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On the existence of $(K_{1,3}, \lambda)$-frames of type g^u

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Abstract

A $(K_{1,3}, \lambda)$-frame of type g^u is a $K_{1,3}$-decomposition of a complete u-partite graph with u parts of size g into partial parallel classes each of which is a partition of the vertex set except for those vertices in one of the u parts. In this paper, we completely solve the existence of a $(K_{1,3}, \lambda)$-frame of type g^u.

1 Introduction

In this paper, the vertex set and edge set (or edge-multiset) of a graph G (or multigraph) are denoted by $V(G)$ and $E(G)$ respectively. For a graph G, we use λG to represent the multi-graph obtained from G by replacing each edge of G with λ copies of it. A graph G is called a *complete u-partite graph* if $V(G)$ can be partitioned into *u parts* M_i, $1 \leq i \leq u$, such that two vertices of G, say x and y, are adjacent if and only if $x \in M_i$ and $y \in M_j$ with $i \neq j$. We use $\lambda K(m_1, m_2, \ldots, m_u)$ for the λ-fold of the complete u-partite graph with m_i vertices in the group M_i.

Given a collection of graphs $\mathcal{H}$, an *$\mathcal{H}$-decomposition* of a graph G is a set of subgraphs (*blocks*) of G whose edge sets partition $E(G)$, and each subgraph is isomorphic to a graph from $\mathcal{H}$. When $\mathcal{H} = \{H\}$, we write $\mathcal{H}$-decomposition as H-decomposition for the sake of brevity. A *parallel class* of a graph G is a set of subgraphs whose vertex sets partition $V(G)$. A parallel class is called *uniform* if each block of the parallel class is isomorphic to the same graph. An $\mathcal{H}$-decomposition of a graph G is called (uniformly) *resolvable* if the blocks can be partitioned into (uniform) parallel classes. Recently, a lot of results have been obtained on uniformly resolvable $\mathcal{H}$-decompositions of K_v, especially on uniformly resolvable $\mathcal{H}$-decompositions with $\mathcal{H} = \{G_1, G_2\}$ ([6, 7, 11, 15, 18–21, 23–26]) and with $\mathcal{H} = \{G_1, G_2, G_3\}$ ([8]). For the graphs related to this paper, the reader is referred to [3, 17].

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A (resolvable) $\mathcal{H}$-decomposition of $\lambda K(m_1, m_2, \ldots, m_u)$ is called a (resolvable) *group divisible design*, denoted by $(\mathcal{H}, \lambda)$-(R)GDD. When $\lambda = 1$, we usually omit λ in the notation. The *type* of an $\mathcal{H}$-GDD is the multiset of group sizes $|M_i|$, $1 \le i \le u$, and we usually use the "exponential" notation for its description: type $1^i 2^j 3^k \ldots$ denotes i occurrences of groups of size 1, j occurrences of groups of size 2, and so on. In this paper, we will use $K_{1,3}$-RGDDs as input designs for recursive constructions. There are some known results on the existence of $K_{1,3}$-RGDDs. For example, $K_{1,3}$-RGDDs of types 2^4 and 4^4 have been constructed in [17], and the existence of a $K_{1,3}$-RGDD of type 12^u for any $u \ge 2$ has been solved in [3].

Let K be a set of positive integers. If $\mathcal{H} = \{K_1, K_2, \ldots, K_t\}$ with $|V(K_i)| \in K$ $(1 \le i \le t)$, then $\mathcal{H}$-GDD is also denoted by K-GDD, and an K-GDD of type 1^v is called a *pairwise balanced design*, denoted by (K, v)-PBD. It is usual to write k rather than $\{k\}$ when $K = \{k\}$ is a singleton.

A set of subgraphs of a complete multipartite graph covering all vertices except those belonging to one part M is said to be a *partial parallel class* missing M. A partition of an $(\mathcal{H}, \lambda)$-GDD of type g^u into *partial parallel classes* is said to be a $(\mathcal{H}, \lambda)$-*frame*. Frames were firstly introduced in [1]. Frames are important combinatorial structures used in graph decompositions. Stinson [27] solved the existence of a $(K_3, 1)$-frame of type g^u. For the existence of a (K_4, λ)-frame of type g^u, see [10, 12–14, 22, 28, 29]. Cao et al. [5] started the research of a $(C_k, 1)$-frame of type g^u. Buratti et al. [2] have completely solved the existence of a (C_k, λ)-frame of type g^u recently. Here we focus on the existence of a $(K_{1,3}, \lambda)$-frame of type g^u which can be used in uniformly resolvable $\mathcal{H}$-decompositions with $K_{1,3} \in \mathcal{H}$ in [3]. It is easy to see that the number of partial parallel classes missing a specified group is $\frac{2g\lambda}{3}$. So we have the following necessary conditions for the existence of a $(K_{1,3}, \lambda)$-frame of type g^u.

Theorem 1.1. *The necessary conditions for the existence of a $(K_{1,3}, \lambda)$-frame of type g^u are $\lambda g \equiv 0 \pmod 3$, $g(u-1) \equiv 0 \pmod 4$, $u \ge 3$ and $g \equiv 0 \pmod 4$ when $u = 3$.*

Not many results have been known for the existence of a $(K_{1,3}, \lambda)$-frame of type g^u.

Theorem 1.2. [3] *There exists a $K_{1,3}$-frame of type 12^u for $u \ge 3$.*

In this paper, we will prove the following main result.

Theorem 1.3. *The necessary conditions for the existence of a $(K_{1,3}, \lambda)$-frame of type g^u are also sufficient with the definite exception of $(\lambda, g, u) = (6t+3, 4, 3)$, $t \ge 0$.*

2 Recursive constructions

For brevity, we use I_k to denote the set $\{1, 2, \ldots, k\}$, and use $(a; b, c, d)$ to denote the 3-star $K_{1,3}$ with vertex set $\{a, b, c, d\}$ and edge set $\{\{a, b\}, \{a, c\}, \{a, d\}\}$. Now we state two basic recursive constructions for $(K_{1,3}, \lambda)$-frames. Similar proofs of these constructions can be found in [9] and [27].

Construction 2.1. *If there exists a $(K_{1,3}, \lambda)$-frame of type $g_1^{u_1} g_2^{u_2} \dots g_t^{u_t}$, then there is a $(K_{1,3}, \lambda)$-frame of type $(mg_1)^{u_1}(mg_2)^{u_2} \dots (mg_t)^{u_t}$ for any $m \geq 1$.*

Construction 2.2. *If there exist a (K, v)-GDD of type $g_1^{t_1} g_2^{t_2} \dots g_m^{t_m}$ and a $(K_{1,3}, \lambda)$-frame of type h^k for each $k \in K$, then there exists a $(K_{1,3}, \lambda)$-frame of type $(hg_1)^{t_1}(hg_2)^{t_2} \dots (hg_m)^{t_m}$.*

Definition 2.1. *Let G be a λ-fold complete u-partite graph with u groups $M_1, M_2, \dots, M_u$ such that $|M_i| = g$ for each $1 \leq i \leq u$. Suppose $N_i \subset M_i$ and $|N_i| = h$ for any $1 \leq i \leq u$. Let H be a λ-fold complete u-partite graph with u groups (called holes) $N_1, N_2, \dots, N_u$. An incomplete resolvable $(K_{1,3}, \lambda)$-group divisible design of type g^u with a hole of size h in each group, denoted by $(K_{1,3}, \lambda)$-IRGDD of type $(g, h)^u$, is a resolvable $(K_{1,3}, \lambda)$-decomposition of $G - E(H)$ in which there are $\frac{2\lambda(g-h)(u-1)}{3}$ parallel classes of G and $\frac{2\lambda h(u-1)}{3}$ partial parallel classes of $G - H$.*

Lemma 2.3. *There exists a $(K_{1,3}, 3)$-IRGDD of type $(12, 4)^2$.*

Proof: Let the vertex set be $Z_{16} \cup \{a_0, a_1, a_2, a_3\} \cup \{b_0, b_1, b_2, b_3\}$, and let the two groups be $\{0, 2, \dots, 14\} \cup \{a_0, a_1, a_2, a_3\}$ and $\{1, 3, \dots, 15\} \cup \{b_0, b_1, b_2, b_3\}$. The required 8 partial parallel classes can be generated from two partial parallel classes Q_1, Q_2 by $+4j \pmod{16}$, $j = 0, 1, 2, 3$. The required 16 parallel classes can be generated from four parallel classes P_i, $i = 1, 2, 3, 4$, by $+4j \pmod{16}$, $j = 0, 1, 2, 3$. The blocks in Q_1, Q_2 and P_i are listed below.

Q_1	$(4; 1, 3, 5)$	$(9; 0, 6, 8)$	$(12; 7, 11, 15)$	$(13; 2, 10, 14)$		
Q_2	$(0; 5, 7, 11)$	$(3; 6, 10, 14)$	$(12; 1, 9, 15)$	$(13; 2, 4, 8)$		
P_1	$(0; 3, 7, 15)$	$(1; 2, 10, 14)$	$(a_0; 5, 9, 13)$	$(b_0; 4, 8, 12)$	$(11; a_1, a_2, a_3)$	$(6; b_1, b_2, b_3)$
P_2	$(6; 3, 7, 15)$	$(9; 2, 10, 12)$	$(a_1; 1, 5, 13)$	$(b_1; 0, 4, 8)$	$(11; a_0, a_2, a_3)$	$(14; b_0, b_2, b_3)$
P_3	$(14; 3, 7, 15)$	$(1; 6, 8, 10)$	$(a_2; 5, 9, 13)$	$(b_2; 0, 4, 12)$	$(11; a_0, a_1, a_3)$	$(2; b_0, b_1, b_3)$
P_4	$(4; 5, 11, 15)$	$(3; 2, 6, 14)$	$(a_3; 1, 9, 13)$	$(b_3; 0, 8, 12)$	$(7; a_0, a_1, a_2)$	$(10; b_0, b_1, b_2)$

□

A k-GDD of type n^k is called a *transversal design*, denoted by TD(k, n). A TD(k, n) is *idempotent* if it contains a parallel class of blocks. A resolvable TD(k, n) is denoted by RTD(k, n). If we can select a block from each parallel class of an RTD(k, n), and all these n blocks form a new parallel class, then this RTD(k, n) is denoted by RTD*(k, n).

Construction 2.4. *Suppose there exist an RTD*(u, n), a $(K_{1,3}, \lambda)$-IRGDD of type $(g + h, h)^u$, a $(K_{1,3}, \lambda)$-RGDD of type g^u, and a $(K_{1,3}, \lambda)$-RGDD of type $(g + h)^u$, then there exists a $(K_{1,3}, \lambda)$-RGDD of type $(gn + h)^u$.*

Proof: We start with an RTD*(u, n) with n parallel classes $P_i = \{B_{i1}, B_{i2}, \dots, B_{in}\}$, $1 \leq i \leq n$, and a parallel class $Q = \{B_{11}, B_{21}, \dots, B_{n1}\}$. Give each vertex weight g. For each block B_{ij} in $P_i \setminus Q$, place a $(K_{1,3}, \lambda)$-RGDD of type g^u whose $t = \frac{2\lambda g(u-1)}{3}$ parallel classes are denoted by F_{ij}^s, $1 \leq s \leq t$. For each block B_{i1} in Q with $1 \leq i \leq n-1$, place a $(K_{1,3}, \lambda)$-IRGDD of type $(g+h, h)^u$ on the vertices of the weighted block B_{i1} and hu new common vertices (take them as u holes). Denote its t parallel classes by F_{i1}^s, $1 \leq s \leq t$, and its $w = \frac{2\lambda h(u-1)}{3}$ partial parallel classes by Q_{i1}^s, $1 \leq s \leq w$.

Further, place on the vertices of the weighted block B_{n1} and these hu new vertices a $(K_{1,3}, \lambda)$-RGDD of type $(g+h)^u$ whose $t+w$ parallel classes are denoted by F^s_{n1}, $1 \le s \le t+w$.

Let $F^s_i = \cup_{j=1}^n F^s_{ij}$, $1 \le s \le t$, $1 \le i \le n$, and $T_j = F^{t+j}_{n1} \cup (\cup_{i=1}^{n-1} Q^j_{i1})$, $1 \le j \le w$. It is easy to see F^s_i and T_j are parallel classes of the required $(K_{1,3}, \lambda)$-RGDD of type $(gn+h)^u$. ∎

Construction 2.5. *If there is a $(K_{1,3}, \lambda)$-RGDD of type g^2, then there exists a $(K_{1,3}, \lambda)$-frame of type g^{2u+1} for any $u \ge 1$.*

Proof: We start with a K_2-frame of type 1^{2u+1} in [4]. Suppose its vertex set is I_{2u+1}. Denote its $2u+1$ partial parallel classes by F_i $(i \in I_{2u+1})$ which is with respect to the group $\{i\}$. The required $(K_{1,3}, \lambda)$-frame of type g^{2u+1} will be constructed on $I_{2u+1} \times I_g$. For any $B = \{a, b\} \in F_i$, place on $B \times I_g$ a copy of a $(K_{1,3}, \lambda)$-RGDD of type g^2, whose $\frac{2\lambda g}{3}$ parallel classes are denoted by $P_j(B)$, $1 \le j \le \frac{2\lambda g}{3}$. Let $P^j_i = \bigcup_{B \in F_i} P_j(B), i \in I_{2u+1}, 1 \le j \le \frac{2\lambda g}{3}$. Then each P^j_i is a partial parallel class with respect to the group $\{i\} \times I_g$. Thus we have obtained a $(K_{1,3}, \lambda)$-frame of type g^{2u+1} for any $u \ge 1$. ∎

Note that if there exists a $(K_{1,3}, \lambda)$-frame of type g^3, then it is easy to see that these $2\lambda g/3$ partial parallel classes missing the same group form a $(K_{1,3}, \lambda)$-RGDD of type g^2. Combining with Construction 2.5, we have the following conclusion.

Lemma 2.6. *The existence of a $(K_{1,3}, \lambda)$-frame of type g^3 is equivalent to the existence of a $(K_{1,3}, \lambda)$-RGDD of type g^2.*

Construction 2.7. *If there exist a $(K_{1,3}, \lambda)$-frame of type $(m_1 g)^{u_1} (m_2 g)^{u_2} \dots (m_t g)^{u_t}$ and a $(K_{1,3}, \lambda)$-frame of type $g^{m_i + \varepsilon}$ for any $1 \le i \le t$, then there exists a $(K_{1,3}, \lambda)$-frame of type $g^{\sum_{i=1}^t m_i u_i + \varepsilon}$, where $\varepsilon = 0, 1$.*

Proof: If there exists a $(K_{1,3}, \lambda)$-frame of type $(m_1 g)^{u_1} (m_2 g)^{u_2} \dots (m_t g)^{u_t}$, there are $\frac{2\lambda |G_j|}{3}$ partial parallel classes missing G_j, $1 \le j \le u_1 + u_2 + \dots u_t$. Add $g\varepsilon$ new common vertices (if $\varepsilon > 0$) to the vertex set of G_j and form a new vertex set G'_j. Then break up G'_j with a $(K_{1,3}, \lambda)$-frame of type $g^{|G_j|/g + \varepsilon}$ with groups $G^1_j, G^2_j, \dots, G^{|G_j|/g}_j, M$, where the $g\varepsilon$ common vertices (if $\varepsilon > 0$) are viewed as a new group M. It has $\frac{2\lambda |G_j|}{3} + \frac{2\lambda g \varepsilon}{3}$ partial parallel classes.

Next match up the $\frac{2\lambda |G_j|}{3}$ partial parallel classes missing G_j with $\frac{2\lambda |G^i_j|}{3}$ partial parallel classes missing G^i_j to get the required partial parallel classes with respect to the group G^i_j (note that $\frac{2\lambda |G_j|}{3} = \sum_{i=1}^{|G^i_j|/g} \frac{2\lambda |G^i_j|}{3}$), $1 \le i \le |G_j|/g$.

Finally, combine these $\frac{2\lambda g \varepsilon}{3}$ partial parallel classes (if $\varepsilon > 0$) from all the groups to get $\frac{2\lambda g \varepsilon}{3}$ partial parallel classes missing M. ∎

3 $\lambda = 1$

By Theorem 1.1, it is easy to see that the two cases $\lambda = 1$ and $\lambda = 3$ are crucial for the whole problem. In this section we first consider the case $\lambda = 1$.

Lemma 3.1. *For each* $u \equiv 1 \pmod 4$, $u \geq 5$, *there exists a* $K_{1,3}$*-frame of type* 3^u.

Proof: For $u = 5, 9$, let the vertex set be Z_{3u}, and let the groups be $M_i = \{i, i+u, i+2u\}$, $0 \leq i \leq u-1$. The required 2 partial parallel classes with respect to the group M_i are $\{Q_1+i, Q_1+i+u, Q_1+i+2u\}$ and $\{Q_2+i, Q_2+i+u, Q_2+i+2u\}$. The blocks in Q_1 and Q_2 are listed below.

$u=5$	Q_1	$(1;2,3,4)$	Q_2	$(2;6,8,9)$		
$u=9$	Q_1	$(1;2,3,4)$	$(5;15,16,17)$	Q_2	$(1;5,6,7)$	$(4;11,12,17)$

For $u \geq 13$, we start with a $K_{1,3}$-frame of type $12^{(u-1)/4}$ from Theorem 1.2 and apply Construction 2.7 with $\varepsilon = 1$ to get the required $K_{1,3}$-frame of type 3^u, where the input design, a $K_{1,3}$-frame of type 3^5, is constructed above. □

Lemma 3.2. *For each* $u \equiv 1 \pmod 2$, $u \geq 5$, *there exists a* $K_{1,3}$*-frame of type* 6^u.

Proof: For $u \equiv 1 \pmod 4$, apply Construction 2.1 with $m = 2$ to get a $K_{1,3}$-frame of type 6^u, where the input design a $K_{1,3}$-frame of type 3^u exists by Lemma 3.1.

For $u \equiv 3 \pmod 4$, when $u = 7, 11, 15$, let the vertex set be Z_{6u}, and let the groups be $M_i = \{i + ju : 0 \leq j \leq 5\}, 0 \leq i \leq u-1$. Three of the four required partial parallel classes P_0, P_1, P_2 with respect to the group M_0 can be generated from an initial partial parallel class P by $+i \pmod{6u}$, $i = 0, 2u, 4u$. The last partial parallel class missing M_0 is $P_3 = Q \cup \{Q+2u\} \cup \{Q+4u\}$. All these required partial parallel classes can be generated from P_0, P_1, P_2, P_3 by $+2j \pmod{6u}$, $0 \leq j \leq u-1$. For each u, the blocks in P and Q are listed below.

$u=7$	P	$(1;2,3,4)$	$(5;9,10,11)$	$(6;8,12,15)$	$(13;22,23,24)$	$(16;17,19,20)$
		$(18;29,34,36)$	$(25;33,37,41)$	$(26;31,38,39)$	$(40;27,30,32)$	
	Q	$(1;16,19,23)$	$(3;20,22,26)$	$(10;25,27,32)$		
$u=11$	P	$(41;60,61,65)$	$(5;9,10,12)$	$(6;7,8,13)$	$(14;17,18,19)$	$(15;21,23,24)$
		$(16;25,26,28)$	$(20;34,35,36)$	$(27;37,39,40)$	$(29;43,45,46)$	$(30;38,47,48)$
		$(31;52,54,56)$	$(3;1,50,51)$	$(32;53,57,59)$	$(42;2,4,62)$	$(64;49,58,63)$
	Q	$(1;4,27,28)$	$(2;15,25,38)$	$(7;36,39,43)$	$(18;42,52,53)$	$(19;54,56,57)$
$u=15$	P	$(66;79,83,86)$	$(2;1,58,70)$	$(69;11,67,74)$	$(73;8,10,12)$	$(14;17,18,19)$
		$(16;23,24,25)$	$(26;36,37,38)$	$(27;39,40,41)$	$(28;42,44,46)$	$(29;47,48,49)$
		$(31;52,53,54)$	$(32;51,55,56)$	$(33;43,50,57)$	$(34;59,61,62)$	$(35;63,71,81)$
		$(3;64,77,89)$	$(4;6,87,88)$	$(13;5,7,82)$	$(20;9,68,84)$	$(21;72,78,85)$
		$(22;65,76,80)$				
	Q	$(1;4,10,32)$	$(3;36,37,38)$	$(5;42,43,53)$	$(9;48,49,50)$	$(14;51,52,58)$
		$(16;17,56,57)$	$(24;55,59,71)$			

For $u = 19$, apply Construction 2.1 with $m = 3$ to get a $K_{1,3}$-frame of type 36^3, where the input design a $K_{1,3}$-frame of type 12^3 exists by Lemma 1.2. Further, applying Construction 2.7 with $\varepsilon = 1$ and a $K_{1,3}$-frame of type 6^7 constructed above, we can obtain a $K_{1,3}$-frame of type 6^{19}.

For $u = 23$, start with a TD(4, 3) in [16]. Delete a vertex from the last group to obtain a {3, 4}-GDD of type $3^3 2^1$. Give each vertex weight 12, and apply Construction 2.2 to get a $K_{1,3}$-frame of type $36^3 24^1$. Applying Construction 2.7 with $\varepsilon = 1$, we can obtain a $K_{1,3}$-frame of type 6^{23}.

For $u = 35$, apply Construction 2.1 with $m = 5$ to obtain a $K_{1,3}$-frame of type 30^7. Then apply Construction 2.7 with $\varepsilon = 0$ to get a $K_{1,3}$-frame of type 6^{35}.

For $u = 47$, start with a TD(5, 5) in [16]. Delete two vertices from the last group to obtain a {4, 5}-GDD of type $5^4 3^1$. Give each vertex weight 12, and apply Construction 2.2 to get a $K_{1,3}$-frame of type $60^4 36^1$. Applying Construction 2.7 with $\varepsilon = 1$, we can obtain a $K_{1,3}$-frame of type 6^{47}.

For all other values of u, we can always write u as $u = 2t+6n+1$ where $0 \le t \le n$, $t \ne 2$, $n \ge 4$ and $n \ne 6$. From [16], there is an idempotent TD$(4, n)$ with n blocks $B_1, B_2, \ldots, B_n$ in a parallel class. Delete $n - t$ vertices in the last group that lie in $B_{t+1}, B_{t+2}, \ldots, B_n$. Taking the truncated blocks $B_1, B_2, \ldots, B_n$ as groups, we have formed a $\{t, n, 3, 4\}$-GDD of type $4^t 3^{n-t}$ when $t \ge 3$, or a $\{n, 3, 4\}$-GDD of type $4^t 3^{n-t}$ when $t = 0, 1$. Then give each vertex weight 12, and use Construction 2.2 to get a $K_{1,3}$-frame of type $48^t 36^{n-t}$. Further, we use Construction 2.7 with $\varepsilon = 1$ to obtain a $K_{1,3}$-frame of type 6^u. The proof is complete. □

4 $\lambda = 3$

In this section we continue to consider the case $\lambda = 3$.

Lemma 4.1. *For each $u \equiv 1 \pmod 4$, $u \ge 5$, there is a $(K_{1,3}, 3)$-frame of type 1^u.*

Proof: For $u = 5, 9, 13, 17, 29, 33$, let the vertex set be Z_u, and let the groups be $M_i = \{i\}$, $i \in Z_u$. The two partial parallel classes are $P_1 + i$ and $P_2 + i$ with respect to the group M_i. The blocks in P_1 and P_2 are listed below.

$u = 5$	P_1	$(1; 2, 3, 4)$			
	P_2	$(2; 1, 3, 4)$			
$u = 9$	P_1	$(1; 2, 3, 4)$	$(5; 6, 7, 8)$		
	P_2	$(1; 2, 4, 6)$	$(3; 5, 7, 8)$		
$u = 13$	P_1	$(1; 2, 3, 4)$	$(5; 6, 7, 8)$	$(9; 10, 11, 12)$	
	P_2	$(1; 5, 7, 9)$	$(2; 8, 10, 11)$	$(12; 3, 4, 6)$	
$u = 17$	P_1	$(1; 2, 3, 4)$	$(5; 6, 7, 8)$	$(9; 10, 12, 14)$	$(11; 13, 15, 16)$
	P_2	$(1; 5, 6, 7)$	$(2; 8, 9, 10)$	$(3; 11, 13, 16)$	$(4; 12, 14, 15)$
$u = 29$	P_1	$(1; 2, 3, 4)$	$(5; 6, 7, 8)$	$(9; 10, 11, 12)$	$(13; 17, 18, 19)$
		$(14; 20, 21, 22)$	$(15; 23, 24, 25)$	$(16; 26, 27, 28)$	
	P_2	$(1; 5, 6, 7)$	$(2; 9, 10, 11)$	$(3; 8, 13, 16)$	$(4; 19, 20, 21)$
		$(12; 23, 24, 26)$	$(18; 22, 25, 27)$	$(28; 14, 15, 17)$	
$u = 33$	P_1	$(1; 2, 3, 4)$	$(5; 6, 7, 8)$	$(9; 10, 11, 12)$	$(13; 17, 18, 19)$
		$(14; 20, 21, 22)$	$(15; 23, 24, 25)$	$(16; 27, 28, 29)$	$(26; 30, 31, 32)$
	P_2	$(1; 5, 6, 8)$	$(2; 9, 10, 11)$	$(3; 12, 13, 14)$	$(4; 17, 21, 22)$
		$(7; 23, 24, 26)$	$(15; 25, 29, 30)$	$(16; 27, 28, 31)$	$(32; 18, 19, 20)$

For all other values of u, apply Construction 2.2 with a $(\{5, 9, 13, 17, 29, 33\}, u)$-PBD from [4] to obtain the conclusion. □

Lemma 4.2. *For each $u \in \{7, 11, 15, 23, 27\}$, there is a $(K_{1,3}, 3)$-frame of type 2^u.*

Proof: Let the vertex set be Z_{2u}, and let the groups be $M_i = \{i, i+u\}$, $0 \le i \le u-1$. The 4 partial parallel classes missing the group M_i are $P_j + i$, $1 \le j \le 4$. For each u, the blocks in P_j are listed below.

$u = 7$	P_1	(1; 2, 3, 4)	(5; 6, 8, 9)	(10; 11, 12, 13)		
	P_2	(1; 2, 3, 4)	(5; 9, 10, 11)	(8; 6, 12, 13)		
	P_3	(1; 3, 5, 6)	(2; 4, 10, 12)	(8; 9, 11, 13)		
	P_4	(2; 5, 10, 11)	(9; 1, 3, 4)	(12; 6, 8, 13)		
$u = 11$	P_1	(1; 2, 3, 4)	(5; 6, 7, 8)	(9; 10, 12, 13)	(14; 15, 16, 17)	(18; 19, 20, 21)
	P_2	(1; 3, 5, 6)	(2; 4, 7, 8)	(9; 13, 14, 15)	(10; 16, 17, 18)	(12; 19, 20, 21)
	P_3	(1; 6, 7, 8)	(2; 3, 5, 9)	(4; 12, 17, 19)	(14; 10, 18, 20)	(21; 13, 15, 16)
	P_4	(1; 8, 9, 13)	(3; 12, 16, 20)	(6; 10, 15, 18)	(7; 17, 19, 21)	(14; 2, 4, 5)
$u = 15$	P_1	(1; 2, 3, 4)	(5; 6, 7, 8)	(9; 10, 11, 12)	(13; 17, 18, 19)	(14; 16, 20, 21)
		(22; 23, 24, 25)	(26; 27, 28, 29)			
	P_2	(1; 5, 6, 7)	(2; 3, 8, 9)	(4; 10, 11, 12)	(13; 17, 18, 19)	(14; 22, 23, 24)
		(16; 21, 26, 27)	(20; 25, 28, 29)			
	P_3	(1; 8, 9, 10)	(2; 5, 6, 7)	(3; 11, 12, 13)	(4; 14, 22, 23)	(16; 20, 25, 27)
		(17; 24, 26, 28)	(29; 18, 19, 21)			
	P_4	(1; 8, 11, 12)	(2; 6, 14, 16)	(5; 17, 18, 19)	(9; 21, 22, 26)	(10; 23, 24, 28)
		(13; 25, 27, 29)	(20; 3, 4, 7)			
$u = 23$	P_1	(18; 8, 21, 38)	(19; 24, 39, 44)	(14; 2, 7, 20)	(4; 29, 37, 45)	(15; 31, 34, 41)
		(36; 1, 17, 28)	(33; 11, 22, 32)	(13; 9, 10, 40)	(30; 6, 26, 27)	(16; 3, 5, 42)
		(43; 12, 25, 35)				
	P_2	(8; 29, 40, 43)	(22; 6, 20, 36)	(2; 26, 28, 45)	(25; 11, 39, 42)	(21; 10, 13, 31)
		(17; 15, 27, 32)	(12; 5, 16, 30)	(4; 33, 38, 44)	(35; 9, 18, 41)	(7; 19, 24, 37)
		(3; 1, 14, 34)				
	P_3	(24; 16, 37, 45)	(12; 11, 30, 34)	(18; 5, 8, 9)	(27; 3, 20, 39)	(6; 22, 38, 42)
		(41; 28, 32, 35)	(44; 7, 40, 43)	(21; 25, 29, 33)	(2; 14, 15, 17)	(19; 1, 4, 10)
		(31; 13, 26, 36)				
	P_4	(31; 6, 14, 41)	(33; 3, 26, 42)	(28; 1, 27, 36)	(4; 7, 22, 43)	(21; 16, 24, 25)
		(17; 12, 19, 39)	(10; 8, 11, 40)	(32; 13, 34, 38)	(9; 2, 15, 30)	(37; 5, 18, 20)
		(44; 29, 35, 45)				
$u = 27$	P_1	(35; 13, 18, 24)	(52; 40, 46, 49)	(28; 11, 17, 26)	(41; 15, 31, 47)	(42; 3, 6, 48)
		(10; 2, 8, 34)	(7; 19, 30, 32)	(4; 12, 16, 29)	(45; 14, 25, 38)	(36; 1, 50, 51)
		(44; 22, 23, 37)	(20; 5, 9, 43)	(39; 21, 33, 53)		
	P_2	(35; 32, 36, 42)	(19; 10, 12, 52)	(9; 13, 34, 39)	(1; 20, 21, 48)	(25; 11, 14, 43)
		(45; 8, 44, 46)	(2; 38, 47, 50)	(40; 6, 24, 53)	(3; 23, 26, 31)	(15; 16, 17, 37)
		(49; 28, 30, 33)	(5; 7, 22, 29)	(18; 4, 41, 51)		
	P_3	(39; 17, 23, 52)	(28; 29, 44, 50)	(19; 6, 18, 30)	(43; 5, 34, 53)	(2; 31, 32, 46)
		(22; 13, 14, 33)	(1; 9, 42, 47)	(24; 20, 36, 38)	(37; 25, 41, 51)	(7; 3, 4, 12)
		(11; 16, 21, 40)	(10; 8, 15, 49)	(48; 26, 35, 45)		
	P_4	(16; 13, 36, 39)	(50; 12, 37, 46)	(51; 15, 25, 32)	(20; 1, 10, 35)	(33; 9, 40, 41)
		(5; 3, 11, 42)	(48; 18, 52, 53)	(8; 34, 43, 47)	(31; 6, 19, 22)	(44; 7, 23, 49)
		(24; 4, 14, 45)	(2; 28, 30, 38)	(26; 17, 21, 29)		

□

Lemma 4.3. *There exists a $(K_{1,3}, 3)$-frame of type 2^u for each $u \equiv 1 \pmod 6$ and $u \ge 19$.*

Proof: For each u, we start with a $K_{1,3}$-frame of type $12^{\frac{u-1}{6}}$ by Lemma 1.2, and apply Construction 2.7 with $\varepsilon = 1$ to get a $(K_{1,3}, 3)$-frame of type 2^u, where the input design a $(K_{1,3}, 3)$-frame of type 2^7 comes from Lemma 4.2. □

Lemma 4.4. *There exists a $(K_{1,3}, 3)$-RGDD of type g^2, $g = 8, 20, 52$.*

Proof: Let the vertex set be Z_{2g}, and let the groups be $\{0, 2, \ldots, 2g-2\}$ and $\{1, 3, \ldots, 2g-1\}$. The required $2g$ parallel classes can be generated from P by $+1 \pmod{2g}$. The blocks in P are listed below.

$g = 8$	$(0; 1, 3, 5)$	$(2; 7, 9, 13)$	$(11; 4, 8, 10)$	$(15; 6, 12, 14)$	
$g = 20$	$(0; 1, 3, 5)$	$(2; 7, 9, 11)$	$(4; 13, 15, 17)$	$(6; 19, 21, 23)$	$(8; 25, 27, 29)$
	$(31; 10, 18, 20)$	$(33; 22, 24, 26)$	$(35; 28, 30, 32)$	$(37; 12, 34, 36)$	$(39; 14, 16, 38)$
$g = 52$	$(89; 68, 84, 102)$	$(15; 14, 46, 96)$	$(37; 56, 60, 72)$	$(26; 59, 67, 77)$	$(4; 3, 61, 73)$
	$(43; 0, 6, 10)$	$(12; 19, 51, 57)$	$(50; 1, 7, 53)$	$(86; 11, 99, 101)$	$(74; 9, 25, 69)$
	$(16; 71, 93, 103)$	$(23; 30, 44, 82)$	$(95; 32, 52, 90)$	$(62; 5, 33, 81)$	$(34; 41, 47, 85)$
	$(87; 42, 88, 98)$	$(58; 29, 31, 35)$	$(39; 22, 36, 92)$	$(91; 8, 18, 76)$	$(2; 49, 65, 97)$
	$(24; 13, 21, 63)$	$(55; 20, 40, 80)$	$(75; 38, 66, 100)$	$(45; 28, 64, 78)$	$(79; 48, 54, 70)$
	$(94; 17, 27, 83)$				

□

Lemma 4.5. *There exists a $(K_{1,3}, 3)$-frame of type l^3 for any $l > 4$ and $l \equiv 0 \pmod 4$.*

Proof: We distinguish two cases.

1. $l \equiv 0 \pmod 8$. Applying Construction 2.5 with a $(K_{1,3}, 3)$-RGDD of type 8^2 from Lemma 4.4, we can obtain a $(K_{1,3}, 3)$-frame of type 8^3. Then apply Construction 2.1 with $m = l/8$ to get a $(K_{1,3}, 3)$-frame of type l^3.

2. $l \equiv 4 \pmod 8$. Let $l = 8k + 4$, $k \geq 1$. For $l = 12$, take a $K_{1,3}$-frame of type 12^3 from Theorem 1.2 and repeat each block 3 times to get a $(K_{1,3}, 3)$-frame of type 12^3. For $l = 20, 52$, the conclusion comes from Lemmas 2.6 and 4.4. For all other values of l, applying Construction 2.4 with $u = 2$, $n = k$, $g = 8$ and $h = 4$, we can obtain a $(K_{1,3}, 3)$-RGDD of type $(8k + 4)^2$, where the input designs an RTD*$(2, k)$ can be obtained from an idempotent TD$(3, k)$ in [16], a $(K_{1,3}, 3)$-IRGDD of type $(12, 4)^2$ exists by Lemma 2.3, a $(K_{1,3}, 3)$-RGDD of type 8^2 comes from Lemma 4.4, and a $K_{1,3}$-RGDD of type 12^2 comes from Lemma 1.2. Then apply Construction 2.5 to get a $(K_{1,3}, 3)$-frame of type $(8k + 4)^3$. □

Lemma 4.6. *For any $t \geq 0$, a $(K_{1,3}, 6t + 3)$-frame of type 4^3 can not exist.*

Proof: By Lemma 2.6 we only need to prove there doesn't exist a $(K_{1,3}, 6t+3)$-RGDD of type 4^2. Assume there exists a $(K_{1,3}, 6t + 3)$-RGDD of type 4^2. Without lose of generality, we suppose the vertex set is Z_8, and the two groups are $\{0, 2, 4, 6\}$ and $\{1, 3, 5, 7\}$. There are $16t + 8$ parallel classes. For each vertex v, suppose there are exactly x parallel classes in which the degree of v is 3. Then we have $3x + (16t + 8 - x) = 4(6t + 3)$. So $x = 4t + 2$.

Now we consider two vertices 0 and 1. The edge $\{0, 1\}$ appears exactly in $3 + 6t$ parallel classes. Suppose there are exactly a parallel classes in which the degree of 0 is 3, and b parallel classes in which the degree of 0 is 1. Then the vertex 1 has degree 3 in the later b parallel classes. So there are $4t+2-b$ parallel classes in which 0 and 1 are not adjacent and the degree of 1 is 3. Thus in these $4t+2-b$ parallel classes the degree of 0 is 3. So we have $4t+2-b+a \leq 4t+2$. That is $a \leq b$. Similarly, we can prove $b \leq a$. Now we have $a = b$. Note that $a+b = 6t+3$. Thus we obtain a contradiction. □

Lemma 4.7. *There exists a* $(K_{1,3}, 6t)$*-frame of type* 4^3*,* $t \geq 1$*.*

Proof: We first construct a $(K_{1,3}, 6)$-RGDD of type 4^2. Let the vertex be Z_8, and let the two groups be $\{0, 2, 4, 6\}$ and $\{1, 3, 5, 7\}$. The required 16 parallel classes are $P_{ij} = \{(0+i; 1+j, 3+j, 5+j), (7+j; 2+i, 4+i, 6+i)\}$, $i = 0, 2, 4, 6$, $j = 0, 2, 4, 6$. By Lemma 2.6 there exists a $(K_{1,3}, 6)$-frame of type 4^3. Repeat each block t times to get the conclusion. □

Lemma 4.8. *For each* $u \geq 4$*, there exists a* $(K_{1,3}, 3)$*-frame of type* 4^u*.*

Proof: For $u = 5, 9$, apply Construction 2.1 with $m = 4$ to get a $(K_{1,3}, 3)$-frame of type 4^u, where the input design a $(K_{1,3}, 3)$-frame of type 1^u exists by Lemma 4.1.

For $u = 7, 11, 15, 19, 23$, apply Construction 2.1 with $m = 2$ to get a $(K_{1,3}, 3)$-frame of type 4^u, where the input designs $(K_{1,3}, 3)$-frames of type 2^u exist by Lemmas 4.2 and 4.3.

When $u = 4, 6, 8, 10, 14$, let the vertex set be $4u$, and let the groups be $M_i = \{i, i+u, i+2u, i+3u\}$, $0 \leq i \leq u-1$. With respect to the group M_i, $0 \leq i \leq u-1$, the 8 partial parallel classes are $P_j + i + uk$, $j = 1, 2$, $0 \leq k \leq 3$. The blocks in P_1 and P_2 are listed below.

$u=4$	P_1	$(1;2,3,6)$	$(5;7,10,11)$	$(14;9,13,15)$		
	P_2	$(1;7,10,14)$	$(2;5,9,15)$	$(13;3,6,11)$		
$u=6$	P_1	$(1;2,3,4)$	$(5;7,8,9)$	$(10;11,13,14)$	$(15;19,20,22)$	$(16;17,21,23)$
	P_2	$(1;3,8,9)$	$(2;7,10,11)$	$(4;14,15,17)$	$(13;21,22,23)$	$(5;16,19,20)$
$u=8$	P_1	$(1;2,3,4)$	$(5;6,7,9)$	$(10;11,12,13)$	$(14;17,18,19)$	
		$(15;20,21,26)$	$(23;28,29,30)$	$(31;22,25,27)$		
	P_2	$(1;10,11,12)$	$(2;9,13,14)$	$(3;15,17,18)$	$(4;19,22,23)$	
		$(6;25,28,31)$	$(7;21,26,27)$	$(20;5,29,30)$		
$u=10$	P_1	$(1;2,3,4)$	$(5;6,7,8)$	$(9;11,12,13)$	$(14;15,18,19)$	$(16;21,22,23)$
		$(17;24,25,26)$	$(27;31,33,36)$	$(29;35,37,38)$	$(39;28,32,34)$	
	P_2	$(1;9,12,13)$	$(2;14,15,16)$	$(3;17,18,19)$	$(4;21,22,23)$	$(5;24,26,28)$
		$(6;29,31,34)$	$(8;32,35,37)$	$(11;27,33,36)$	$(25;7,38,39)$	
$u=14$	P_1	$(4;19,38,52,)$	$(5;27,32,36)$	$(10;33,34,55)$	$(11;6,9,18)$	$(13;7,12,17)$
		$(21;15,25,26)$	$(23;8,16,20)$	$(24;37,51,54)$	$(39;29,45,49)$	$(40;22,31,35)$
		$(46;1,30,48)$	$(47;2,3,50)$	$(53;41,43,44)$		
	P_2	$(2;1,11,36)$	$(3;18,19,20,)$	$(4;21,22,23)$	$(5;24,25,26)$	$(6;27,29,30)$
		$(7;31,32,33)$	$(8;34,35,45)$	$(9;17,52,53)$	$(12;47,48,55)$	$(13;15,46,49)$
		$(41;10,40,44)$	$(43;39,50,51)$	$(54;16,37,38)$		

For $u = 12, 18$, apply Construction 2.1 with $m = \frac{u}{6}$ and a $(K_{1,3}, 3)$-frame of type 8^3 from Lemma 4.5 to get a $(K_{1,3}, 3)$-frame of type $(\frac{4u}{3})^3$. Applying Construction 2.7 with $\varepsilon = 0$ and a $(K_{1,3}, 3)$-frame of type $4^{\frac{u}{3}}$, we can get a $(K_{1,3}, 3)$-frame of type 4^u.

For all other values of u, take a $(\{4,5,6,7,8,9,10,11,12,14,15,18,19,23\}, u)$-PBD from [4], then apply Construction 2.2 to obtain the conclusion. ∎

Lemma 4.9. *For each* $u \equiv 1 \pmod 2$, $u \geq 5$, *there is a* $(K_{1,3}, 3)$*-frame of type* 2^u.

Proof: For $u \equiv 1 \pmod 4$, apply Construction 2.1 with $m = 2$ to get a $(K_{1,3}, 3)$-frame of type 2^u, where the input design a $(K_{1,3}, 3)$-frame of type 1^u exists by Lemma 4.1.

For $u \equiv 3 \pmod 4$, when $u \in \{7, 11, 15, 19, 23, 27, 31, 55\}$, a $(K_{1,3}, 3)$-frame of type 2^u exists by Lemmas 4.2 and 4.3.

For $u = 35, 63$, we start with a $(K_{1,3}, 3)$-frame of type 1^5 or 1^9 from Lemma 4.1, and apply Construction 2.1 with $m = 14$ to get a $(K_{1,3}, 3)$-frame of type 14^5 or 14^9. Applying Construction 2.7 with $\varepsilon = 0$ and a $(K_{1,3}, 3)$-frame of type 2^7, we can get a $(K_{1,3}, 3)$-frame of type 2^u.

For $u = 39$, start with a TD$(5, 4)$ in [16]. Delete a vertex from the last group to obtain a $\{4, 5\}$-GDD of type $3^1 4^4$. Give each vertex weight 4, and apply Construction 2.2 to get a $(K_{1,3}, 3)$-frame of type $12^1 16^4$, where the input design $(K_{1,3}, 3)$-frames of type 4^4 and 4^5 exist by Lemma 4.8. Applying Construction 2.7 with $\varepsilon = 1$ and $(K_{1,3}, 3)$-frames of type 2^7 and 2^9, we can obtain a $(K_{1,3}, 3)$-frame of type 2^{39}.

For $u = 47$, start with a TD$(5, 5)$ in [16]. Delete 2 vertices from the last group to obtain a $\{4, 5\}$-GDD of type $3^1 5^4$. Give each vertex weight 4, and apply Construction 2.2 to get a $(K_{1,3}, 3)$-frame of type $12^1 20^4$. Applying Construction 2.7 with $\varepsilon = 1$, we can obtain a $(K_{1,3}, 3)$-frame of type 2^{47}.

For $u = 95$, we start with a $(K_{1,3}, 3)$-frame of type 1^5 from Lemma 4.1, and apply Construction 2.1 with $m = 38$ to get a $(K_{1,3}, 3)$-frame of type 38^5. Applying Construction 2.7 with $\varepsilon = 0$ and a $(K_{1,3}, 3)$-frame of type 2^{19}, we can get a $(K_{1,3}, 3)$-frame of type 2^{95}.

For all other values of u, we can always write u as $u = 2t+8n+1$ where $0 \leq t \leq n$, $t \neq 2, 3$, $n \geq 4$ and $n \neq 6, 10$. We start with an idempotent TD$(5, n)$ in [16] with n blocks $B_1, B_2, \cdots, B_n$ in a parallel class. Delete $n - t$ vertices in the last group that lie in $B_{t+1}, B_{t+2}, \cdots, B_n$. Taking the truncated blocks $B_1, B_2, \cdots, B_n$ as groups, we have formed a $\{t, n, 4, 5\}$-GDD of type $5^t 4^{n-t}$ when $t \geq 4$, or a $\{n, 4, 5\}$-GDD of type $5^t 4^{n-t}$ when $t = 0, 1$. Give each vertex weight 4, and apply Construction 2.2 to get a $(K_{1,3}, 3)$-frame of type $20^t 16^{n-t}$. Applying Construction 2.7 with $\varepsilon = 1$ and $(K_{1,3}, 3)$-frames of types 2^9 and 2^{11}, we can obtain a $(K_{1,3}, 3)$-frame of type 2^u. The proof is complete. ∎

5 Proof of Theorem 1.3

Now we are in the position to prove our main result.

Proof of Theorem 1.3: We distinguish two cases.

1. $\lambda \equiv 1, 2 \pmod 3$. In this case we have three subcases.

(1) $g \equiv 3 \pmod{12}$. By Theorem 1.1 we have $u \equiv 1 \pmod 4$, $u \geq 5$. There exists a $K_{1,3}$-frame of type 3^u by Lemma 3.1. Repeat each block λ times to get a $(K_{1,3}, \lambda)$-frame of type 3^u. Apply Construction 2.1 with $m = g/3$ to get a $(K_{1,3}, \lambda)$-frame of type g^u.

(2) $g \equiv 6 \pmod{12}$. By Theorem 1.1 we have $u \equiv 1 \pmod 2$, $u \geq 5$. Similarly we can obtain a $(K_{1,3}, \lambda)$-frame of type 6^u from a $K_{1,3}$-frame of type 6^u which exists by Lemma 3.2. Then we apply Construction 2.1 with $m = g/6$ to get a $(K_{1,3}, \lambda)$-frame of type g^u.

(3) $g \equiv 0 \pmod{12}$. By Theorem 1.1 we have $u \geq 3$. Similarly we can use Construction 2.1 with $m = g/12$ and a $K_{1,3}$-frame of type 12^u from Lemma 1.2 to obtain a $(K_{1,3}, \lambda)$-frame of type g^u.

2. $\lambda \equiv 0 \pmod 3$. In this case we also have three subcases.

(1) $g \equiv 1, 3 \pmod 4$. By Theorem 1.1 we have $u \equiv 1 \pmod 4$, $u \geq 5$. Similarly we can use Construction 2.1 with $m = g$ and a $(K_{1,3}, 3)$-frame of type 1^u from Lemma 4.1 to obtain a $(K_{1,3}, \lambda)$-frame of type g^u.

(2) $g \equiv 2 \pmod 4$. By Theorem 1.1 we have $u \equiv 1 \pmod 2$, $u \geq 5$. Similarly we can use Construction 2.1 with $m = g/2$ and a $(K_{1,3}, 3)$-frame of type 2^u from Lemma 4.9 to obtain a $(K_{1,3}, \lambda)$-frame of type g^u.

(3) $g \equiv 0 \pmod 4$. Let $g = 4s$, $s \geq 1$. By Theorem 1.1 we have $u \geq 3$. When $u = 3$ and $s = 1$, by Lemma 4.6 a $(K_{1,3}, 6t + 3)$-frame of type 4^3 can not exist for any $t \geq 0$, and by Lemma 4.7 there exists a $(K_{1,3}, 6t)$-frame of type 4^3 for any $t \geq 1$. When $u = 3$ and $s > 1$, a $(K_{1,3}, \lambda)$-frame of type g^u can be obtained from a $(K_{1,3}, 3)$-frame of type g^u which exists by Lemma 4.5. When $u \geq 4$, there exists a $(K_{1,3}, 3)$-frame of type 4^u by Lemma 4.8. Apply Construction 2.1 with $m = s$ to get a $(K_{1,3}, \lambda)$-frame of type g^u. ∎

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References

[1] B. Alspach, P. J. Schellenberg, D. R. Stinson and D. Wagner, The Oberwolfach problem and factors of uniform odd length cycles, *J. Combin. Theory Ser. A* **52** (1989), 20–43.

[2] M. Buratti, H. Cao, D. Dai and T. Traetta, A complete solution to the existence of (k, λ)-cycle frames of type g^u, *J. Combin. Des.* **25** (2017), 197–230.

[3] F. Chen and H. Cao, Uniformly resolvable decompositions of K_v into K_2 and $K_{1,3}$ graphs, *Discrete Math.* **339** (2016), 2056–2062.

[4] C. J. Colbourn and J. H. Dinitz, *Handbook of Combinatorial Designs, 2nd Ed.* Chapman & Hall/CRC, 2007.

[5] H. Cao, M. Niu and C. Tang, On the existence of cycle frames and almost resolvable cycle systems, *Discrete Math.* **311** (2011), 2220–2232.

[6] J. H. Dinitz, A. C. H. Ling and P. Danziger, Maximum uniformly resolvable designs with block sizes 2 and 4, *Discrete Math.* **309** (2009), 4716–4721.

[7] P. Danziger, G. Quattrocchi and B. Stevens, The Hamilton-Waterloo problem for cycle sizes 3 and 4, *J. Combin. Des.* **12** (2004), 221–232.

[8] G. Lo Faro, S. Milici and A. Tripodi, Uniformly resolvable decompositions of K_v into paths on two, three and four vertices, *Discrete Math.* **338** (2015), 2212–2219.

[9] S. Furino, Y. Miao and J. Yin, *Frames and resolvable designs: Uses, Constructions and Existence*, CRC Press, Boca Raton, FL, 1996.

[10] S. Furino, S. Kageyama, A. C. H. Ling, Y. Miao and J. Yin, Frames with block size four and index three, *Discrete Math.* **106** (2002), 117–124.

[11] M. Gionfriddo and S. Milici, On the existence of uniformly resolvable decompositions of K_v and $K_v - I$ into paths and kites, *Discrete Math.* **313** (2013), 2830–2834.

[12] G. Ge, Uniform frames with block size four and index one or three, *J. Combin. Des.* **9** (2001), 28–39.

[13] G. Ge and A. C. H. Ling, A symptotic results on the existience of 4-RGDDs and uniform 5-GDDs, *J. Combin. Des.* **13** (2005), 222–237.

[14] G. Ge, C. W. H. Lam and A. C. H. Ling, Some new uniform frames with block size four and index one or three, *J. Combin. Des.* **12** (2004), 112–122.

[15] P. Hell and A. Rosa, Graph decompositions, handcuffed prisoners and balanced P-designs, *Discrete Math.* **2** (1972), 229–252.

[16] R. Julian R. Abel, C. J. Colbourn and J. H. Dintz, in: *Handbook of Combinatorial Designs, 2nd Ed.* (C. J. Colbourn and J. H. Dinitz, Eds.), Chapman & Hall/CRC, 2007.

[17] S. Küçükçifçi, G. Lo Faro, S. Milici and A. Tripodi, Resolable 3-star designs, *Discrete Math.* **338** (2015), 608–614.

[18] S. Küçükçifçi, S. Milici and Z. Tuza, Maximum uniformly resolvable decompositions of K_v and $K_v - I$ into 3-stars and 3-cycles, *Discrete Math.* **338** (2015), 1667–1673.

[19] S. Milici, A note on uniformly resolvable decompositions of K_p and $K_v - I$ into 2-stars and 4-cycles, *Australas. J. Combin.* **56** (2013), 195–200.

[20] S. Milici and Z. Tuza, Uniformly resolvable decompositions of K_v into P_3 and K_3 graphs, *Discrete Math.* **331** (2014), 137–141.

[21] R. Rees, Uniformly resolvable pairwise balanced designs with block sizes two and tree, *J. Combin. Theory Ser. A* **5** (1987), 207–225.

[22] R. Rees and D. R. Stinson, Frames with block size four, *Canad. J. Math.* **44** (1992), 1030–1049.

[23] E. Schuster, Uniformly resolvable designs with index one and block sizes three and four—with three or five parallel classes of block size four, *Discrete Math.* **309** (2009), 2452–2465.

[24] E. Schuster, Uniformly resolvable designs with index one and block sizes three and five and up to five with blocks of size five, *Discrete Math.* **309** (2009), 4435–4442.

[25] E. Schuster, Small uniformly sesolvable designs for block sizes 3 and 4, *J. Combin. Des.* **21** (2013), 481–523.

[26] E. Schuster and G. Ge, On uniformly resolvable designs with block sizes 3 and 4, *Des. Codes Cryptogr.* **57** (2010), 47–69.

[27] D. R. Stinson, Frames for Kirkman triple systems, *Discrete Math.* **65** (1987), 289–300.

[28] H. Wei and G. Ge, Some more 5-GDDs, 4-frames and 4-RGDDs, *Discrete Math.* **336** (2014), 7–21.

[29] X. Zhang and G. Ge, On the existence of partitionable skew Room frames, *Discrete Math.* **307** (2007), 2786–2807.

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