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On graphs having a unique minimum independent dominating set

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Abstract

In this paper, we consider graphs having a unique minimum independent dominating set. We first discuss the effects of deleting a vertex, or the closed neighborhood of a vertex, from such graphs. We then discuss five operations which, in certain circumstances, can be used to combine two graphs, each having a unique minimum independent dominating set, to produce a new graph also having a unique minimum independent dominating set. Using these operations, we characterize the set of trees having a unique minimum independent dominating set.

1 Introduction

In this paper, we consider graphs having a unique minimum independent dominating set. Unique minimum dominating sets, both independent and otherwise, have been much studied. For example, unique minimum vertex dominating sets were first considered in [7] where trees were the class of graphs primarily considered. Since then, unique minimum dominating sets have been studied in block graphs, cactus graphs, and Cartesian products (see [1, 3, 9, 10]). The maximum number of edges contained in a graph having a unique minimum dominating set of a specified cardinality was considered in [2] and [6].

Graphs containing a unique minimum independent dominating set have received less attention. In [5], the authors discussed a hereditary class of graphs containing all graphs G for which every induced subgraph of G has a unique minimum independent dominating set if and only if it has a unique minimum dominating set. Unique minimum independent dominating sets were also considered in trees T satisfying

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$\gamma(T) = i(T)$. In [11], the maximum number of edges in a graph having a unique minimum independent dominating set of cardinality 2 was considered. We note that minimum independent dominating sets can also be viewed as maximal independent sets of minimum cardinality. Quite a bit of work has been done on graphs having a unique maximum independent set, and, in general, the total number of maximal independent sets in a given graph. We direct the reader towards [4, 12–15] for just a few examples of such work.

Subsequently, we begin in Section 3 by discussing the effects of deleting a vertex, or the closed neighborhood of a vertex, from a graph having a unique minimum independent dominating set. We then turn our attention to trees in Section 4, where we strengthen some of our earlier results. In Section 5, we consider a collection of operations which can be used to combine two graphs having a unique minimum independent dominating set to produce a new graph also having a unique minimum independent dominating set. Finally, in Section 6, we use these operations to characterize those trees having a unique minimum independent dominating set.

2 Notation and Definitions

In this paper, we consider only finite, simple graphs. Given a graph G, we let $V(G)$ denote the vertex set of G and $E(G)$ denote the edge set of G. If $v \in V(G)$, the *open neighborhood of* v, denoted $N(v)$, is defined by $N(v) = \{u : vu \in E(G)\}$ while the *closed neighborhood of* v, denoted $N[v]$, is defined by $N[v] = N(v) \cup \{v\}$. When required, we may write $N_G[v]$ to indicate the closed neighborhood of v in G. Given $S \subseteq V(G)$, the open and closed neighborhoods of S, denoted $N(S)$ and $N[S]$ respectively, are defined by $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup \{S\}$. We say that S *dominates* every vertex in its closed neighborhood. If $S \subseteq V(G)$ with $v \in S$, a *private neighbor of* v *with respect to* S is any vertex u such that $N[u] \cap S = \{v\}$. We note that it is possible for v to be a *self-private neighbor*. An *external private neighbor of* v *with respect to* S is any vertex belonging to the set $\{u \in V(G) - S : N[u] \cap S = \{v\}\}$. We let $epn(v, S)$ denote the set of external private neighors of v with respect to S. A subset of vertices D is a *dominating set* if $N[D] = V(G)$. The minimum cardinality of a dominating set in G, called the *domination number of* G, is denoted $\gamma(G)$, and any dominating set whose cardinality equals $\gamma(G)$ is a γ-set. A subset of vertices I is independent if no two vertices in I share an edge. The minimum cardinality of an independent dominating set in G is called the *independent domination number of* G, and is denoted by $i(G)$. Any independent dominating set of cardinality $i(G)$ is an i-set. As notational conventions, we let $\mathcal{UI}$ represent the class of graphs having a unique minimum independent dominating set. If $G \in \mathcal{UI}$, we let $I(G)$ denote the unique i-set of G. For other terminology and notation not explicitly mentioned, we follow [8].

3 Deleting vertices and closed neighborhoods

In [5], the authors prove the following.

Lemma 1. *[5] If any graph G has a unique i-set $I(G)$, then every vertex in $I(G)$ fullfills either $|epn(x, I(G))| = 0$ or $|epn(x, I(G))| \geq 2$.*

We are thus motivated to make the following definitions.

Definition 1. *Given a graph $G \in \mathcal{UI}$ and its unique i-set $I(G)$, we define the following sets.*

$$\begin{aligned}\mathcal{A}(I(G)) &= \{v \in I(G) : |epn(v, I(G))| \geq 2\}\\ \mathcal{B}(I(G)) &= \{v \in I(G) : |epn(v, I(G))| = 0\}\end{aligned}$$

We see that if $G \in \mathcal{UI}$, then $V(G)$ can be partitioned as $V(G) = \mathcal{A}(I(G)) \cup \mathcal{B}(I(G)) \cup (V(G) - I(G))$. Bearing this is mind, we now consider the implications of deleting a vertex, or the closed neighborhood of a vertex, chosen from each of these sets.

We begin with the following.

Lemma 2. *Let $G \in \mathcal{UI}$. For any $v \in V(G) - I(G)$, $i(G - v) = i(G)$.*

Proof. Since $v \notin I(G)$, we see that $I(G)$ dominates $G - v$. Hence, $i(G - v) \leq i(G)$. Suppose that $i(G - v) < i(G)$, and let D be an i-set for $G - v$. Consider then D in G. If D dominates G, then we arive at a contradiction since this implies that $I(G)$ is not a *minimum* independent dominating set. Thus, D fails to dominate v. In this case, $D \cup \{v\}$ is an independent dominating set of cardinality at most $|I(G)|$. This contradicts the *uniqueness* of $I(G)$. Our result is shown. □

We briefly note that if $G \in \mathcal{UI}$ and we delete a vertex $v \in V(G) - I(G)$, it is not guaranteed that $G - v \in \mathcal{UI}$. For example, $P_3 \in \mathcal{UI}$, but if we delete a leaf from P_3, the resulting graph, P_2, is not in $\mathcal{UI}$.

We note here that the conditions in Lemma 1, while necessary, are not sufficient to imply that a general graph G is a member of $\mathcal{UI}$ (take C_6 for example). They are, however, sufficient for trees T satisfying $\gamma(T) = i(T)$ as illustrated in [5]. For an arbitrary graph G, the following conditions are necessary and sufficient for $G \in \mathcal{UI}$.

Lemma 3. *For an arbitrary graph G, $G \in \mathcal{UI}$ if and only if there exists an i-set D of G such that for all $v \in V(G) - D$, $i(G - N[v]) \geq i(G)$.*

Proof. First, suppose that $G \in \mathcal{UI}$. In this case, let $D = I(G)$, and consider $v \in V(G) - D$. Observe that $N[v] \neq V(G)$ since otherwise $\{v\}$ is a minimum independent dominating set distinct from D, a contradiction. Thus, we may assume that $V(G - N[v])$ is nonempty. Suppose, then, that $i(G - N[v]) < i(G)$ and let D' be an i-set for $G - N[v]$. We see that $D' \cup \{v\}$ is an independent dominating set for G of cardinality at most $|I(G)|$, a contradiction. Thus, $i(G - N[v]) \geq i(G)$ as claimed.

Now suppose that G has an i-set D such that for all $v \in V(G) - D$, $i(G - N[v]) \geq i(G)$. For the sake of contradiction, suppose that $G \notin \mathcal{UI}$. Let D' be an i-set of G distinct from D, and let $v \in D' - D$. We see that $D' - \{v\}$ is an i-set for $G - N[v]$. Thus, $i(G - N[v]) = |D' - \{v\}| = |D'| - 1 = |D| - 1 < i(G)$. This, however, contradicts the assumed property of D. □

We now consider deleting a vertex from $I(G)$.

Lemma 4. *Let $G \in \mathcal{UI}$. For any $v \in \mathcal{A}(I(G))$, $i(G - v) \geq i(G)$.*

Proof. For the sake of contradiction, suppose that $i(G - v) < i(G)$, and let D be an i-set for $G - v$. Consider D in G. Since $v \in \mathcal{A}(I(G))$, v has at least two external private neighbors in G with respect to $I(G)$. Thus, D dominates every vertex in $epn(v, I(G))$. If D dominates G, then $I(G)$ is not a *minimum* independent dominating set, a contradiction. Hence, D fails to dominate v. In this case, $D \cup \{v\}$ is an independent dominating set of cardinality at most $|I(G)|$. Furthermore, since $epn(v, D \cup \{v\}) \neq epn(v, I(G))$, we see that $D \cup \{v\}$ is distinct from $I(G)$. Thus, the *uniqueness* of $I(G)$ has been contradicted. □

We briefly note that it is possible for $i(G - v) = i(G)$ for some $v \in \mathcal{A}(I(G))$ as the following example illustrates.

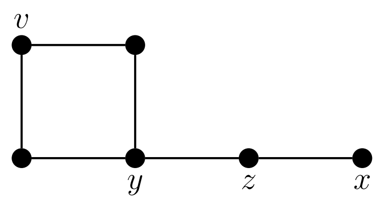

Figure 1: $i(G - v) = i(G)$

In this example, $i(G) = 2$, $I(G) = \{v, z\}$, and $i(G - v) = 2$ with an i-set given by $\{x, y\}$. We also note that if $v \in \mathcal{A}(I(G))$, then $G - v$ is not guaranteed to be in $\mathcal{UI}$. This is in contrast to the following result.

Lemma 5. *Let $G \in \mathcal{UI}$. For any $v \in \mathcal{B}(I(G))$, $G - v \in \mathcal{UI}$, and $I(G - v) = I(G) - \{v\}$.*

Proof. Since $v \in \mathcal{B}(I(G))$, v has no external private neighbors with respect to $I(G)$. Thus, $I(G) - \{v\}$ dominates $G - v$. Hence, $i(G - v) \leq i(G) - 1$. By similar logic as applied in the proof of Lemma 4, we see that $i(G - v) = i(G) - 1$.

Moreover, we also see that $I(G) - \{v\}$ is an i-set for $G - v$. Suppose $G - v$ has another i-set, call it D'. Note that D' dominates $G - v$ but does not dominate G, else we would have $i(G) = i(G) - 1$. Thus, in G, D' fails to dominate v. In particular, this implies that no neighbor of v is in D'. Hence, $D' \cup \{v\}$ is an independent dominating set of G of cardinality at most $|I(G)|$. Since $D' \neq I(G) - \{v\}$ we see that $D' \cup \{v\} \neq I(G)$, a contradiction. Thus, $G - v \in \mathcal{UI}$ with $I(G - v) = I(G) - \{v\}$. □

The sets $\mathcal{A}(I(G))$ and $\mathcal{B}(I(G))$ are similar in the following respect.

Lemma 6. *Let $G \in \mathcal{UI}$. For any $v \in I(G)$, $i(G-N[v]) = i(G)-1$, $G-N[v] \in \mathcal{UI}$, and $I(G-N[v]) = I(G)-\{v\}$.*

Proof. First note that $I(G)-\{v\}$ is an independent dominating set for $G-N[v]$. Thus, $i(G-N[v]) \leq i(G)-1$. Assuming $i(G-N[v]) < i(G)-1$ results in a contradiction as in the proof of Lemma 4. Thus, we have $i(G-N[v]) = i(G)-1$. If $G-N[v]$ has an i-set distinct from $I(G)-\{v\}$, call it D', then $D' \cup \{v\}$ is an i-set of G distinct from $I(G)$, a contradiction. Thus, we see that $G-N[v] \in \mathcal{UI}$ with $I(G-N[v]) = I(G)-\{v\}$. □

Our last lemma in this section does not concern deleting a vertex or a private neighbor. Since we use these techniques when proving the result, we present it here. We will make use of this result in Theorem 1 to come.

Lemma 7. *If $T \in \mathcal{UI}$ is a tree with $v \in V(G)-I(G)$, then $N[v] \cap \mathcal{A}(I(G)) \neq \emptyset$.*

Proof. Note that since $I(T)$ is a dominating set, $|N(v) \cap I(T)| \geq 1$. For the sake of contradiction, suppose that $(N(v) \cap I(T)) \subseteq \mathcal{B}(I(T))$ with $N(v) \cap I(T) = \{b_1, b_2, \ldots, b_k\}$. Consider then $T-N[v]$. Since T is a tree, b_i and b_j have no common neighbors when $i \neq j$. This, together with the fact that each b_j has no external private neighbors with respect to $I(T)$, implies that $I(T)-\{b_1, b_2, \ldots, b_k\}$ is an independent dominating set for $T-N[v]$. Thus, $i(T-N[v]) \leq i(T)-k$ for some $k \geq 1$. This, however, contradicts Lemma 3. Thus, v has a neighor in $\mathcal{A}(I(T))$. □

4 Trees

In this section, we seek to improve upon Lemma 4 in the case when G is a tree. Our proofs will take advantage of rooted trees. Thus, for notational convenience, given a rooted tree T, we let T_v denote the subgraph of T induced by v and all of its descendants.

We begin with the following.

Lemma 8. *Let $T \in \mathcal{UI}$ be a tree rooted at a vertex $v \in \mathcal{A}(I(T))$ with $epn(v, I(T)) = \{p_1, p_2, \ldots, p_k\}$. For $1 \leq j \leq k$, $i(T_{p_j}) = |I(T) \cap V(T_{p_j})|+1$ and $\{p_j\} \cup (I(T) \cap V(T_{p_j}))$ is a a minimum independent dominating set for T_{p_j}.*

Proof. For $j \in \{1, 2, \ldots, k\}$, consider T_{p_j}, the subtree of T induced by p_j and all of its descendants. By Lemma 6, $T-N[v] \in \mathcal{UI}$ with $I(T-N[v]) = I(T)-\{v\}$. This implies that $T_{p_j}-p_j \in \mathcal{UI}$ with $I(T_{p_j}-p_j) = V(T_{p_j}) \cap I(T)$. Notice that $V(T_{p_j}) \cap I(T)$ does not dominate p_j in T_{p_j} since p_j is an external private neighbor of v with respect to $I(T)$ in T. In particular, this implies that none of the descendants of p_j are contained in $I(T)$. Thus, let D be an i-set of T_{p_j}. There are two cases to consider.

- First, suppose that $p_j \notin D$. In this case, some descendant of p_j is contained in D, and D is an independent dominating set for $T_{p_j} - p_j$. Since $T_{p_j} - p_j \in \mathcal{UI}$ and no descendant of p_j is contained in $I(T_{p_j} - p_j)$, we see that $|D| > |I(T_{p_j} - p_j)| = |I(T) \cap V(T_{p_j})|$.

- Now, suppose that $p_j \in D$. In this case, no descendant of p_j is contained in D. Let $d_1, d_2, \ldots, d_n$ denote the descendants of p_j. Observe that if we delete p_j from T_{p_j}, we are left with a forest whose components, namely $T_{d_1}, T_{d_2}, \ldots, T_{d_n}$, are found in $T - N[v]$. Hence, by Lemma 6, the components of $T_{p_j} - p_j$ are each graphs in $\mathcal{UI}$. Thus, we see that

$$\begin{aligned} |D| &= 1 + |D \cap V(T_{p_j} - p_j)| \\ &= 1 + \sum_{m=1}^{n} |D \cap V(T_{d_m})| \\ &= 1 + \sum_{m=1}^{n} |I(T) \cap V(T_{d_m})| \quad \text{by Lemma 2} \\ &= 1 + |I(T) \cap V(T_{p_j} - p_j)| \\ &= 1 + |I(T) \cap V(T_{p_j})|. \end{aligned}$$

Thus, in either case, we see that $i(T_{p_j}) > |I(T) \cap V(T_{p_j})|$. Moreover, we also see that $\{p_j\} \cup (I(T) \cap V(T_{p_j}))$ is a minimum independent dominating set for T_{p_j}. Thus, our result is proven. □

This lemma is particularly nice since it implies the following.

Proposition 1. *Let $T \in \mathcal{UI}$ be a tree. For all $v \in \mathcal{A}(I(T))$, $i(T - v) > i(T)$.*

Proof. Root T at v. Let $epn(v, I(T)) = \{p_1, p_2, \ldots, p_k\}$ and let $N(v) - epn(v, I(T)) = \{n_1, n_2, \ldots, n_m\}$. If we delete v from T, we are left with $k + m$ components, namely

$$T_{p_1}, T_{p_2}, \ldots, T_{p_k}, T_{n_1}, T_{n_2}, \ldots, T_{n_m}.$$

Thus, we see that

$$i(T - v) = \sum_{s=1}^{k} i(T_{p_s}) + \sum_{t=1}^{m} i(T_{n_t}).$$

By Lemma 8, $\{p_s\} \cup (I(T) \cap V(T_{p_s}))$ is an i-set for T_{p_s} for $1 \leq s \leq k$. Let F denote the subforest of $T - v$ given by $T_{n_1} \cup T_{n_2} \cup \cdots \cup T_{n_m}$. Let $\alpha = |I(T) \cap V(F)|$. Consider $i(F)$. We see that if $i(F) \geq \alpha - k + 1$, then our result is shown.

Thus, suppose that $i(F) \leq \alpha - k$ and let D be an i-set for F. We see that

$$D \cup \bigcup_{s=1}^{k} (\{p_s\} \cup (I(T) \cap V(T_{p_s})))$$

is an independent dominating set of T of cardinality at most $|I(T)|$ distinct from $I(T)$, a contradiction.

Thus, we see that $i(F) \geq \alpha - k + 1$, in which case $i(T - v) > i(T)$. □

Thus, we see that when we consider trees in $\mathcal{UI}$, the result of Lemma 4 can be improved upon.

Continuing on, our next result will be used in Section 5.

Lemma 9. *If $T \in \mathcal{UI}$ is a tree with $v \in V(T) - I(T)$ a shared neighbor of at least two vertices in $I(T)$, then $T - v \in \mathcal{UI}$ with $I(T - v) = I(T)$.*

Proof. Let $T_1, T_2, \ldots, T_k$ be the components of $T - v$, and let $I_j = I(T) \cap V(T_j)$ for $1 \leq j \leq k$. Note that for each j, I_j is an independent dominating set for T_j. Since v has at least two neighbors in $I(T)$, we can alter the minimum dominating set $I(T)$ on one of the components, say T_j, and create an independent dominating set for all of T. That is, if D is any i-set for T_j, then

$$D \cup \bigcup_{s \neq j} I_s$$

is an independent dominating set for T. This observation implies that I_j is, in fact, an i-set for T_j, and that each $T_j \in \mathcal{UI}$. Since each $T_j \in \mathcal{UI}$, $T - v \in \mathcal{UI}$ as well. Our result is shown. □

5 Operations

Using our observations above, we now illustrate a collection of operations which allow us to construct a new graph in $\mathcal{UI}$ by combining two graphs in $\mathcal{UI}$. In particular, throughout this section, G_1 and G_2 are assumed to both be graphs in $\mathcal{UI}$. We let I_j denote the unique i-set of G_j for $j = 1, 2$.

Operation 1. *For $j = 1, 2$, choose $u_j \in V(G_j) - I_j$. If G is the graph defined by $G = (G_1 \cup G_2) + u_1u_2$, then G has the unique i-set $I_1 \cup I_2$.*

Proof. First, observe that $I_1 \cup I_2$ is an independent dominating set for G. Thus, $i(G) \leq |I_1 \cup I_2| = |I_1| + |I_2|$. Suppose that $i(G) < |I_1| + |I_2|$, and let D be an i-set of G. In particular, this implies that $D \neq I_1 \cup I_2$. Let $D_1 = D \cap V(G_1)$ and $D_2 = D \cap V(G_2)$. Note that if $D_1 = I_1$, then $D_2 = I_2$ since $u_1 \notin I_1$ and $G_2 \in \mathcal{UI}$. Similarly, if $D_2 = I_2$, then $D_1 = I_1$ since $u_2 \notin D_2$ and $G_1 \in \mathcal{UI}$. Thus, we have $D_1 \neq I_1$ and $D_2 \neq I_2$.

Without loss of generality, suppose that $|D_1| \leq |I_1|$. First note that D_1 does not dominate G_1, since otherwise I_1 is not the unique i-set of G_1. Since the only vertex of $V(G_1)$ that can be dominated from outside of $V(G_1)$ by D is u_1, we see that D_1 fails to dominate u_1. Hence, $u_2 \in D_2$. This implies each of the following.

- D_2 independently dominates $V(G_2)$. Since I_2 is the unique i-set of G_2, and since $u_2 \notin I_2$, we see that $|D_2| > |I_2|$.
- D_1 independently dominates $G - u_1$. Thus, by Lemma 2, $|D_1| \geq |I_1|$.

Hence, we see that $|D| = |D_1| + |D_2| > |I_1| + |I_2|$, a contradiction.

Thus, we see that $i(G) = |I_1 \cup I_2|$. By the logic applied above, if D is any i-set of G containing one of u_1 or u_2, then $|D| > |I_1 \cup I_2|$. This implies that $I_1 \cup I_2$ is the unique i-set of G. □

Operation 2. *For $j = 1, 2$, let $v_j \in \mathcal{A}(I_j)$. Let u be a new vertex that is neither in G_1 nor G_2. If G is the graph defined by $V(G) = V(G_1) \cup V(G_2) \cup \{u\}$ and $E(G) = E(G_1) \cup E(G_2) \cup \{v_1u, uv_2\}$, then G has the unique i-set $I_1 \cup I_2$.*

Proof. First, observe that $I_1 \cup I_2$ is an independent dominating set for G. Thus, $i(G) \leq |I_1 \cup I_2| = |I_1| + |I_2|$. Let D be an i-set for G. Once again, let $D_1 = D \cap V(G_1)$ and let $D_2 = D \cap V(G_2)$. There are two cases to consider.

- First, suppose that $u \in D$. Since D is independent, this implies that $v_1 \notin D$ and that $v_2 \notin D$. Hence, D_1 is an independent dominating set for $G_1 - v_1$ and D_2 is an independent dominating set for $G_2 - v_2$. By Lemma 4, this implies that $|D_1| \geq |I_1|$ and $|D_2| \geq |I_2|$. Hence, we see that $|D| = |D_1 \cup D_2 \cup \{u\}| = |D_1| + |D_2| + 1 \geq |I_1| + |I_2| + 1 > |I_1| + |I_2|$, a contradiction.
- Now suppose that $u \notin D$. In this case, D_1 is an independent dominating set for G_1 and D_2 is an independent dominating set for G_2. This implies that $D_1 = I_1$ and $D_2 = I_2$. Thus, $D = I_1 \cup I_2$.

Hence, we see that G has a unique i-set given by $I_1 \cup I_2$. □

Operation 3. *Let $G_1 \in \mathcal{UI}$ be a tree and let $G_2 \in \mathcal{UI}$. Let $v_1 \in \mathcal{A}(I_1)$ and $v_2 \in \mathcal{B}(I_2)$. Let u be a new vertex that is neither in G_1 nor G_2. If G is the graph defined as in Operation 2, then G has the unique i-set $I_1 \cup I_2$.*

Proof. First, observe that $I_1 \cup I_2$ is an independent dominating set for G. Thus, $i(G) \leq |I_1 \cup I_2| = |I_1| + |I_2|$. Let D be an i-set for G. Let $D_1 = D \cap V(G_1)$ and let $D_2 = D \cap V(G_2)$. Once again, we consider two cases.

- First, suppose that $u \in D$. Since D is independent, this implies that $v_1 \notin D$ and that $v_2 \notin D$. Hence, D_1 is an independent dominating set for $G_1 - v_1$ and D_2 is an independent dominating set for $G_2 - v_2$. By Proposition 1 and Lemma 5, we see that

$$\begin{aligned} |D| &= 1 + |D_1| + |D_2| \\ &\geq 1 + |D_1| + |I_2| - 1 \\ &= |D_1| + |I_2| \\ &> |I_1| + |I_2| \\ &= |I_1 \cup I_2|. \end{aligned}$$

  Thus, we have arrived at a contradiction. Hence, u is not a member of any i-set of G.

- Now suppose that $u \notin D$. In this case, D_1 is an independent dominating set for G_1 and D_2 is an independent dominating set for G_2. This implies that $D_1 = I_1$ and $D_2 = I_2$. Thus, $D = I_1 \cup I_2$.

Thus, we see that G has a unique i-set given by $I_1 \cup I_2$. □

We note that if G_1 is not a tree, then Operation 3 is not guaranteed to produce a graph in $\mathcal{UI}$. For example, if we let G_1 be the graph from Figure 1 with $v_1 = v$, and let $G_2 = K_1$, then Operation 3 will produce the graph below, which does not have a unique i-set.

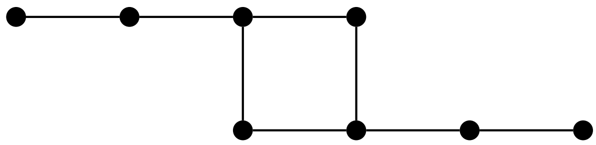

Figure 2: Operation 3 requires G_1 to be a tree

Operation 4. *Let $G_1 \in \mathcal{UI}$ be a tree and let $G_2 \in \mathcal{UI}$. Let $v_1 \in V(G_1) - I_1$ be a common neighbor of at least two vertices in I_1, and let $v_2 \in \mathcal{A}(I_2)$. If G is the graph formed by joining G_1 and G_2 with the new edge v_1v_2, then G has the unique i-set $I_1 \cup I_2$.*

Proof. Once again, we see that $I_1 \cup I_2$ is an independent dominating set for G. Thus, $i(G) \leq |I_1| + |I_2|$. Let D be an i-set for G. Let $D_1 = D \cap V(G_1)$ and let $D_2 = D \cap V(G_2)$. We consider two cases.

- First, suppose that $v_1 \in D$. In this case, D_1 is an independent dominating set for G_1. Since $v_1 \notin I_1$, this implies that $|D_1| > |I_1|$. Additionally, if $v_1 \in D$ then $v_2 \notin D$. Hence, D_2 is an independent dominating set for $G_2 - v_2$. By Lemma 4, we see that $|D_2| \geq |I_2|$. Hence, we see that $|D| = |D_1| + |D_2| > |I_1| + |I_2|$, a contradiction.

- Now suppose that $v_1 \notin D$. This implies that D_2 is a minimum independent dominating set for G_2. Thus, $D_2 = I_2$. This implies that D_1 is a minimum independent dominating set for $G_1 - v_1$. By Lemma 9, we see that $D_1 = I_1$ and thus $D = I_1 \cup I_2$.

Hence, we see that $G \in \mathcal{UI}$ and that $I(G) = I_1 \cup I_2$. □

In the operation above, if $v_2 \in \mathcal{B}(I_2)$, then the resulting graph G is not guaranteed to have a unique i-set. For example, in the figure below, if we add in the dashed edge, the resulting graph will not have a unique i-set.

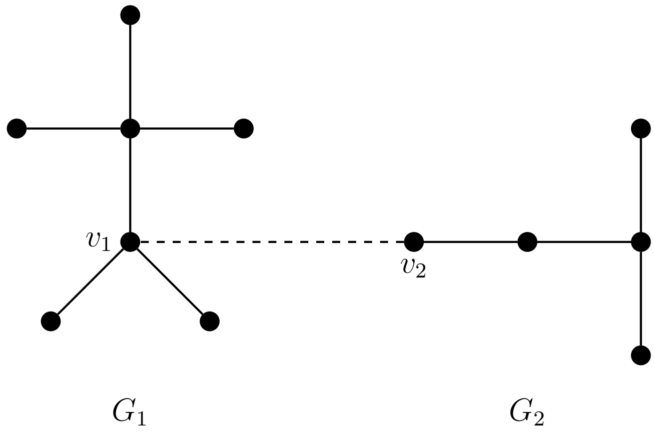

Figure 3: Operation 4 requires $v_2 \in \mathcal{A}(I_2)$

The reason Operation 4 failed to produce a graph in $\mathcal{UI}$ in the example above is that $i(G_1 - N[v_1]) = i(G_1)$. In an attempt to circumvent this problem, we make the following definition. Given a graph $G \in \mathcal{UI}$, let

$$\mathcal{C}(G) = \{v \in V(G) - I(G) : |N(v) \cap I(G)| \geq 2 \text{ and } i(G - N[v]) > i(G)\}.$$

With this notation established, we present the following operation.

Operation 5. *Let $G_1 \in \mathcal{UI}$ be a tree and let $G_2 \in \mathcal{UI}$. Let $v_1 \in \mathcal{C}(G_1)$ and let $v_2 \in \mathcal{B}(I_2)$. If G is formed by joining G_1 and G_2 with the new edge v_1v_2, then G has the unique i-set $I_1 \cup I_2$.*

Proof. Let D be an i-set for G. Let $D_1 = D \cap V(G_1)$ and let $D_2 = D \cap V(G_2)$. We consider two cases.

- First, suppose that $v_1 \in D$. In this case, D_1 is an independent dominating set for G_1. Note that $|D_1| = 1 + |D_1 - \{v_1\}|$. Since $D_1 - \{v_1\}$ independently dominates $G_1 - N[v]$, and since $v_1 \in \mathcal{C}(G_1)$, we see that $|D_1 - \{v\}| > i(G) = |I_1|$. Thus, $|D_1| \geq |I_1| + 2$. Additionally, if $v_1 \in D$, then $v_2 \notin D$. Hence, D_2 is an independent dominating set for $G_2 - v_2$. By Lemma 5, we see that $|D_2| \geq |I_2| - 1$. Hence, we see that $|D| = |D_1| + |D_2| \geq |I_1| + 2 + |I_2| - 1 > |I_1| + |I_2|$, a contradiction.

- Now suppose that $v_1 \notin D$. This implies that D_2 is a minimum independent dominating set for G_2. Thus, $D_2 = I_2$. This implies that D_1 is a minimum independent dominating set for $G_1 - v_1$. Lemma 9 then implies that $D_1 = I_1$. Thus, $D = I_1 \cup I_2$.

Hence, we see that $G \in \mathcal{UI}$ and that $I(G) = I_1 \cup I_2$. □

Note that after performing each of these five operations, $\mathcal{A}(I(G)) = \mathcal{A}(I_1) \cup \mathcal{A}(I_2)$ and that $\mathcal{B}(I(G)) = \mathcal{B}(I_1) \cup \mathcal{B}(I_2)$.

6 Characterizing Trees

In this section, we utilize the operations discussed in the previous section to characterize the trees T having a unique minimum independent dominating set.

Theorem 1. *Let T be a tree. $T \in \mathcal{UI}$ if and only if T can be constructed from a disjoint union of isolated vertices and stars, each with at least two leaves, by a finite sequence of Operations 1 through 5.*

Proof. Given our work in the previous section, if T can be constructed from a disjoint union of isolated vertices and stars, each with at least two leaves, by a finite sequence of Operations 1 through 5, then $T \in \mathcal{UI}$. Thus, it remains to show that if $T \in \mathcal{UI}$, then T can be constructed in this manner.

We proceed by induction on $i(T)$. If $i(T) = 1$, then, by Lemma 1, T is either K_1 or a star with at least 2 leaves. In either case, the result holds.

Assume the result holds for all trees T in $\mathcal{UI}$ satisfying $i(T) < k$, $k \geq 2$. Let $T \in \mathcal{UI}$ be a tree satisfying $i(T) = k$. We consider two cases, each with two subcases.

Case One: T has a leaf in $I(T)$.

Suppose that T has a leaf, call it l, in $I(T)$. Notice that $l \in \mathcal{B}(I(T))$. Let v denote the single neighbor of l. Since $I(T)$ is independent, $v \notin I(T)$. Additionally, by Lemma 7, some neighbor of v, distinct from l, is in $\mathcal{A}(I(T))$. Let $a_1 \in N(v) \cap \mathcal{A}(I(T))$. We consider the following two subcases.

Subcase One: $|N(v) \cap I(T)| = 2$.

First suppose that $|N(v) \cap I(T)| = 2$. Let $N(v) = \{l, a_1, o_1, o_2, \ldots, o_k\}$. Observe that $o_1, o_2, \ldots, o_k$ are not in $I(T)$. Root T at v. By Lemma 9, each of T_{a_1}, $T_{o_1}, T_{o_2}, \ldots, T_{o_k}$ has a unique i-set. Thus, by our induction hypothesis, each of these subtrees can be constructed from a disjoint union of isolates and stars by a finite sequence of Operations 1 through 5. To construct T, first note that since $a_1 \in \mathcal{A}(I(T))$, we also have $a_1 \in \mathcal{A}(I(T_{a_1}))$. Thus, we can connect l, v and T_{a_1} by applying Operation 3. Call this resulting graph F. From there, we can reconstruct T by connecting $T_{o_1}, T_{o_2}, \ldots, T_{o_k}$ to F by performing Operation 1 k-times.

Subcase Two: $|N(v) \cap I(T)| > 2$.

Once again, root T at v. Let

$$N(v) = \{l, a_1, a_2, \ldots, a_j, b_1, b_2, \ldots, b_k, o_1, o_2, \ldots, o_m\}$$

where $a_1, a_2, \ldots, a_j \in \mathcal{A}(I(T))$, $b_1, b_2, \ldots, b_k \in \mathcal{B}(I(T))$ and $o_1, o_2, \ldots, o_m \in V(T) - I(T)$. Let $T' = T - l$. Recall that since $T \in \mathcal{UI}$, Lemma 3 implies that $i(T - N[v]) \geq$

$i(T)$. Thus, in particular, we have that

$$\begin{aligned} i(T' - N_{T'}[v]) &= i(T - N[v]) \\ &\geq i(T) \\ &> i(T) - 1 \\ &= i(T - l) \\ &= i(T'). \end{aligned}$$

Thus, we see that $i(T' - N_{T'}[v]) > i(T')$. Thus, $v \in \mathcal{C}(T')$. Recall that $T' \in \mathcal{UI}$ by Lemma 5. Thus, by our induction hypothesis, T' can be constructed from a disjoint union of isolated vertices and stars by a finite sequence of Operations 1 through 5. We can then reconstruct T from T' and l by applying Operation 5.

Case Two: No leaf of T is in $I(T)$.

Consider a diametral path $v_1v_2 \cdots v_{k-2}v_{k-1}v_kv_{k+1}$ in T. Since $i(T) \geq 2$, and since no leaf of T is in $I(T)$, we see that $k \geq 4$. Observe that $v_{k+1} \notin I(T)$ in which case $v_k \in I(T)$. This further implies that $v_k \in \mathcal{A}(I(T))$. We once again consider two subcases.

Subcase One: $v_{k-1} \in epn(v_k, I(T))$.

In this case, observe that $N(v_{k-1}) = \{v_{k-2}, v_k\}$ since otherwise either $I(T)$ contains a leaf or $v_1v_2 \cdots v_{k+1}$ is not a diametral path. Moreover, since $v_{k-1} \in epn(v_k, I(T))$, we see that $v_{k-2} \notin I(T)$. Thus, consider $T - N[v_k]$. By Lemma 6, $T - N[v_k] \in \mathcal{UI}$ and $i(T - N[v_k]) = i(T) - 1$. Thus, we can apply our induction hypothesis to $T - N[v_k]$. We can then reconstruct T from $T - N[v_k]$ and the subgraph induced by $N[v_k]$ by applying Operation 1.

Subcase Two: $v_{k-1} \notin epn(v_k, I(T))$.

Since $v_k \in \mathcal{A}(I(T))$, this implies that v_k has at least two leaf neighbors. Consider $N(v_{k-1})$. We see that $|N(v_{k-1}) \cap I(T)| \geq 2$, and that v_{k-1} has no leaf neighbors.

First suppose $N(v_{k-1}) = \{v_{k-2}, v_k\}$. In this case, $v_{k-2} \in I(T)$. Since $T - N[v_k] \in \mathcal{UI}$ by Lemma 6, we can apply our induction hypothesis to $T - N[v_k]$. We can then reconstruct T from $T - N[v_k]$, v_{k-1}, and $\{v_k\} \cup epn(v_k, I(T))$ by applying either Operation 2 or Operation 3.

Suppose now that $N(v_{k-1}) = \{v_{k-2}, v_k, o_1, o_2, \ldots, o_r\}$. Since $I(T)$ contains no leaves, we see that $o_1, o_2, \ldots, o_r$ are each in $\mathcal{A}(I(T))$. In particular, this implies that each has at least two leaf neighbors. Root T at v_{k-1}. By Lemma 9, $T_{v_{k-2}} \in \mathcal{UI}$ in which case we can apply the induction hypothesis to construct it from Operations 1 through 5. We can then reconstruct T as follows. First, combine $\{v_k\} \cup epn(v_k, I(T))$, $\{o_1\} \cup epn(o_1, I(T)), \ldots, \{o_r\} \cup epn(o_r, I(T))$ through one Operation 2 followed by Operation 4 $(r-1)$-times. From there, we can reconstruct T by performing Operation 5 if $v_{k-2} \in \mathcal{B}(I(T))$, Operation 4 if $v_{k-2} \in \mathcal{A}(I(T))$, or Operation 1 if $v_{k-2} \notin I(T)$. □

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