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A new family of orthogonal Latin hypercube designs

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Abstract

We construct a new family of orthogonal Latin hypercube designs having second order property with n rows and $m = 4$ columns, where $n \equiv 3 \pmod 4$. In particular, if $n \equiv 3 \pmod{16}$, then we also report a family of such designs with $m = 6$ columns.

1 Introduction

Latin hypercube designs are widely used for computer experiments. A Latin hypercube design LH(n, m), is an $n \times m$ matrix whose columns are permutations of the column vector $(1, 2, \ldots, n)'$. In the context of computer experiments, the columns of a Latin hypercube design represent the input factors, and the rows the experimental runs. It is sometimes convenient to visualize a Latin hypercube design in its centered form. For a positive integer n, let g_n be an $n \times 1$ vector with its ith element equal to $(i - (n+1)/2)$, $1 \le i \le n$, and let G_n be the set of all permutations of g_n. A centered Latin hypercube design is an $n \times m$ matrix with columns from G_n. Henceforth we consider Latin hypercube designs in the centered form only. A (centered) Latin hypercube design L is called orthogonal if the columns of L are mutually orthogonal.

We shall denote an *orthogonal* Latin hypercube design with n rows and m columns as OLH$_1(n, m)$. A subclass of OLH$_1(n, m)$ designs consists of those that satisfy the following two conditions:

(a) the entry-wise square of each column is orthogonal to all columns in the design;

(b) the entry-wise product of any two distinct columns is orthogonal to all columns in the design.

An orthogonal Latin hypercube design with n rows and m columns satisfying the conditions (a) and (b) will be called orthogonal Latin hypercube designs with *second order property* and denoted as OLH$_2(n, m)$. Whereas an OLH$_1(n, m)$ design ensures that the estimates of linear effects are mutually uncorrelated, an $\mathrm{OLH}_2(n, m)$ design ensures that not only the estimates of linear effects are mutually uncorrelated but they are also uncorrelated with the estimates of quadratic and interaction effects in a second order model.

In this paper, we focus on orthogonal Latin hypercube designs with second order property, i.e., on $\mathrm{OLH}_2(n, m)$ designs. Several families of $\mathrm{OLH}_2(n, m)$ designs are known (see [2–4, 6–9]). However, there exist values of n for which such designs with more than two columns are not known. The known families of $\mathrm{OLH}_2(n, m)$ designs cover the cases $n \equiv 0, 1 \pmod 4$. Also, as shown by Lin et al. [5], no $\mathrm{OLH}_1(n, m)$ (and hence, no $\mathrm{OLH}_2(n, m)$) design can exist if $n \equiv 2 \pmod 4$. Not much seems to be known about $\mathrm{OLH}_2(n, m)$ designs when $n \equiv 3 \pmod 4$ and $m > 3$. In this paper we give a method of construction of $\mathrm{OLH}_2(n, m)$ designs with $n \equiv 3 \pmod 4$ rows and $m = 4$ columns. In particular, when $n \equiv 3 \pmod{16}$, we are also able to construct $\mathrm{OLH}_2(n, m)$ designs with $m = 6$ columns. The proposed method of construction of $\mathrm{OLH}_2(n, m)$ designs uses an orthogonal design in conjunction with an existing $\mathrm{OLH}_2(n', m)$ design, where $n' < n$.

2 The method of construction

Throughout, a prime over a vector or a matrix will denote its transpose. For an integer u, let $\boldsymbol{a}_u = (-x_u, -x_{u-1}, \ldots, -x_2, -x_1, x_1, x_2, \ldots, x_{u-1}, x_u)'$, where the x_i's are real numbers. For our purpose, we assume that no x_i equals zero. Let A be a $2u \times m$ matrix whose columns are permutations of $\boldsymbol{a}_u$. The matrix A is called an orthogonal design if the columns of A are mutually orthogonal. Such orthogonal designs are useful in constructing orthogonal Latin hypercube designs. Four orthogonal designs with $u = 1, 2, 4, 8$ are displayed in Lin et al. [5].

An $\mathrm{OLH}_2(15, 4)$ design was reported in [3]. We have now found an $\mathrm{OLH}_2(19, 6)$ design via a computer search. For ease of reference, we display both these designs in Table 1.

Remark. Note that in the orthogonal designs given in [5], the last u rows are negatives of the first u rows. Similarly, in both the designs in Table 1, if we exclude the row of all zeros, then the last $(n-1)/2$ rows are negatives of the first $(n-1)/2$ rows. These facts are useful in the construction described below.

We first state the following result for subsequent use. The proof of the result is easy and follows from the well known fold-over principle of Box and Wilson [1].

Lemma 2.1 *Let A_1 be an $a \times b$ matrix and $A = [A_1', \ -A_1']'$. Then (i) the entry-wise square of each column of A is orthogonal to all columns of A and (ii) the entry-wise product of any two distinct columns of A is orthogonal to all columns of A.*

We now give a method of construction of $\mathrm{OLH}_2(n, m)$ designs where $n \equiv 3 \pmod 4$.

Theorem 2.1 *Let $n = 4s+3$. Then there exists an* $\mathrm{OLH}_2(4s+3, 4)$ *design for each integer $s \geq 3$.*

Proof. A computer search reveals that there does not exist an $\mathrm{OLH}_2(7, m)$ design with $m \geq 2$ columns. For $s = 2$, Dey and Sarkar [3] obtained an $\mathrm{OLH}_2(11, 3)$ design, and also observed that no $\mathrm{OLH}_2(11, m)$ design with $m \geq 4$ columns exists. Attention is thus confined to the values of $s \geq 3$. We distinguish two cases according as s is odd or even.

Case 1: s odd, $s \geq 3$: When $s = 3$, we have an $\mathrm{OLH}_2(15, 4)$ shown in Table 1. Now, let $s \geq 5$. For $j = 1, 2, \ldots, (s-3)/2$, let D_j be an 8×4 matrix obtained by replacing the elements x_1, x_2, x_3 and x_4 by $8 + 4(j-1), 9 + 4(j-1), 10 + 4(j-1)$ and $11 + 4(j-1)$, respectively, in the orthogonal design with $u = 4$ in [5]. Also, let d_{15} be the $\mathrm{OLH}_2(15, 4)$ design displayed in Table 1. Let

$$d^{(1)} = \begin{bmatrix} d_{15} \\ D_1 \\ D_2 \\ \vdots \\ D_{\frac{s-3}{2}} \end{bmatrix}.$$

Then using the lemma and the facts in the remark, it is easily seen that $d^{(1)}$ is an $\mathrm{OLH}_2(4s+3, 4)$ design, where $s \geq 5$ is an odd integer.

Case 2: s even, $s \geq 4$: For $s = 4$, we have an $\mathrm{OLH}_2(19, 4)$ obtained by deleting any two columns of the $\mathrm{OLH}_2(19, 6)$ displayed in Table 1. Call this design $d_{19,4}$. Now, let $s \geq 6$. For $j = 1, 2, \ldots, (s-4)/2$, let E_j be an 8×4 matrix obtained by replacing the elements x_1, x_2, x_3 and x_4 by $10+4(j-1), 11+4(j-1), 12+4(j-1)$ and $13+4(j-1)$, respectively, in the orthogonal design with $u = 4$ in [5]. Let

$$d^{(2)} = \begin{bmatrix} d_{19,4} \\ E_1 \\ E_2 \\ \vdots \\ E_{\frac{s-4}{2}} \end{bmatrix}.$$

Then it is easily seen that $d^{(2)}$ is an $\mathrm{OLH}_2(4s+3, 4)$ design, where $s \geq 6$ is an even integer. $\square$

In a special case, we can obtain an $\mathrm{OLH}_2(n, m)$ design with $m = 6$ columns. Suppose $n = 4s + 3$ and $s \equiv 0 \pmod 4$. This implies that $n \equiv 3 \pmod{16}$. For $s = 4$, we have an $\mathrm{OLH}_2(19, 6)$ design, displayed in Table 1. We denote this design by $d_{19,6}$. Let $s \equiv 0 \pmod 4$, $s \geq 8$. For $j = 1, 2, \ldots, (s-4)/4$, replace the elements $x_1, x_2, \ldots, x_8$ in any 6 columns of the 16×8 orthogonal design ($u = 8$) in [5] by $10 + 4(j-1), 11 + 4(j-1), \ldots, 17 + 4(j-1)$, respectively, to obtain the 16×6

matrices $F_j,\ j = 1, 2, \ldots, \frac{s-4}{4}$. Let $d^{(3)}$ be the design

$$d^{(3)} = \begin{bmatrix} d_{19,6} \\ F_1 \\ F_2 \\ \vdots \\ F_{\frac{s-4}{4}} \end{bmatrix}.$$

Then $d^{(3)}$ is an $\mathrm{OLH}_2(4s + 3, 6)$ design where $s \equiv 0 \pmod 4$. We thus have the following result.

Theorem 2.2 *Let* $n = 4s+3$, *where* $s \equiv 0 \pmod 4$. *Then there exists an* $\mathrm{OLH}_2(4s+3, 6)$ *design.*

TABLE 1
Orthogonal Latin hypercube designs $\mathrm{OLH}_2(15, 4)$ *and* $\mathrm{OLH}_2(19, 6)$

$\mathrm{OLH}_2(15, 4)$

−7	−7	−1	−3
−6	6	−4	−4
−5	5	6	6
−4	−4	5	1
−3	3	−2	−2
−2	−2	−3	5
−1	−1	−7	7
0	0	0	0
7	7	1	3
6	−6	4	4
5	−5	−6	−6
4	4	−5	−1
3	−3	2	2
2	2	3	−5
1	1	7	−7

$\mathrm{OLH}_2(19, 6)$

−9	2	7	1	8	4
−8	5	−6	−5	4	−3
−7	−3	−2	9	−6	−9
−6	1	5	−4	−3	−5
−5	−6	4	−7	−9	6
−4	7	−9	2	−5	8
−3	−8	−3	3	7	2
−2	−4	1	8	−1	7
−1	−9	−8	−6	2	−1
0	0	0	0	0	0
1	9	8	6	−2	1
2	4	−1	−8	1	−7
3	8	3	−3	−7	−2
4	−7	9	−2	5	−8
5	6	−4	7	9	−6
6	−1	−5	4	3	5
7	3	2	−9	6	9
8	−5	6	5	−4	3
9	−2	−7	−1	−8	−4

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