It easily follows that Σ is an $8CS$ of order $v = 48h + 17$. We can determine a 3-tricolouring of Σ in the following way:

- assign the colour 1 to the blocks of $\mathcal{C}_1$, $\mathcal{D}_1$, $\mathcal{E}_{1,1}$, $\mathcal{E}_{2,2}$ and $\mathcal{E}_{3,3}$;
- assign the colour 2 to the blocks of $\mathcal{C}_2$, $\mathcal{D}_2$, $\mathcal{E}_{1,2}$, $\mathcal{E}_{2,3}$ and $\mathcal{E}_{3,1}$;
- assign the colour 3 to the blocks of $\mathcal{C}_3$, $\mathcal{D}_3$, $\mathcal{E}_{1,3}$, $\mathcal{E}_{2,1}$ and $\mathcal{E}_{3,2}$.

In particular, any vertex is contained in $24h+8$ blocks, $8h+3$ colored with one color, another $8h + 3$ colored with a second color and the remaining $8h + 2$ colored with the third color. This proves the statement in the case $k \equiv 1 \mod 3$.

(3) Let $k \equiv 2 \mod 3$, so that $k = 3h + 2$ for some $h \geq 0$, and let $v = 48h + 33$. By Theorem 3.3 it is possible to suppose that $h \geq 1$. Let us consider pairwise disjoint sets X_1, X_2, X_3, Y_1, Y_2, Y_3, with $|X_1| = 12$, $|X_2| = |X_3| = 10$, $|Y_1| = |Y_2| = |Y_3| = 16h$, and consider an element $\infty \notin \bigcup X_i \cup \bigcup Y_j$.

According to Theorem 3.3, we can consider an $8CS$ $\Sigma_1 = (X_1 \cup X_2 \cup X_3 \cup \{\infty\}, \mathcal{B}_1)$ with a 3-tricolouring. The blocks of $\mathcal{B}_1$ are divided into three subsets $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, where the blocks of $\mathcal{C}_i$ are colored with the colour i.

Similarly, as seen in the case $k \equiv 0 \mod 3$, we can consider an $8CS$ $\Sigma_2 = (Y_1 \cup Y_2 \cup Y_3 \cup \{\infty\}, \mathcal{B}_2)$ with a 3-tricolouring. The blocks of $\mathcal{B}_2$ are divided into three subsets $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, where the blocks of $\mathcal{D}_i$ are colored i. Moreover, by Theorem 2.5 we can decompose the bipartite graphs K_{X_i,Y_j}, for any $i, j = 1, 2, 3$, into a family $\mathcal{E}_{ij}$ of 8-cycles.

Now let us consider the system

$$\Sigma = (\bigcup_{i=1}^{3} X_i \cup \bigcup_{j=1}^{3} Y_j \cup \{\infty\}, \bigcup_{i=1}^{3} \mathcal{C}_i \cup \bigcup_{j=1}^{3} \mathcal{D}_j \cup \bigcup_{i,j=1,2,3} \mathcal{E}_{ij}).$$

It easily follows that Σ is an $8CS$ of order $v = 48h + 33$. We can determine a 3-tricolouring of Σ in the following way:

- assign the colour 1 to the blocks of $\mathcal{C}_1$, $\mathcal{D}_1$, $\mathcal{E}_{1,1}$, $\mathcal{E}_{2,2}$ and $\mathcal{E}_{3,3}$;
- assign the colour 2 to the blocks of $\mathcal{C}_2$, $\mathcal{D}_2$, $\mathcal{E}_{1,2}$, $\mathcal{E}_{2,3}$ and $\mathcal{E}_{3,1}$;
- assign the colour 3 to the blocks of $\mathcal{C}_3$, $\mathcal{D}_3$, $\mathcal{E}_{1,3}$, $\mathcal{E}_{2,1}$ and $\mathcal{E}_{3,2}$.

In particular, any vertex is contained in $24h + 16$ blocks, $8h + 6$ colored with one color, another $8h + 5$ colored with a second color and the final $8h + 5$ colored with the third color. The result is now proved in the case $k \equiv 2 \mod 3$. □

4 Upper 3-chromatic index

This section indicates upper bounds for the upper 3-chromatic index. Our aim is to find its exact value in the case $v = 16k + 1$ and $k \equiv 0 \bmod 3$.

Let $\Sigma = (X, \mathcal{B})$ be an $8CS$ of order v and suppose that a c-colouring of type s of Σ is given. Let us denote by $\mathcal{B}_i$ the set of blocks colored i and by X_i the set of vertices belonging to blocks of $\mathcal{B}_i$.

Lemma 4.1. *Let $\Sigma = (X, \mathcal{B})$ be an $8CS$ of order v with a c-tricolouring. Then:*

1. $c \leq 8$ *if* $k \equiv 0 \mod 3$;
2. $c \leq 9$ *if* $k \equiv 1, 2 \mod 3$, *with* $k > 1$;
3. $c \leq 10$ *if* $k = 1$.

Proof. Let $v = 16k + 1$. Then any $x \in X$ belongs to $8k$ blocks. Then, following the notation, $|X_i| \geq 2\lfloor \frac{8k}{3} \rfloor + 1$. So we must have

$$c\left(2\left\lfloor \frac{8k}{3} \right\rfloor + 1\right) \leq 3(16k + 1).$$

This inequality implies the lemma. □

Using the previous notation we need the following technical lemma, which will determine an upper bound for $\overline{\chi}_3^{(8)}(16k + 1)$ for any k. The idea comes from [8, Lemma 5.3].

Lemma 4.2. *Let $\Sigma = (X, \mathcal{B})$ be an $8CS$ of order v with a c-colouring of type s, for some $s \geq 2$. Then*

$$|X_i \cup X_j| \geq 4\left\lfloor \frac{8k}{s} \right\rfloor + 1$$

for any $i \neq j$.

Proof. Let $v = 16k+1$, for some $k \geq 1$. Then $|X_i| \geq 2\lfloor \frac{8k}{s} \rfloor + 1$ for any i. Let $|X_i| = 2\lfloor \frac{8k}{s} \rfloor + 1 + k_i$ for some $k_i \geq 0$ and for any i.

Let $x \in X_i \cap X_j$ for $i \neq j$. Let us suppose that $y \in X_i \cap X_j$ for $y \neq x$. Either y is not adjacent to x in the blocks of $\mathcal{B}_i$ (which are at most k_i) or in the blocks of $\mathcal{B}_j$ (which are at most k_j). This means that $|X_i \cap X_j| \leq k_i + k_j + 1$. So

$$|X_i \cup X_j| \;=\; 4\left\lfloor \frac{8k}{s} \right\rfloor + 2 + k_i + k_j - |X_i \cap X_j| \;\geq\; 4\left\lfloor \frac{8k}{s} \right\rfloor + 1.$$

□

It is now possible to prove the first of the two main results of this section.

Theorem 4.3. *$\overline{\chi}_3^{(8)}(16k+1) \leq 7$ for any $k \geq 2$ and $\overline{\chi}_3^{(8)}(17) \leq 6$.*

Proof. Let us use the fixed notation. Given $v = 16k+1$, we consider an $8CS$ $\Sigma = (X, \mathcal{B})$ of order v with a c-tricolouring.

Now let $v = 17$, so that $k = 1$. Clearly we must have $|X_i| \geq 8$ for any i. So

$$51 = 3 \cdot |X| = \sum_{i=1}^{c} |X_i| \geq 8c.$$

This implies that $c \leq 6$ and so $\overline{\chi}_3^{(8)}(17) \leq 6$. We can now suppose that $k \geq 2$. By Lemma 4.2,

$$|X_i \cup X_j| \geq 4\left\lfloor \frac{8k}{3} \right\rfloor + 1,$$

for any $i \neq j$. Since any vertex belongs to three of the sets $X_1, \ldots, X_c$, we get

$$\left[\binom{c}{2} - \binom{c-3}{2}\right](16k+1) = \sum_{i \neq j} |X_i \cup X_j| \geq \binom{c}{2}\left(4\left\lfloor \frac{8k}{3} \right\rfloor + 1\right)$$

and so

$$(3c-6)(16k+1) \geq \frac{c(c-1)}{2}\left(4\left\lfloor \frac{8k}{3} \right\rfloor + 1\right). \tag{1}$$

Let $c = 9$. Then by (1) we get:

$$112k - 5 \geq 48\left\lfloor \frac{8k}{3} \right\rfloor \geq 48\frac{8k-2}{3} \Rightarrow 16k - 27 \leq 0.$$

The only possibility is that $k = 1$, so that $|\mathcal{B}| = 17$. However, since $c = 9$, we would have $|\mathcal{B}_i| = 1$ for some i, which is not possible, because we should have $|\mathcal{B}_i| \geq 2$ for any i. Together with Lemma 4.1 this proves that $c \leq 8$ for any k. It must be noted that if $c = 8$ a contradiction is obtained, impling that we must have $c \leq 7$.

Let us suppose $c = 8$. By (1) we have

$$\begin{aligned} 9(16k+1) &\geq 14\left(4\left\lfloor\frac{8k}{3}\right\rfloor+1\right) \\ &\geq 14\left(4\frac{8k-2}{3}+1\right), \\ \text{which implies } 16k-97 &\leq 0. \end{aligned}$$

It is clear that the only possibility is that $k \leq 6$.

If $k = 2$, then any $x \in X$ belongs to 16 blocks, six coloured with a first colour, five coloured with a second colour and five coloured with a third colour. So $|X_i| = 11 + k_i$ for any $i = 1, \ldots, 8$, and

$$3 \cdot 33 = \sum_{i=1}^{8} |X_i| \Rightarrow \sum_{i=1}^{8} k_i = 11.$$

If $k_i = 0$ for some i, then the blocks of $\mathcal{B}_i$ are a decomposition of the complete graph on X_i in 8-cycles. However, this is not possible, because $|\mathcal{B}_i| = \frac{5\cdot 11}{8} \notin \mathbb{N}$. Then $k_i = 1$ for at least one i, so that $|X_i| = 12$. In this case any element of X_i must belong to five blocks of $\mathcal{B}_i$. Therefore this is not possible, because $|\mathcal{B}_i| = \frac{5\cdot 12}{8} \notin \mathbb{N}$. So $k \neq 2$.

If $k = 3, 5, 6$, by (1) and by the fact that $c = 8$, we obtain a contradiction.

If $k = 4$, then $|X| = 65$ and any $x \in X$ belongs to 32 blocks, 11 coloured with a first colour, 11 coloured with a second colour and 10 coloured with a third colour. We know that $|X_i| = 21 + k_i$, where $k_i \geq 0$, for any i and moreover:

$$\begin{aligned} 3 \cdot 65 &= \sum_{i=1}^{8} |X_i| \\ \text{which implies } \sum_{i=1}^{8} k_i &= 27. \end{aligned} \tag{2}$$

Let us denote by Y_i the set of elements of X_i belonging to 11 blocks of $\mathcal{B}_i$ and by Z_i the set of of elements of X_i belonging to 10 blocks of $\mathcal{B}_i$. For any i, j, with $i \neq j$, we have

$$X_i \cap X_j = (Y_i \cap Y_j) \cup (Y_i \cap Z_j) \cup (Z_i \cap Y_j).$$

Taking any $x \in X_i \cap X_j$, either x belongs to Y_i or to Y_j. It is possible to suppose that $x \in Y_i$, which implies that $k_i \geq 2$. Taking any $y \in X_i \cap X_j$, $y \neq x$, either $\{x, y\}$ does not belong to any block in $\mathcal{B}_i$ or does not belong to any block in $\mathcal{B}_j$. So y either is one of the $k_i - 2$ elements of X_i not adjacent to x in any of the blocks of $\mathcal{B}_i$ or is one of the k_j elements of X_j not adjacent to x in any of the blocks of $\mathcal{B}_j$. This shows that

$$|X_i \cap X_j| \leq k_i + k_j - 1,$$

for any i, j, with $i \neq j$. Therefore

$$3|X| = \sum_{1 \leq i < j \leq 8} |X_i \cap X_j| \leq \sum_{1 \leq i < j \leq 8} (k_i + k_j - 1)$$

which implies that

$$195 \leq -\binom{8}{2} + 7\sum_{i=1}^{8} k_i.$$

By (2) we obtain a contradiction, and this proves the theorem. □

In the next result we determine the exact value of $\overline{\chi}_3^{(8)}(16k+1)$ in the case that $k \equiv 0 \mod 3$.

Theorem 4.4. $\overline{\chi}_3^{(8)}(16k+1) = 7$ *for any* $k \equiv 0 \mod 3$.

Proof. Let $k = 3h$ for some $h \geq 1$. Then, if $v = 16k+1 = 48h+1$, let us consider six pairwise disjoints sets A_i, for $i = 1, \ldots, 6$, so that $|A_i| = 8h$ for any i, and take $\infty \notin A_i$ for any i. According to Theorem 1.1 this can be considered as three $8CS$, $\Sigma_j = (A_{2j-1} \cup A_{2j} \cup \{\infty\}, \mathcal{B}_j)$ for $j = 1, 2, 3$. By [12, Theorem 2.15] it is possible to decompose the complete equipartite graphs K_{A_1,A_3,A_5} into 8-cycles C_i, $i = 1, \ldots, 24h^2$; K_{A_1,A_4,A_6} into 8-cycles D_i, $i = 1, \ldots, 24h^2$; K_{A_2,A_3,A_6} into 8-cycles E_i, $i = 1, \ldots, 24h^2$; and K_{A_2,A_4,A_5} into 8-cycles F_i, $i = 1, \ldots, 24h^2$. If:

$$\mathcal{B} \ = \ \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \bigcup_i (C_i \cup D_i \cup E_i \cup F_i),$$

then the system $\Sigma = (\bigcup_{i=1}^{6} A_i \cup \{\infty\}, \mathcal{B})$ is an $8CS$ of order v.

Let $\phi \colon \mathcal{B} \to \{1, \ldots, 7\}$ be the colouring assigning the colour 1 to the blocks of $\mathcal{B}_1$, the colour 2 to the blocks of $\mathcal{B}_2$, the colour 3 to the blocks of $\mathcal{B}_3$, the color 4 to the blocks C_i, the colour 5 to the blocks D_i, the colour 6 to the blocks E_i, and the colour 7 to the blocks F_i. Then it follows easily that ϕ is 7-tricolouring of Σ. □

5 Quadricolourings for $8CS$

This section deals with quadricolourings, determining the exact value of $\chi_4^{(8)}(16k+1)$ in the case that $k \equiv 0 \mod 4$ and giving an upper bound for $\overline{\chi}_4^{(8)}(16k+1)$.

By using the previous notation let us consider an $8CS$, $\Sigma = (X, \mathcal{B})$ of order v, with a c-colouring of type s. We will denote by $\mathcal{B}_i$ the set of blocks colored i and by X_i the set of vertices belonging to blocks of $\mathcal{B}_i$.

Proposition 5.1. $\overline{\chi}_4^{(8)}(16k+1) \leq 14$ *for* $k \geq 6$, *and* $\overline{\chi}_4^{(8)}(16k+1) \leq 13$ *for* $k \leq 5$.

Proof. Let $\Sigma = (X, \mathcal{B})$ be an $8CS$ of order $v = 16k+1$ and let $\phi \colon \mathcal{B} \to \{1, \ldots, c\}$ be a colouring. Then any $x \in X$ belongs to $8k$ blocks and $|X_i| \geq 4k+1$. It is easy to check that

$$c(4k+1) \ \leq \ 4(16k+1).$$

This implies that $c \leq 15$. Since $|X_i| = 4k+1+k_i$ for any i and for some $k_i \geq 0$, we have

$$4|X| \ = \ \sum_{i=1}^{c} |X_i|, \text{ which implies } \sum_{i=1}^{c} k_i = 64k + 4 - 4ck - c.$$

Similarly, as in the proof of Lemma 4.2 we obtain

$$|X_i \cap X_j| \leq k_i + k_j + 1$$

which implies

$$6|X| = \sum_{i<j} |X_i \cap X_j| \leq \binom{c}{2} + (c-1)\sum_{i=1}^{c} k_i,$$

which implies

$$96k + 6 \ \leq \ \binom{c}{2} + (c-1)(64k + 4 - 4ck - c). \tag{3}$$

Let us consider $c = 15$. Then by (3) it can be noted that $40k + 55 \leq 0$, which is not possible. It follows that $c \leq 14$. We now suppose that $c = 14$; then by (3) we find that $8k \geq 45$, so that $k \geq 6$. This proves the statement. □

Theorem 5.2. $\chi_4^{(8)}(16k+1) = 4$ *if and only if* $k \equiv 0 \mod 4$.

Proof. Let $\Sigma = (X, \mathcal{B})$ be an $8CS$ of order v with a 4-quadricolouring. Then $X_i = X$ and

$$|\mathcal{B}_i| \ = \ \frac{|X_i| \cdot 2k}{8} \ = \ \frac{k(16k+1)}{4}$$

for any i. Then we must have $k \equiv 0 \mod 4$.

Let us now consider $k \equiv 0 \mod 4$, so that $k = 4h$ for some $h \geq 0$. If $v = 16k + 1 = 64h + 1$, let us consider, on $\mathbb{Z}_v$, the following blocks:

$$A_i = (0, i + 4h, 48h + 1, i + 12h, 32h + 1, i + 16h, 28h + 1, i)$$

for $i = 1, \ldots, 4h$. Let $\mathcal{B}$ be the set of all the blocks A_i and their translated forms. Then $\Sigma = (\mathbb{Z}_v, \mathcal{B})$ is an $8CS$ of order v. Let $\phi\colon \mathcal{B} \to \{1, 2, 3, 4\}$ be the colouring assigned in the following way:

- the blocks A_i, for $i = 1, \ldots, h$, and all their translated forms are colored 1;
- the blocks A_i, for $i = h + 1, \ldots, 2h$, and all their translated forms are colored 2;
- the blocks A_i, for $i = 2h + 1, \ldots, 3h$, and all their translated forms are colored 3;
- the blocks A_i, for $i = 3h + 1, \ldots, 4h$, and all their translated forms are colored 4.

It follows immediately that this is a 4-quadricolouring of Σ and it shows that $4 \in \Omega_4^{(8)}(16k+1)$ for $k \equiv 0 \mod 4$. □

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