AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **69(1)** (2017), Pages 18–28

Group divisible designs $\mathsf{GDD}(n, n, n, 1; \lambda_1, \lambda_2)$

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Abstract

We give a complete solution for the existence problem of group divisible designs (or PBIBDs) with blocks of size $k = 3$, four groups of size $(n, n, n, 1)$, and any two indices (λ_1, λ_2). Moreover, we introduce a construction of infinitely many group divisible designs with t groups of size n and one group of size 1. The construction technique utilizes our main result, together with some other known designs.

1 Introduction

A *group divisible design* $\mathsf{GDD}(g = g_1 + g_2 + \cdots + g_s, s, k; \lambda_1, \lambda_2)$ is an ordered pair $(G, \mathcal{B})$ where G is a g-set of symbols that is partitioned into s sets, called *groups*, of size $g_1, g_2, \ldots, g_s$, and $\mathcal{B}$ a collection of k-subsets of G, called *blocks*, such that each pair of symbols from the same group appear together in λ_1 blocks and each pair of symbols from distinct groups appear together in λ_2 blocks. λ_1 and λ_2 are called the *indices* of the design. A group divisible design is a partially balanced incomplete block design (PBIBD) where the set of symbols are partitioned into groups with two different associates. Symbols occurring together in the same group are called *first associates*, and symbols occurring in different groups are called *second associates*. (See [2, 3].)

Many papers in the literature have focused on the designs with $k = 3$. Fu, Rodger, and Sarvate [2, 3] completely solved the existence of group divisible designs where all groups have the same size, namely $\mathsf{GDD}(g = n + n + \cdots + n, m, 3; \lambda_1, \lambda_2)$. In 1992, Colbourn, Hoffman, and Rees [1] showed a necessary and sufficient condition for the existence of a $\mathsf{GDD}(g = n + n + \cdots + n + u, t + 1, 3; 0, 1)$. Later, Pabhapote and Punnim [9] investigated all triples of positive integers (n, m, λ) for which a $\mathsf{GDD}(g = n + m, 2, 3; \lambda, 1)$ exists. Pabhapote [8] proved the existence of $\mathsf{GDD}(g =$

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$n+m, 2, 3; \lambda_1, \lambda_2)$ for all $m \neq 2$ and $n \neq 2$ in which $\lambda_1 \geq \lambda_2$. In 2014, Lapchinda et al. [5, 6] worked on GDDs with three groups; they gave a complete solution for the existence problem of group divisible designs with block size $k = 3$ and three groups of size $(n, n, 1)$ in two separated cases $\lambda_1 \geq \lambda_2$ and $\lambda_1 < \lambda_2$. Here we give a complete solution for the group divisible designs with block size $k = 3$ and four groups of size $(n, n, n, 1)$ for any two indices (λ_1, λ_2). Having three groups of the same size allows us to utilize latin squares in our construction technique. In the last section, we extend the main result to construct infinitely many $\mathsf{GDD}(g = n+\cdots+n+1, t+1, 3; \lambda_1, \lambda_2)$s.

Since for the main result we are dealing with GDDs with four groups and block size 3, the notation $\mathsf{GDD}(n, n, n, 1; \lambda_1, \lambda_2)$ is used for $\mathsf{GDD}(g = n+n+n+1, 4, 3; \lambda_1, \lambda_2)$ from this point forward, and we refer to blocks as *triples*. Our necessary conditions for the existence problem of a $\mathsf{GDD}(n, n, n, 1; \lambda_1, \lambda_2)$ can be easily obtained from a graph model by describing a $\mathsf{GDD}(n, n, n, 1; \lambda_1, \lambda_2)$ graphically as follows. Let λK_v denote the graph of v vertices where each pair of vertices is joined by λ edges. Let G_1 and G_2 be graphs. The graph $G_1 \vee_\lambda G_2$ is obtained from the union of G_1 and G_2 by joining each vertex in G_1 to each vertex in G_2 with λ edges. A *G-decomposition of a graph H* is a partition of the edges of H such that each element of the partition induces a copy of G. The existence of a $\mathsf{GDD}(n, n, n, 1; \lambda_1, \lambda_2)$ is easily seen to be equivalent to the existence of a K_3-decomposition of $\lambda_1 K_n \vee_{\lambda_2} (\lambda_1 K_n \vee_{\lambda_2} (\lambda_1 K_n \vee_{\lambda_2} K_1))$. As in [2], edges joining vertices in the same group are called *pure edges*, otherwise, they are *cross edges*.

Theorem 1.1. (Necessary Conditions) *Let $n \geq 1$ and $\lambda_1, \lambda_2 \geq 0$ be integers. If there exists a $\mathsf{GDD}(n, n, n, 1; \lambda_1, \lambda_2)$, then*

(i) $2 \mid (n-1)\lambda_1 + \lambda_2$, *and*
(ii) *if $n = 2$ then $\lambda_1 \leq 3\lambda_2$.*

Proof. Let $G = \lambda_1 K_n \vee_{\lambda_2} (\lambda_1 K_n \vee_{\lambda_2} (\lambda_1 K_n \vee_{\lambda_2} K_1))$. Since there exists a K_3-decomposition of G, each vertex must have even degree. Vertices of G are of degree $3n\lambda_2$ or $(n-1)\lambda_1 + (2n+1)\lambda_2$, so this yields (i). (Note that the vertex degree also yields $2 \mid 3n\lambda_2$. However, (i) has already implied $2 \mid 3n\lambda_2$.) When $n = 2$, any pure edge must be contained in a triple which contains exactly two cross edges. Thus the number of pure edges is at most half of the number of cross edges, and so (ii) holds. □

2 Preliminary Background

This section includes the major tools that are used in our construction of GDDs, namely latin squares, triple systems, and packings.

Our latin squares of order n are always based on the symbol set $\{1, 2, 3, \ldots, n\}$. If $L = \{l_{ij}\}$ is a latin square, we refer to l_{ij} as the symbol in the cell (i, j) of L. $L = \{l_{ij}\}$ is *idempotent* if $l_{ii} = i$ for all i. Two latin squares $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$ of the same order n are called *orthogonal* if the n^2 ordered pairs (a_{ij}, b_{ij}), the pairs formed by superimposing one square on the other, are all different. The existence of idempotent latin squares and orthogonal latin squares are well-known, as given in Theorem 2.1; see [10].

Theorem 2.1. [10]

(*i*) *For all positive integers* $n \neq 2$*, there exists an idempotent latin square of order* n*.*

(*ii*) *For all positive integers* $n \neq 2, 6$*, there exists a pair of orthogonal latin squares of order* n*.*

A *triple system* $\mathsf{TS}(n, \lambda)$ *of index* λ *and order* n is an ordered pair $(S, \mathcal{T})$, where S is an n-set, and $\mathcal{T}$ is a collection of 3-subsets of S called triples or blocks, such that each pair of distinct elements of S appear together in λ triples. We can consider a triple system $\mathsf{TS}(n, \lambda)$ to be a $\mathsf{GDD}(g = n, 1, 3; \lambda, \lambda_2)$ or a $\mathsf{GDD}(g = 1 + 1 + 1 + \cdots + 1, n, 3; \lambda_1, \lambda_2)$ where λ_1 and λ_2 are any nonnegative integers. The existence of triple systems is concluded in Theorem 2.2. See more details in [7].

Theorem 2.2. [7] *A* $\mathsf{TS}(n, \lambda)$ *exists if and only if* λ *and* n *satisfy one of the following cases:*

(*i*) $\lambda \equiv 0 \pmod 6$ *for all positive integers* $n \neq 2$*,*

(*ii*) $\lambda \equiv 1$ *or* $5 \pmod 6$ *for all positive integers* $n \equiv 1$ *or* $3 \pmod 6$*,*

(*iii*) $\lambda \equiv 2$ *or* $4 \pmod 6)$ *for all positive integers* $\equiv 0$ *or* $1 \pmod 3$*, and*

(*iv*) $\lambda \equiv 3 \pmod 6$ *for all odd positive integers.*

A *packing with triangles* of the complete graph K_n is a 3-tuple $(S, \mathcal{T}, \mathcal{L})$, where S is the vertex set of K_n, $\mathcal{T}$ is a collection of edge disjoint complete subgraphs K_3 of K_n, and $\mathcal{L}$ is the collection of edges in K_n not belonging to one of the K_3 of $\mathcal{T}$. The collection of edges $\mathcal{L}$ is called the *leave*. If $|\mathcal{L}|$ is as small as possible, then $(S, \mathcal{T}, \mathcal{L})$ is called a *maximum packing of order* n; see [7].

Theorem 2.3. [7] *Let* n *be any positive integer. If* $(S, \mathcal{T}, \mathcal{L})$ *is a maximum packing of order* n*, then the leave is*

(*i*) *a* 1*-factor if* $n \equiv 0$ *or* $2 \pmod 6$*,*

(*ii*) *a* 4*-cycle if* $n \equiv 5 \pmod 6$*,*

(*iii*) *a tripole, that is a spanning graph with each vertex having odd degree and containing* $\frac{n+2}{2}$ *edges, if* $n \equiv 4 \pmod 6$*, and*

(*iv*) *the empty set if* $n \equiv 1$ *or* $3 \pmod 6$*.*

3 Sufficiency

When $\lambda_2 = 0$, a $\mathsf{GDD}(n, n, n, 1; \lambda_1, 0)$ exists if and only if there exists a $\mathsf{TS}(n, \lambda_1)$. Hence we focus only on GDDs with $\lambda_2 \geq 1$. When $n = 2$, the extra necessary condition (*iii*) in Theorem 1.1 suggests that we construct a $\mathsf{GDD}(2, 2, 2, 1; \lambda_1, \lambda_2)$ separately.

Throughout the rest of the paper, for any positive integer n we let $X_n = \{x_1, x_2, \ldots, x_n\}$, $Y_n = \{y_1, y_2, \ldots, y_n\}$, $Z_n = \{z_1, z_2, \ldots, z_n\}$ and $W = \{w\}$ be disjoint sets and let $V_n = X_n \cup Y_n \cup Z_n \cup W$.

Lemma 3.1. *Let $n, \lambda_2 \geq 1$ and $\lambda_1 \geq 0$ be integers. There exists a* $\mathsf{GDD}(2, 2, 2, 1; \lambda_1, \lambda_2)$ *where* $2 \mid (\lambda_1 + \lambda_2)$ *and* $\lambda_1 \leq \lambda_2$.

Proof. Let

$$\begin{aligned}\mathcal{B}_1 = \{&\{w, x_1, y_2\}, \{w, y_2, z_1\}, \{w, z_1, x_2\}, \{w, x_2, y_1\}, \{w, y_1, z_2\}, \{w, z_2, x_1\},\\ &\{x_1, y_1, z_1\}, \{x_2, y_2, z_2\}, \{x_1, y_1, z_1\}, \{x_1, y_2, z_2\}, \{x_2, y_1, z_2\}, \{x_2, y_2, z_1\}\}.\end{aligned}$$

Then $(V_2, \mathcal{B}_1)$ is a $\mathsf{GDD}(2, 2, 2, 1; 0, 2)$. Since $2 \mid (\lambda_2 + \lambda_1)$ and $\lambda_1 \leq \lambda_2$, we have that $\lambda_2 - \lambda_1$ is an even nonnegative integer. Let $\mathcal{B}$ be $\frac{1}{2}(\lambda_2 - \lambda_1)$ copies of $\mathcal{B}_1$; then $(V_2, \mathcal{B})$ is a $\mathsf{GDD}(2, 2, 2, 1; 0, \lambda_2 - \lambda_1)$. By Theorem 2.2, we can let $(V_2, \mathcal{T})$ be a $\mathsf{TS}(7, \lambda_1)$, and thus $(V_2, \mathcal{B} \cup \mathcal{T})$ is our desired GDD. □

Lemma 3.2. *Let $n, \lambda_2 \geq 1$ and $\lambda_1 \geq 0$ be integers. There exists a* $\mathsf{GDD}(2, 2, 2, 1; \lambda_1, \lambda_2)$ *where* $2 \mid (\lambda_1 + \lambda_2)$ *and* $\lambda_2 < \lambda_1 \leq 3\lambda_2$.

Proof. Let

$$\begin{aligned}\mathcal{B} =\{&\{w, x_1, x_2\}, \{w, y_1, y_2\}, \{w, z_1, z_2\}, \{x_1, x_2, y_1\}, \{x_1, x_2, y_2\},\\ &\{y_1, y_2, z_1\}, \{y_1, y_2, z_2\}, \{z_1, z_2, x_1\}, \{z_1, z_2, x_2\}\}.\end{aligned}$$

Then $(V_2, \mathcal{B})$ is a $\mathsf{GDD}(2, 2, 2, 1; 3, 1)$. By Theorem 2.2, let $(V_2, \mathcal{T})$ is a $\mathsf{TS}(7, 1)$. Since $2 \mid (\lambda_1 + \lambda_2)$ and $\lambda_2 < \lambda_1 \leq 3\lambda_2$, write $\lambda_1 - \lambda_2 = 2q$ where $0 \leq q \leq \lambda_2$. Then $(V_2, \mathcal{C})$ where $\mathcal{C}$ is the union of q copies of $\mathcal{B}$ and $\lambda_2 - q$ copies of $\mathcal{T}$ is a GDD(2,2,2,1;λ_1, λ_2). □

For $n \neq 2$, we first construct $\mathsf{GDD}(n, n, n, 1; \lambda_1, \lambda_2)$s where $\lambda_1 \leq \lambda_2$, and use them to construct the GDDs for the remaining case $\lambda_1 > \lambda_2$. Lemma 3.3 constructs the designs containing only cross edges.

Lemma 3.3. *Let $n, \lambda_2 \geq 1$ and $\lambda_1 \geq 0$ be integers. If $n \neq 2$ and λ_2 is even, then there exists a* $\mathsf{GDD}(n, n, n, 1; 0, \lambda_2)$.

Proof. It suffices to show only when $\lambda_2 = 2$. For $i \in \{1, 2, \ldots, n\}$, let

$$\mathcal{B}_i = \{\{w, x_i, y_i\}, \{w, x_i, z_i\}, \{w, y_i, z_i\}, \{x_i, y_i, z_i\}\}.$$

By Theorem 2.1, there is an idempotent latin square $L = \{l_{ij}\}$ of order n. Let

$$\mathcal{B} = \{\{x_i, y_j, z_k\} : i, j, k \in \{1, 2, \ldots, n\}, l_{ij} = k \text{ and } i \neq j\}.$$

It is easily seen that pairs of elements from different groups occur either twice in $\mathcal{B}_i$ or precisely once in $\mathcal{B}$. Then $(V_n, \mathcal{B}^*)$ is a $\mathsf{GDD}(n, n, n, 1; 0, 2)$, where $\mathcal{B}^*$ is the union of $\mathcal{B}_i$, for all $i \in \{1, 2, \ldots, n\}$ and two copies of $\mathcal{B}$. □

Lemmas 3.4, 3.5 and 3.7 provide the construction of a $\mathsf{GDD}(n, n, n, 1; \lambda_1, \lambda_2)$ where $n \neq 2$ and $\lambda_1 \leq \lambda_2$.

Lemma 3.4. *Let $n, \lambda_2 \geq 1$ and $\lambda_1 \geq 0$ be integers. If $n \neq 2$ is even, $\lambda_1 \leq \lambda_2$ and $\lambda_1 \equiv \lambda_2 \pmod 2$, then there exists a* $\mathsf{GDD}(n, n, n, 1; \lambda_1, \lambda_2)$.

Proof. Since n is even, then, by Theorem 2.2, there exists a $\mathsf{TS}(3n + 1, \lambda_1)$, say $(V_n, \mathcal{T})$. Since $\lambda_1 \equiv \lambda_2 \pmod 2$, $\lambda_2 - \lambda_1$ is even. Then, by Lemma 3.3, let $(V_n, \mathcal{B})$ be a $\mathsf{GDD}(n, n, n, 1; 0, \lambda_2 - \lambda_1)$. Therefore $(V_n, \mathcal{B} \cup \mathcal{T})$ is a desired GDD. □

Lemma 3.5. *Let $n, \lambda_2 \geq 1$ and $\lambda_1 \geq 0$ be integers. If $n \equiv 1, 3 \pmod 6$, and $\lambda_2 \equiv 0 \pmod 2$, then there exists a* $\mathsf{GDD}(n, n, n, 1; \lambda_1, \lambda_2)$.

Proof. We can decompose pure edges and cross edges separately. For pure edges, since $n \equiv 1, 3 \pmod 6$, by Theorem 2.2, there exists a $\mathsf{TS}(n, \lambda_1)$. Let $(X_n, \mathcal{T}_1)$, $(Y_n, \mathcal{T}_2)$ and $(Z_n, \mathcal{T}_3)$ be those $\mathsf{TS}(n, \lambda_1)$s which exist from the theorem. For cross edges, since λ_2 is even, by Lemma 3.3, there exists a $\mathsf{GDD}(n, n, n, 1; 0, \lambda_2)$, say $(V_n, \mathcal{B})$. Then $(V_n, \mathcal{B} \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3)$ is a desired GDD. □

Lemma 3.6. *Let $n > 0$ be an integer. If $n \equiv 5 \pmod 6$, then there exists a* $\mathsf{GDD}(n, n, n, 1; 1, 2)$.

Proof. Let $(X_n, \mathcal{T}_1, \mathcal{L}_1)$, $(Y_n, \mathcal{T}_2, \mathcal{L}_2)$ and $(Z_n, \mathcal{T}_3, \mathcal{L}_3)$ be maximum packings with triangles of order n. In addition, by Theorem 2.3 (*ii*), since $n \equiv 5 \pmod 6$, the leaves $\mathcal{L}_i$ are 4-cycles, say

$$\begin{aligned}
\mathcal{L}_1 &= \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_1\}\},\\
\mathcal{L}_2 &= \{\{y_1, y_2\}, \{y_2, y_3\}, \{y_3, y_4\}, \{y_4, y_1\}\},\\
\mathcal{L}_3 &= \{\{z_1, z_2\}, \{z_2, z_3\}, \{z_3, z_4\}, \{z_4, z_1\}\}.
\end{aligned}$$

Let

$$\begin{aligned}
\mathcal{C}_1 &= \{\{w, x_1, x_2\}, \{w, x_2, x_3\}, \{w, x_3, x_4\}, \{w, x_4, x_1\}\},\\
\mathcal{C}_2 &= \{\{w, y_1, y_2\}, \{w, y_2, y_3\}, \{w, y_3, y_4\}, \{w, y_4, y_1\}\},\\
\mathcal{C}_3 &= \{\{w, z_1, z_2\}, \{w, z_2, z_3\}, \{w, z_3, z_4\}, \{w, z_4, z_1\}\}.
\end{aligned}$$

For $i \in \{5, 6, \ldots, n\}$, let

$$\mathcal{B}_{ii} = \{\{x_i, y_i, z_i\}, \{w, x_i, y_i\}, \{w, y_i, z_i\}, \{w, x_i, z_i\}\}.$$

By Theorem 2.1, there is an idempotent latin square $L = \{l_{ij}\}$ of order n.
For $i \neq j \in \{1, 2, \ldots, n\}$ or $i = j \in \{1, 2, 3, 4\}$, let

$$\mathcal{B}_{ij} = \{\{x_i, y_j, z_{l_{ij}}\}, \{x_i, y_j, z_{l_{ij}}\}\}.$$

Let $\mathcal{T}$ be $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, $\mathcal{B}$ the union of $\mathcal{B}_{ij}$, for $i, j \in \{1, 2, \ldots, n\}$, and $\mathcal{C}$ the union of $\mathcal{C}_i$, for $i \in \{1, 2, 3\}$. Thus $(V_n, \mathcal{T} \cup \mathcal{B} \cup \mathcal{C})$ is a $\mathsf{GDD}(n, n, n, 1; 1, 2)$. □

Lemma 3.7. *Let $n, \lambda_2 \geq 1$ and $\lambda_1 \geq 0$ be integers. If $n \equiv 5 \pmod 6$, $\lambda_1 \leq \lambda_2$ and $\lambda_2 \equiv 0 \pmod 2$, then there exists a* $\mathsf{GDD}(n, n, n, 1; \lambda_1, \lambda_2)$.

Proof. If $\lambda_1 \equiv 0 \pmod 3$ then a $\mathsf{TS}(n, \lambda_1)$ exists. Let $(X_n, \mathcal{T}_1)$, $(Y_n, \mathcal{T}_2)$ and $(Z_n, \mathcal{T}_3)$ be $\mathsf{TS}(n, \lambda_1)$s. Since λ_2 is even, by Lemma 3.3, let $(V_n, \mathcal{B})$ be a $\mathsf{GDD}(n, n, n, 1; 0, \lambda_2)$. Then $(V_n, \mathcal{B} \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3)$ is the desired GDD. If $\lambda_1 \equiv 1 \pmod 3$ then $\lambda_1 - 1 \equiv 0 \pmod 3$. Then there exists a $\mathsf{GDD}(n, n, n, 1; \lambda_1 - 1, \lambda_2 - 2)$; together with a $\mathsf{GDD}(n, n, n, 1; 1, 2)$ in Lemma 3.6, we have the desired GDD. Similarly, a $\mathsf{GDD}(n, n, n, 1; \lambda_1 - 2, \lambda_2 - 2)$ exists where $\lambda_1 \equiv 2 \pmod 3$, and by Theorem 2.2, there always exists a $\mathsf{TS}(3n+1, 2)$ which is a $\mathsf{GDD}(n, n, n, 1; 2, 2)$; combining them together yields our desired GDD. □

Theorem 3.8. *Let $n, \lambda_2 \geq 1$ and $\lambda_1 \geq 0$ be integers. If $(n, \lambda_1, \lambda_2)$ satisfy the necessary conditions in Theorem* 1.1 *and $\lambda_1 \leq \lambda_2$, then there exists a* $\mathsf{GDD}(n, n, n, 1; \lambda_1, \lambda_2)$.

Proof. If $(n, \lambda_1, \lambda_2)$ satisfies the necessary conditions in Theorem 1.1, then $\lambda_1 \equiv \lambda_2 \pmod 2$ where n is even, and $\lambda_2 \equiv 0 \pmod 2$ where n is odd. Therefore, by Lemmas 3.1, 3.4, 3.5 and 3.7, there exists a $\mathsf{GDD}(n, n, n, ; \lambda_1, \lambda_2)$ where $\lambda_1 \leq \lambda_2$ □

Lemma 3.9. *Let n be any positive integer. If $n \neq 2$ is even, then there exists a* $\mathsf{GDD}(n, n, n, 1; 3, 1)$.

Proof. If $n \equiv 0, 4 \pmod 6$ then, by Theorem 2.2, let $(X_n, \mathcal{T}_1)$, $(Y_n, \mathcal{T}_2)$ and $(Z_n, \mathcal{T}_3)$ be $\mathsf{TS}(n, 2)$s and $(V_n, \mathcal{T})$ be a $\mathsf{TS}(3n+1, 1)$. Then $(V_n, \mathcal{T} \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3)$ is a desired GDD.

If $n \equiv 2 \pmod 6$ then, by Theorem 2.3, let $(X_n, \mathcal{T}_1, \mathcal{L}_1)$, $(Y_n, \mathcal{T}_2, \mathcal{L}_2)$ and $(Z_n, \mathcal{T}_3, \mathcal{L}_3)$ be maximum packings with triangles of order n; the leaves $\mathcal{L}_t$, $t \in \{1, 2, 3\}$ are 1-factors. Let

$$\begin{aligned}
\mathcal{L}_1 &= \{\{x_1, x_{(1+\frac{n}{2})}\}, \{x_2, x_{(2+\frac{n}{2})}\}, \ldots, \{x_{(\frac{n}{2})}, x_n\}\},\\
\mathcal{L}_2 &= \{\{y_1, y_{(1+\frac{n}{2})}\}, \{y_2, y_{(2+\frac{n}{2})}\}, \ldots, \{y_{(\frac{n}{2})}, y_n\}\},\\
\mathcal{L}_3 &= \{\{z_1, z_{(1+\frac{n}{2})}\}, \{z_2, z_{(2+\frac{n}{2})}\}, \ldots, \{z_{(\frac{n}{2})}, z_n\}\}.
\end{aligned}$$

Since $\frac{n}{2} \neq 2$, by Theorem 2.1, there is an idempotent latin square of order $\frac{n}{2}$, say $L = \{l_{ij}\}$. For $i \in \{1, 2, \ldots, \frac{n}{2}\}$, let

$$\begin{aligned}
\mathcal{B}_{ii} = \{&\{w, x_i, x_{(i+\frac{n}{2})}\}, \{w, y_i, y_{(i+\frac{n}{2})}\}, \{w, z_i, z_{(i+\frac{n}{2})}\},\\
&\{x_i, y_i, z_i\}, \{x_i, y_{(i+\frac{n}{2})}, z_{(i+\frac{n}{2})}\},\\
&\{x_{(i+\frac{n}{2})}, y_i, z_{(i+\frac{n}{2})}\}, \{x_{(i+\frac{n}{2})}, y_{(i+\frac{n}{2})}, z_i\}\},
\end{aligned}$$

and for $i \neq j \in \{1, 2, \ldots, \frac{n}{2}\}$, let

$$\begin{aligned}
\mathcal{B}_{ij} = \{&\{x_i, y_j, z_{l_{ij}}\}, \{x_i, y_{(j+\frac{n}{2})}, z_{(l_{ij}+\frac{n}{2})}\},\\
&\{x_{(i+\frac{n}{2})}, y_j, z_{(l_{ij}+\frac{n}{2})}\}, \{x_{(i+\frac{n}{2})}, y_{(j+\frac{n}{2})}, z_{l_{ij}}\}\}.
\end{aligned}$$

Thus $(V_n, \mathcal{T} \cup \mathcal{B})$, where $\mathcal{T}$ is the union of three copies of $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, and $\mathcal{B}$ is the union of $\mathcal{B}_{ij}$ for all $i, j \in \{1, 2, \ldots, \frac{n}{2}\}$, is a $\mathsf{GDD}(n, n, n, 1; 3, 1)$. □

Lemma 3.10. *Let $n > 0$ be an integer. If $n \neq 2$, then there exists a* GDD $(n, n, n, 1; 5, 1)$.

Proof. If $n \equiv 0, 4 \pmod 6$ then, by Theorem 2.2, let $(X_n, \mathcal{T}_1)$, $(Y_n, \mathcal{T}_2)$ and $(Z_n, \mathcal{T}_3)$ be $\mathsf{TS}(n, 4)$s and $(V_n, \mathcal{T})$ be a $\mathsf{TS}(3n + 1, 1)$. Then $(V_n, \mathcal{T} \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3)$ is a desired GDD.

If $n \equiv 2 \pmod 6$, then let $(X, \mathcal{T}_1, \mathcal{L}_1)$, $(Y, \mathcal{T}_2, \mathcal{L}_2)$ and $(Z, \mathcal{T}_3, \mathcal{L}_3)$ be maximum packings with triangles of order n. Then by Theorem 2.3, the leaves $\mathcal{L}_t$, $t \in \{1, 2, 3\}$ are 1-factors. Let

$$\begin{aligned}
\mathcal{L}_1 &= \{\{x_1, x_{(1+\frac{n}{2})}\}, \{x_2, x_{(2+\frac{n}{2})}\}, \ldots, \{x_{(\frac{n}{2})}, x_n\}\},\\
\mathcal{L}_2 &= \{\{y_1, y_{(1+\frac{n}{2})}\}, \{y_2, y_{(2+\frac{n}{2})}\}, \ldots, \{y_{(\frac{n}{2})}, y_n\}\},\\
\mathcal{L}_3 &= \{\{z_1, z_{(1+\frac{n}{2})}\}, \{z_2, z_{(2+\frac{n}{2})}\}, \ldots, \{z_{(\frac{n}{2})}, z_n\}\}.
\end{aligned}$$

Since $\frac{n}{2}$ is neither 2 nor 6, by Theorem 2.1 we can let $L = \{l_{ij}\}$ and $L^* = \{l^*_{ij}\}$ be two orthogonal latin squares of order $\frac{n}{2}$. For each pair of $i, j \in \{1, 2, \ldots, \frac{n}{2}\}$, we define $\mathcal{B}_{ij}$ in three cases separately.

For i, j such that $l^*_{ij} = 1$, let $\mathcal{B}_{ij}$ be the collection of blocks of a GDD(2,2,2,1;3,1) based on the symbol set $\{x_i, x_{(i+\frac{n}{2})}\} \cup \{y_j, y_{(j+\frac{n}{2})}\} \cup \{z_{l_{ij}}, z_{(l_{ij}+\frac{n}{2})}\} \cup \{w\}$.

For i, j such that $l^*_{ij} = 2$, let

$$\begin{aligned}
\mathcal{B}_{ij} = \{&\{x_i, x_{(i+\frac{n}{2})}, y_j\}, \{x_i, x_{(i+\frac{n}{2})}, y_{(j+\frac{n}{2})}\}, \{y_j, y_{(j+\frac{n}{2})}, z_{l_{ij}}\},\\
&\{y_j, y_{(j+\frac{n}{2})}, z_{(l_{ij}+\frac{n}{2})}\}, \{z_{l_{ij}}, z_{(l_{ij}+\frac{n}{2})}, x_i\}, \{z_{l_{ij}}, z_{(l_{ij}+\frac{n}{2})}, x_{(i+\frac{n}{2})}\}\}.
\end{aligned}$$

For i, j such that $l^*_{ij} \geq 3$, let

$$\begin{aligned}
\mathcal{B}_{ij} = \{&\{x_i, y_j, z_{l_{ij}}\}, \{x_i, y_{(j+\frac{n}{2})}, z_{(l_{ij}+\frac{n}{2})}\}, \{x_{(i+\frac{n}{2})}, y_j, z_{(l_{ij}+\frac{n}{2})}\},\\
&\{x_{(i+\frac{n}{2})}, y_{(j+\frac{n}{2})}, z_{l_{ij}}\}\}.
\end{aligned}$$

Let $\mathcal{T}$ be the union of five copies of $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$ and $\mathcal{B}$ the union of $\mathcal{B}_{ij}$, for all i, j. Thus $(V_n, \mathcal{T} \cup \mathcal{B})$ is a $\mathsf{GDD}(n, n, n, 1; 5, 1)$. ☐

The next main theorem is the conclusion for the existence of our desired GDDs.

Theorem 3.11. *Let $n > 0$ and $\lambda_1, \lambda_2 \geq 0$ be integers. There exists a* GDD*$(n, n, n, 1; \lambda_1, \lambda_2)$ if and only if*

(*i*) $\lambda_2 = 0$, *and there exists a* $\mathsf{TS}(n, \lambda_1)$,
(*ii*) $\lambda_2 \neq 0, n = 2$, $2 \mid (\lambda_1 + \lambda_2)$ *and* $\lambda_1 \leq 3\lambda_2$, *or*
(*iii*) $\lambda_2 \neq 0, n \neq 2$, *and* $2 \mid (n - 1)\lambda_1 + \lambda_2$.

Proof. For necessity, (i) holds by Theorem 2.2, and the other conditions hold by Theorem 1.1. Now we prove the sufficiency. If we assume (i), then the statement is true trivially. If we assume (ii), then the sufficiency has been proved by Lemmas 3.1–3.2.

Next we assume (iii); let $n \neq 2$ and $\lambda_2 \neq 0$. Theorem 3.8 provides the case $\lambda_1 \leq \lambda_2$. Assume that $\lambda_1 > \lambda_2$ and write $\lambda_1 = 6q + r$ where $0 \leq r < 6$. If

$r \leq \lambda_2$, by Theorem 3.8 we can let $(V_n, \mathcal{B})$ be a $\mathsf{GDD}(n, n, n, 1; r, \lambda_2)$. Let $(X_n, \mathcal{T}_1)$, $(Y_n, \mathcal{T}_2)$ and $(Z_n, \mathcal{T}_3)$ be $\mathsf{TS}(n, 6q)$s. Then $(V_n, \mathcal{B} \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3)$ is a desired GDD. Now suppose that $r > \lambda_2$. Then when n is even, (r, λ_2) can only be one of the ordered pairs in $\{(3, 1), (4, 2), (5, 1), (5, 3)\}$; and when n is odd, (r, λ_2) can only be in $\{(3, 2), (4, 2), (5, 2), (5, 4)\}$.

If n is even, a $\mathsf{GDD}(n, n, n, 1; 3, 1)$ and a $\mathsf{GDD}(n, n, n, 1; 5, 1)$ are constructed in Lemmas 3.9 and 3.10, respectively. Furthermore, a $\mathsf{GDD}(n, n, n, 1; 4, 2)$ and a $\mathsf{GDD}(n, n, n, 1; 5, 3)$ can be constructed by adding one and two copies of a $\mathsf{TS}(3n + 1, 1)$ to a $\mathsf{GDD}(n, n, n, 1; 3, 1)$, respectively.

If n is odd, by Lemma 3.5 and 3.6 , let $(V_n, \mathcal{B}_1)$ be a $\mathsf{GDD}(n, n, n, 1; 1, 2)$, and by Lemma 3.3, let $(V_n, \mathcal{B}_2)$ be a $\mathsf{GDD}(n, n, n, 1; 0, 2)$. By Theorem 2.2, let $(X_n, \mathcal{T}_1)$, $(Y_n, \mathcal{T}_2)$ and $(Z_n, \mathcal{T}_3)$ be $\mathsf{TS}(n, 3)$s, and $(V_n, \mathcal{T}_4)$ be a $\mathsf{TS}(3n + 1, 2)$. Therefore $(V_n, \mathcal{B})$ is a $\mathsf{GDD}(n, n, n, 1; r, \lambda_2)$, where $(r, \lambda_2, \mathcal{B})$ is as in the following table.

(r, λ_2)	$\mathcal{B}$
$(3, 2)$	$\mathcal{B}_2 \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$
$(4, 2)$	$\mathcal{B}_1 \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$
$(5, 2)$	$\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4$
$(5, 4)$	$\mathcal{B}_2 \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4$

□

4 $\mathsf{GDD}(g = n + \cdots + n + 1, t + 1, 3; \lambda_1, \lambda_2)$

In this section, we introduce a construction of infinitely many $\mathsf{GDD}(g = n + n + \cdots + n + 1, t + 1, 3; \lambda_1, \lambda_2)$s that utilizes our result and the results from [5] and [6], namely the existence of a $\mathsf{GDD}(g = n + n + n + 1, 4, 3; \lambda_1, \lambda_2)$ and a $\mathsf{GDD}(g = n + n + 1, 3, 3; \lambda_1, \lambda_2)$. The necessary conditions to obtain the desired GDDs are shown in Theorem 4.1 which can be proved by a standard idea, similar to the proof of Theorem 1.1. Theorems 4.2 and 4.3 form a complete solution for the existence of a $\mathsf{GDD}(g = n + n + 1, 3, 3; \lambda_1, \lambda_2)$. Beside these two theorems, our construction also needs Theorems 4.4 and 4.5 to obtain a K_3-decomposition of the graph $\lambda_2 K_{t(n)}$ and $\lambda_2 K_{t(n),m}$ where $K_{t(n)}$ is a complete multipartite graph with t groups of size n and $K_{t(n),m}$ is a complete multipartite graph with t groups of size n and one group of size m; see more details in [1] and [4].

Theorem 4.1. (Necessary Conditions) *Let n, t, λ_1 and λ_2 be integers. If there exists a $\mathsf{GDD}(g = n + n + \cdots + n + 1, t + 1, 3; \lambda_1, \lambda_2)$ if and only if*

(i) $2 \mid \lambda_2 tn$,
(ii) $2 \mid [\lambda_1(n - 1) + \lambda_2((t - 1)n + 1)]$,
(iii) $3 \mid tn[\lambda_1(n - 1) + \lambda_2(n(t - 1) + 2)]$, *and*
(iv) *if* $n = 2$ *then* $\lambda_1 \leq t\lambda_2$.

Theorem 4.2. [5] *Let n, λ_1, λ_2 be positive integers. If $\lambda_1 < \lambda_2$ then there exists a* $\mathsf{GDD}(g = n + n + 1, 3, 3; \lambda_1, \lambda_2)$ *if and only if*

(*i*) $2 \mid [\lambda_1(n-1) + \lambda_2(n+1)]$,
(*ii*) $3 \mid [\lambda_1 n(n-1) + \lambda_2 n(n+2)]$, *and*
(*iii*) $\lambda_2 \leq 2\lambda_1$.

Theorem 4.3. [6] *Let $n \geq 3, \lambda_1, \lambda_2$ be positive integers. If $\lambda_1 \geq \lambda_2$ then there exists a* $\mathsf{GDD}(g = n + n + 1, 3, 3; \lambda_1, \lambda_2)$ *if and only if*

(*i*) $2 \mid [\lambda_1(n-1) + \lambda_2(n+1)]$, *and*
(*ii*) $3 \mid [\lambda_1 n(n-1) + \lambda_2 n(n+2)]$.

Theorem 4.4. [4] *Let n, t and λ_2 be positive integers. If $n \neq 2$ then there exists a* $\mathsf{GDD}(g = n + n + n + \cdots + n, t, 3; 0, \lambda_2)$ *if and only if*

(*i*) $2 \mid \lambda_2(t-1)n$
(*ii*) $3 \mid \lambda_2 t(t-1)n^2$, *and*
(*iii*) $t \geq 3$.

Theorem 4.5. [1] *Let $n, m, t \geq 1$ be integers. There exists a* $\mathsf{GDD}(g = n + n + n + \cdots + n + m, t + 1, 3; 0, 1)$ *if and only if*

(*i*) *If $t = 2$ then $m = n$,*
(*ii*) $m \leq n(t-1)$,
(*iii*) $2 \mid [(t-1)n + m]$
(*iv*) $2 \mid tn$, *and*
(*v*) $3 \mid [t(t-1)n^2 + tmn]$.

Now we will construct infinitely many $\mathsf{GDD}(g = n+n+\cdots+n+1, t+1, 3; \lambda_1, \lambda_2)$s using Theorems 4.2–4.5, together with Theorem 3.11. The construction is separated into two corollaries depending on the parity of the number of groups of size n in the design. For our construction, let $X_i = \{x_{i1}, x_{i2}, \ldots, x_{in}\}$ for $i = 1, 2, \ldots, t$ and $W = \{w\}$ be disjoint sets.

Corollary 4.6. *Let $n \geq 3, \lambda_1$ and λ_2 be positive integers, and let $t \geq 6$ be an even integer. If there exists a* $\mathsf{GDD}(g = n + n + 1, 3, 3; \lambda_1, \lambda_2)$*, and $n, t, \lambda_1, \lambda_2$ satisfy the necessary conditions in* Theorem 4.1*, then there exists a* $\mathsf{GDD}(g = n + n + \cdots + n + 1, t + 1, 3; \lambda_1, \lambda_2)$.

Proof. By the assumption, for $i = 1, 2, \ldots, \frac{t}{2}$, we can construct a $\mathsf{GDD}(g = n + n + 1, 3, 3; \lambda_1, \lambda_2)$ based on the element set $X_{2i-1} \cup X_{2i} \cup W$. Let H be the graph that represents such construction for all i. Suppose that a K_3-decomposition of graph G represents a $\mathsf{GDD}(g = n + n + \cdots + n + 1, t + 1, 3; \lambda_1, \lambda_2)$. Then $G = H + \lambda_2 K_{\frac{t}{2}(2n)}$. By Theorems 4.1-4.3, the degree of each vertex of both graphs G and H is even and the number of edges of graphs G and H are both divisible by three. These yield that the degree of each vertex of the graph $K_{\frac{t}{2}(2n)}$ is also even, and the number of edges of the graph $K_{\frac{t}{2}(2n)}$ is also divisible by three. Moreover, $\frac{t}{2} \geq 3$ since $t \geq 6$. By Theorem 4.4, the graph $K_{\frac{t}{2}(2n)}$ can be decomposed into triangles. Hence there exists a desired GDD. □

Corollary 4.7. *Let $n \geq 3, \lambda_1$ and λ_2 be positive integers, and let $t \geq 9$ be an odd integer. If there exist a* $\mathsf{GDD}(g = n+n+1, 3, 3; \lambda_1, \lambda_2)$ *and a* $\mathsf{GDD}(g = n+n+n+1, 4, 3; \lambda_1, \lambda_2)$*, and $n, t, \lambda_1, \lambda_2$ satisfy the necessary conditions in* Theorem 4.1*, then there exists a* $\mathsf{GDD}(g = n+n+\cdots+n+1, t+1, 3; \lambda_1, \lambda_2)$.

Proof. We construct a $\mathsf{GDD}(g = n+n+1, 3, 3; \lambda_1, \lambda_2)$ based on the element set $X_{2i-1} \cup X_{2i} \cup W$ for $i = 1, 2, \ldots, \frac{t-3}{2}$, represented by a K_3-decomposition of the graph H_1, and a $\mathsf{GDD}(g = n+n+n+1, 4, 3; \lambda_1, \lambda_2)$ based on the element set $X_{t-2} \cup X_{t-1} \cup X_t \cup W$, represented by a K_3-decomposition of the graph H_2. We construct our desired GDD similarly to the previous Corollary that is a K_3-decomposition of the graph $G = H_1 + H_2 + \lambda_2 K_{\frac{t-3}{2}(2n),3n}$ and use of Theorem 4.5 to obtain a K_3-decomposition of the graph $\lambda_2 K_{\frac{t-3}{2}(2n),3n}$. □

Acknowledgements

We would like to thank all the anonymous reviewers for their very helpful insights. In particular, their suggestion to add the last section is very much appreciated. Their comments greatly improved the manuscript.

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(Received 5 Sep 2015; revised 22 Jan 2016, 7 June 2017)