

The k -proper index of complete bipartite and complete multipartite graphs*

WENJING LI XUELIANG LI[†] JINGSHU ZHANG

*Center for Combinatorics and LPMC
Nankai University
Tianjin 300071
China*

linwenjing_61004@163.com lxl@nankai.edu.cn jszhang1992@126.com

Abstract

Let G be an edge-colored graph. A tree T in G is a *proper tree* if no two adjacent edges of it are assigned the same color. Let k be a fixed integer with $2 \leq k \leq n$. For a vertex subset $S \subseteq V(G)$ with $|S| \geq 2$, a tree is called an *S -tree* if it connects the vertices of S in G . A *k -proper coloring* of G is an edge-coloring of G having the property that for every set S of k vertices of G , there exists a proper S -tree T in G . The minimum number of colors that are required in a k -proper coloring of G is defined as the *k -proper index* of G , denoted by $px_k(G)$. In this paper, we determine the 3-proper index of all complete bipartite and complete multipartite graphs and partially determine the k -proper index of them for $k \geq 4$.

1 Introduction

All graphs considered in this paper are simple, finite, undirected and connected. We follow the terminology and notation of Bondy and Murty in [2] for those not defined here. Let G be a graph, we use $V(G)$, $E(G)$, $|G|$, $\Delta(G)$ and $\delta(G)$ to denote the vertex set, edge set, order (number of vertices), maximum degree and minimum degree of G , respectively. For $D \subseteq V(G)$, let $\overline{D} = V(G) \setminus D$, and let $G[D]$ denote the subgraph of G induced by D .

Let G be a nontrivial connected graph with an *edge-coloring* $c: E(G) \rightarrow \{1, \dots, t\}$, $t \in \mathbb{N}$, where adjacent edges may be colored with the same color. If adjacent edges of G receive different colors by c , then c is called a *proper coloring*. The minimum

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[†] Also at School of Mathematics and Statistics, Qinghai Normal University, Xining, Qinghai 810008, China.

number of colors required in a proper coloring of G is referred as the *chromatic index* of G and denoted by $\chi'(G)$. Meanwhile, a path in G is called a *rainbow path* if no two edges of the path are colored with the same color. The graph G is called *rainbow connected* if for any two distinct vertices of G , there is a rainbow path connecting them. For a connected graph G , the *rainbow connection number* of G , denoted by $rc(G)$, is defined as the minimum number of colors that are required to make G rainbow connected. These concepts were first introduced by Chartrand et al. in [6] and have been well-studied since then. For further details, we refer the reader to a book [10].

Motivated by rainbow coloring and proper coloring in graphs, Andrews et al. [1] and, independently, Borozan et al. [3] introduced the concept of proper-path coloring. Let G be a nontrivial connected graph with an edge-coloring. A path in G is called a *proper path* if no two adjacent edges of the path are colored with the same color. The graph G is called *proper connected* if for any two distinct vertices of G , there is a proper path connecting them. The *proper connection number* of G , denoted by $pc(G)$, is defined as the minimum number of colors that are required to make G proper connected. For more details, we refer to a dynamic survey [9].

Chen et al. [7] recently generalized the concept of proper-path to proper tree. A tree T in an edge-colored graph is a *proper tree* if no two adjacent edges of it are assigned the same color. For a vertex subset $S \subseteq V(G)$, a tree is called an *S -tree* if it connects S in G . Let G be a connected graph of order n with an edge-coloring and let k be a fixed integer with $2 \leq k \leq n$. A *k -proper coloring* of G is an edge-coloring of G having the property that for every set S of k vertices of G , there exists a proper S -tree T in G . The minimum number of colors that are required in a k -proper coloring of G is the *k -proper index* of G , denoted by $px_k(G)$. Clearly, $px_2(G)$ is precisely the proper connection number $pc(G)$ of G . For a connected graph G , it is easy to see that $px_2(G) \leq px_3(G) \leq \dots \leq px_n(G)$. The following results are not difficult to obtain.

Proposition 1.1. [7] *If G is a nontrivial connected graph of order $n \geq 3$, and H is a connected spanning subgraph of G , then $px_k(G) \leq px_k(H)$ for any k with $3 \leq k \leq n$. In particular, $px_k(G) \leq px_k(T)$ for every spanning tree T of G .*

Proposition 1.2. [7] *For an arbitrary connected graph G with order $n \geq 3$, we have $px_k(G) \geq 2$ for any integer k with $3 \leq k \leq n$.*

A *Hamiltonian path* in a graph G is a path containing every vertex of G and a graph having a Hamiltonian path is a *traceable graph*.

Proposition 1.3. [7] *If G is a traceable graph with $n \geq 3$ vertices, then $px_k(G) = 2$ for each integer k with $3 \leq k \leq n$.*

Armed with Proposition 1.3, we can easily obtain

$$px_k(K_n) = px_k(P_n) = px_k(C_n) = px_k(W_n) = px_k(K_{s,s}) = 2$$

for each integer k with $3 \leq k \leq n$, where K_n , P_n , C_n and W_n are respectively a complete graph, a path, a cycle and a wheel on $n \geq 3$ vertices and $K_{s,s}$ is a regular complete bipartite graph with $s \geq 2$.

A vertex set $D \subseteq G$ is called an s -dominating set of G if every vertex in \overline{D} is adjacent to at least s distinct vertices of D . If, in addition, $G[D]$ is connected, then we call D a *connected s -dominating set*. Recently, Chang et al. [4] gave an upper bound for the 3-proper index of graphs with respect to the connected 3-dominating set.

Theorem 1.1. [4] *If D is a connected 3-dominating set of a connected graph G with minimum degree $\delta(G) \geq 3$, then $px_3(G) \leq px_3(G[D]) + 1$.*

Using this, we can easily obtain the following.

Theorem 1.2. *For any complete bipartite graph $K_{s,t}$ with $t \geq s \geq 3$, we have $2 \leq px_3(K_{s,t}) \leq 3$.*

Proof. Let U and W be the two partite sets of $K_{s,t}$, where $U = \{u_1, u_2, u_3, \dots, u_s\}$ and $W = \{w_1, w_2, w_3, \dots, w_t\}$. Obviously, $D = \{u_1, u_2, u_3, w_1, w_2, w_3\}$ is a connected 3-dominating set of $K_{s,t}$ and $\delta(K_{s,t}) \geq 3$. It follows from Theorem 1.1 that $px_3(K_{s,t}) \leq px_3(K_{s,t}[D]) + 1 = 3$. By Proposition 1.2, we have $px_3(K_{s,t}) \geq 2$. \square

Naturally, we wonder among these complete bipartite graphs, whose 3-proper index is 2. Moreover, what are the exact values of $px_3(K_{s,t})$ with $s + t \geq 3, t \geq s \geq 1$ and $px_3(K_{n_1, n_2, \dots, n_r})$ with $r \geq 3$? Moreover, what happens when $k \geq 4$? So our paper is organised as follows: In Section 2, we concentrate on all complete bipartite graphs and determine the value of the 3-proper index of each of them. In Section 3, we go on investigating all complete multipartite graphs and obtain the 3-proper index of each of them. In the final section, we turn to the case that $k \geq 4$, and give a partial answer. In the sequel, we use $c(uw)$ to denote the color of the edge uw .

2 The 3-proper index of a complete bipartite graph

In this section, we concentrate on all complete bipartite graphs $K_{s,t}$ with $s + t \geq 3, t \geq s \geq 1$ and obtain a complete answer of the value of $px_3(K_{s,t})$. From [7], we know $px_3(K_{1,t}) = t$. Hence, in the following we assume that $t \geq s \geq 2$. Our result will be divided into three separate theorems depending upon the value of s .

Theorem 2.1. *For any integer $t \geq 2$, we have*

$$px_3(K_{2,t}) = \begin{cases} 2 & \text{if } 2 \leq t \leq 4; \\ 3 & \text{if } 5 \leq t \leq 18; \\ \lceil \sqrt{\frac{t}{2}} \rceil & \text{if } t \geq 19. \end{cases}$$

Proof. Let U, W be the two partite sets of $K_{2,t}$, where $U = \{u_1, u_2\}$ and $W = \{w_1, w_2, \dots, w_t\}$. Suppose that there exists a 3-proper coloring $c : E(K_{2,t}) \rightarrow \{1, 2, \dots, k\}$, $k \in \mathbb{N}$. Corresponding to the 3-proper coloring, there is a color code $code(w)$ assigned to every vertex $w \in W$, consisting of an ordered 2-tuple (a_1, a_2) , where $a_i = c(u_i w) \in \{1, 2, \dots, k\}$ for $i = 1, 2$. In turn, if we give each vertex of W a code, then we can induce the corresponding edge-coloring of $K_{2,t}$.

Claim 1: $px_3(K_{2,t}) = 2$ if $2 \leq t \leq 4$.

Proof. Give the codes $(1, 2), (2, 1), (1, 1), (2, 2)$ to w_1, w_2, w_3, w_4 (if each of these vertices exists). Then it is easy to check that for every 3-subset S of $K_{2,t}$, the edge-colored $K_{2,t}$ has a proper path P connecting S . □

Claim 2: $px_3(K_{2,t}) > 2$ if $t > 4$.

Proof. Otherwise, give $K_{2,t}$ a 3-proper coloring with colors 1 and 2. Then for any 3-subset S of $K_{2,t}$, any proper tree connecting S must actually be a path. For $t > 4$, there are at least two vertices w_p, w_q in W such that $code(w_p) = code(w_q)$. We may assume that $code(w_1) = code(w_2)$. Then for an arbitrary integer i with $3 \leq i \leq t$, let $S = \{w_1, w_2, w_i\}$. There must be a proper path of length 4 connecting S . Suppose that the path is $w_a u_{a'} w_b u_{b'} w_c$, where $\{w_a, w_b, w_c\} = \{w_1, w_2, w_i\}$ and $\{u_{a'}, u_{b'}\} = \{u_1, u_2\}$. By symmetry, we can assume that $u_{a'} = u_1, u_{b'} = u_2$. Then $w_b = w_i$ for otherwise we have $c(w_a u_1) = c(u_1 w_b)$ or $c(w_b u_2) = c(u_2 w_c)$, a contradiction. By symmetry, let $w_a = w_1, w_c = w_2$. Thus $c(w_i u_1) \neq c(w_i u_2)$. Without loss of generality, we can suppose that $c(w_i u_1) = 1$ and $c(w_i u_2) = 2$. Hence, $code(w_i) = (1, 2)$ for each integer $3 \leq i \leq t$. Now let $S = \{w_3, w_4, w_5\}$. It is easy to verify that there is no proper path $w_a u_{a'} w_b u_{b'} w_c$ connecting S , for we always have $c(w_a u_{a'}) = c(u_{a'} w_b), c(w_b u_{b'}) = c(u_{b'} w_c)$. □

Claim 3: Let k be a integer where $k \geq 3$. Then $px_3(K_{2,t}) \leq k$ for $4 < t \leq 2k^2$.

Proof. Set $code(w_1) = (1, 1), code(w_2) = (1, 2), \dots, code(w_k) = (1, k);$

$code(w_{k+1}) = (2, 1), code(w_{k+2}) = (2, 2), \dots, code(w_{2k}) = (2, k);$

...

$code(w_{k(k-1)+1}) = (k, 1), code(w_{k(k-1)+2}) = (k, 2), \dots, code(w_{k^2}) = (k, k)$

(if each of these vertices exists). And let $code(w_{k^2+i}) = code(w_i)$ for $1 \leq i \leq k^2$ (if each of these vertices exists). Now, we prove that this induces a 3-proper coloring of $K_{2,t}$. First of all, we notice that each code appears at most twice. Let S be a 3-subset of $K_{2,t}$. We consider the following two cases.

Case 1: Let $S = \{w_l, w_m, w_n\}$, where $1 \leq l < m < n \leq t$.

Subcase 1.1: If there is a $j \in \{1, 2\}$ such that the colors of $u_j w_l, u_j w_m, u_j w_n$ are pairwise distinct, then the tree $T = \{u_j w_l, u_j w_m, u_j w_n\}$ is a proper S -tree.

Subcase 1.2: If there is no such j , that is, at least two of the edges $u_j w_l, u_j w_m, u_j w_n$ share the same color for both $j = 1$ and $j = 2$.

i) $code(w_l), code(w_m)$ and $code(w_n)$ are pairwise distinct. Without loss of generality, we suppose that $c(u_1w_l) = c(u_1w_m) = a, c(u_2w_l) = c(u_2w_n) = b$ ($1 \leq a, b \leq k^2$). Then $c(u_1w_n) \neq c(u_1w_l), c(u_2w_l) \neq c(u_2w_m)$. If $a = b$, then we have $c(u_1w_n) \neq c(w_nu_2)$. So the path $P = w_lu_1w_nu_2w_m$ is a proper S -tree. Otherwise, the path $P = w_nu_1w_lu_2w_m$ is a proper S -tree.

ii) Two of the codes of the vertices in S are the same. Without loss of generality, we assume that $code(w_l) = code(w_m) = (a, b), code(w_n) = (x, y)$ ($1 \leq a, b, x, y \leq k^2$). Notice that $(x, y) \neq (a, b)$, then suppose that $x \neq a$. Since $k \geq 3$, there are two positive integers $p, q \leq k$ such that $p \neq a, p \neq x$ and $q \neq b, q \neq p$. Pick a vertex w_r whose code is (p, q) (this vertex exists since all of the k^2 codes appear at least once). Then the tree $T = \{u_1w_m, u_1w_n, u_1w_r, w_ru_2, u_2w_l\}$ is a proper S -tree.

Case 2: $S = \{u_r, w_l, w_m\}$, where $1 \leq l < m \leq t$. By symmetry, let $r = 1$.

Suppose that $code(w_l) = (a, b), code(w_m) = (x, y)$ ($1 \leq a, b, x, y \leq k^2$). If $a \neq x$ then the path $P = w_lu_1w_m$ is a proper S -tree. If $a = x$, then we consider whether $b = y$ or not. We discuss two subcases.

i) $b \neq y$, then at least one of them is not equal to a , assume that $b \neq a$. So the path $P = u_1w_lu_2w_m$ is a proper S -tree.

ii) $b = y$, that is $code(w_l) = code(w_m)$, so all of the k^2 codes appear at least at once. Since $k \geq 3$, there are two positive integers $p, q \leq k$ such that $p \neq a$ and $q \neq b, q \neq p$. Pick a vertex w_r whose code is (p, q) . Then the path $P = w_lu_1w_ru_2w_m$ is a proper S -tree.

Case 3: $S = \{u_1, u_2, w_l\}$, where $1 \leq l \leq t$.

Suppose that $code(w_l) = (a, b)$ ($1 \leq a, b \leq k^2$). If $a \neq b$, then the path $P = u_1w_lu_2$ is a proper S -tree. Otherwise, according to our edge-coloring, there exists a vertex w_r of W with the code (p, q) such that $q \neq a$ and $p \neq q$. Then the path $P = w_lu_2w_ru_1$ is a proper S -tree. □

Claim 4: $px_3(K_{2,t}) > k$ for $t > 2k^2$.

Proof. For any edge-coloring of $K_{2,t}$ with k colors, there must be a code which appears at least three times. Suppose that w_1, w_2, w_3 are the vertices with the same code and set $S = \{w_1, w_2, w_3\}$. Then for any tree T connecting S , there is a $j \in \{1, 2\}$ such that $\{u_jw_l, u_jw_m\} \subseteq E(T)$ for some $\{l, m\} \subseteq \{1, 2, 3\}, l \neq m$. But $c(u_jw_l) = c(u_jw_m)$, so T can not be a proper S -tree. Thus $px_3(K_{2,t}) > k$. □

By Claims 2–4, we have the following result: if $5 \leq t \leq 8, px_3(K_{2,t}) = 3$; if $t > 8$, let $k = \left\lceil \sqrt{\frac{t}{2}} \right\rceil$, then $3 \leq \sqrt{\frac{t}{2}} \leq k < \sqrt{\frac{t}{2}} + 1$, i.e., $2(k - 1)^2 + 1 \leq t \leq 2k^2$, so we have $px_3(K_{2,t}) = k = \left\lceil \sqrt{\frac{t}{2}} \right\rceil$. Notice that $px_3(K_{2,t}) = 3$ for $5 \leq t \leq 18$. □

Theorem 2.2. For any integer $t \geq 3$, we have

$$px_3(K_{3,t}) = \begin{cases} 2 & \text{if } 3 \leq t \leq 12; \\ 3 & \text{otherwise.} \end{cases}$$

Proof. Let U, W be the two partite sets of $K_{3,t}$, where $U = \{u_1, u_2, u_3\}$ and $W = \{w_1, w_2, \dots, w_t\}$. Suppose that there exists a 3-proper coloring $c : E(K_{3,t}) \rightarrow \{0, 1, 2, \dots, k - 1\}$, $k \in \mathbb{N}$. Analogously to Theorem 2.1, corresponding to the 3-proper coloring, there is a color code(w) assigned to every vertex $w \in W$, consisting of an ordered 3-tuple (a_1, a_2, a_3) , where $a_i = c(u_i w) \in \{0, 1, 2, \dots, k - 1\}$ for $i = 1, 2, 3$. In turn, if we give each vertex of W a code, then we can induce the corresponding edge-coloring of $K_{3,t}$.

Case 1: $3 \leq t \leq 8$.

In this part, we give the vertices of W the codes which induce a 3-proper coloring of $K_{3,t}$ with colors 0 and 1. And by application of binary system, we can introduce the assignment of the codes in a clear way. Recall the Abelian group \mathbb{Z}_2 . We build a bijection $f : \{w_1, w_2, \dots, w_8\} \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, where $f(w_{4a_1+2a_2+a_3+1}) = (a_1, a_2, a_3)$. For instance, $f(w_3) = (0, 1, 0)$. Under this condition, we use its restriction f_W on W . Now, we prove that f induces a 3-proper coloring of $K_{3,t}$. Let S be an arbitrary 3-subset.

Subcase 1.1: $S = \{w_l, w_m, w_n\}$ for some l, m, n .

Because there is no copy of any code, we can find a vertex in U , say u_1 , such that $u_1 w_l, u_1 w_m, u_1 w_n$ are not all with the same color. We may assume that $c(u_1 w_l) = c(u_1 w_m) = 0$ and $c(u_1 w_n) = 1$.

i) $code(w_l) = (0, 0, 0)$. Then there is a ‘1’ in the code of w_m . By symmetry, assume that $c(u_2 w_m) = 1$. Then there is a proper path $P = w_l u_2 w_m u_1 w_n$ connecting S .

ii) $code(w_l) = (0, 0, 1)$. If $code(w_m) = (0, 0, 0)$, then we return to *i)*. Otherwise, the code of w_m is neither $(0, 0, 0)$ nor $(0, 0, 1)$. So $c(u_2 w_m) = 1$. Then the proper S -tree is the same as that in *i)*.

iii) $code(w_l) = (0, 1, 0)$. It is similar to *ii)*.

iv) $code(w_l) = (0, 1, 1)$. Then either $c(u_2 w_m) = 0$ or $c(u_3 w_m) = 0$. By symmetry, we suppose that $c(u_2 w_m) = 0$. Then the path $P = w_m u_2 w_l u_1 w_n$ is a proper S -tree.

Subcase 1.2: $S = \{u_j, w_l, w_m\}$ for some j, l, m .

If $c(u_j w_l) \neq c(u_j w_m)$, then the path $P = w_l u_j w_m$ is a proper S -tree. Otherwise, by symmetry, we assume that $c(u_j w_l) = c(u_j w_m) = 0$, then there is a $j' \neq j$ such that $c(u_{j'} w_l) \neq c(u_{j'} w_m)$ (otherwise w_l, w_m will have the same code). So one of $c(u_{j'} w_l)$ and $c(u_{j'} w_m)$ equals 1, say $c(u_{j'} w_l) = 1$. Then the path $P = u_{j'} w_l u_{j'} w_m$ is a proper S -tree.

Subcase 1.3: $S = \{u_{j_1}, u_{j_2}, w_l\}$ for some j_1, j_2, l .

If $c(u_{j_1} w_l) \neq c(u_{j_2} w_l)$, then the path $P = u_{j_1} w_l u_{j_2}$ is a proper S -tree. Otherwise, by symmetry, we assume that $c(u_{j_1} w_l) = c(u_{j_2} w_l) = 0$. By the sequence of the codes according to f and $t \geq 3$, we know that for any two vertices $u_{a'}, u_{b'}$ of U , there exists a vertex $w \in W$ such that $c(u_{a'} w) \neq c(u_{b'} w)$. Similar to Subcase 1.2, we can obtain a proper S -tree.

Subcase 1.4: $S = \{u_1, u_2, u_3\}$.

$P = u_1w_3u_2w_2u_3$ is a proper path connecting S .

Case 2: $9 \leq t \leq 12$.

Set $code(w_1) = (0, 0, 1)$, $code(w_2) = (0, 1, 0)$, $code(w_3) = (0, 1, 1)$,
 $code(w_4) = (1, 0, 0)$, $code(w_5) = (1, 0, 1)$, $code(w_6) = (1, 1, 0)$.

And let $code(w_{6+i}) = code(w_i)$ for $1 \leq i \leq 6$ (if each of these vertices exist). For convenience, we denote $w_{6+i} = w'_i$. Now, we claim that this induces a 3-proper coloring of $K_{3,t}$. Let S be an arbitrary 3-subset of $K_{3,t}$. Based on Case 1, we only consider about the case that $\{w_i, w'_i\} \subseteq S$ for some $1 \leq i \leq 6$. By symmetry, we suppose that $i = 1$. First of all, we list three proper paths containing w_1, w'_1 : $P_1 = w_1u_3w_2u_2w'_1$, $P_2 = w_1u_2w_3u_1w_4u_3w'_1$ and $P_3 = w_1u_1w_5u_2w_6u_3w'_1$, in which w_j can be replaced by w'_j for $2 \leq j \leq 6$. Then, we can always find a proper path from $\{P_1, P_2, P_3\}$ connecting S whichever the third vertex of S is.

Case 3: $t \geq 13$.

We claim that $px_3(K_{3,t}) = 3$. We prove it by contradiction. If there is a 3-proper coloring of $K_{3,t}$ with two colors 0 and 1, then any proper tree for an arbitrary 3-subset S is in fact a path. Consider the set $S \subseteq W$. As the graph is bipartite and we just care about the shortest proper path connecting S , there are only two possible types of such a path:

I: $w_a u_{a'} w_b u_{b'} w_c$

II: $w_a u_{a'} w_b u_{b'} w' u_{c'} w_c$

where $\{u_{a'}, u_{b'}, u_{c'}\} = U$ and $\{w_a, w_b, w_c\} = S, w' \in W \setminus S$.

Firstly, as $t \geq 13$, we know that some code appears more than once. But it can not appear more than twice. Otherwise, assume that w_i, w'_i, w''_i are the three vertices with the same code, and let $S = \{w_i, w'_i, w''_i\}$. Whether the proper path connecting S is type I or type II, it should be $c(w_a u_{a'}) \neq c(w_b u_{a'})$, contradicting with the assumption that $code(w_i) = code(w'_i) = code(w''_i)$.

Secondly, we prove the following several claims by contradiction.

Claim 1: The repetitive code can not be $(0, 0, 0)$ or $(1, 1, 1)$.

Proof. Suppose that $code(w_1) = code(w_2) = (0, 0, 0)$. Let $S = \{w_1, w_2, w_3\}$ where $w_3 \in W \setminus \{w_1, w_2\}$, and let P be a proper path connecting S . Then w_1, w_2 are the two end vertices of P , and so the two end edges of it are assigned the same color. However, since the length of P is even, the colors of the end edges can not be the same, a contradiction. Analogously, the code $(1, 1, 1)$ cannot appear more than once. □

Claim 2: If the code $(0, 0, 1)$ is repeated, then there is no vertex in W with $(0, 0, 0)$ as its code.

Proof. Suppose that $code(w_1) = code(w_2) = (0, 0, 1)$, $code(w_3) = (0, 0, 0)$. Let $S = \{w_1, w_2, w_3\}$, and let P be a proper path connecting S . Then w_3 is one of the end vertices of P . Moreover, the path P must be type II, for in type I, we need

$c(w_a u_{a'}) \neq c(w_b u_{a'})$ and $c(w_b u_{b'}) \neq c(w_c u_{b'})$, which is impossible for S . We can also deduce that $u_{a'} = u_3$ because $c(w_a u_{a'}) \neq c(w_b u_{a'})$. And $\{w_1, w_2\} \neq \{w_a, w_b\}$ since they are with the same code. So we have $w_a = w_3$. Thus, $\{w_b, w_c\} = \{w_1, w_2\}$ and $\{u_{b'}, u_{c'}\} = \{u_1, u_2\}$, contradicting with the fact that $c(w_b u_{b'}) \neq c(w_c u_{c'})$. \square

Analogously, we have that the repetitive code $(0, 1, 0)$ or $(1, 0, 0)$ can not exist along with the code $(0, 0, 0)$, respectively. Symmetrically, the repetitive code $(0, 1, 1)$, $(1, 0, 1)$ or $(1, 1, 0)$ can not exist along with the code $(1, 1, 1)$, respectively.

Finally, as $t \geq 13$ and no code could appear more than twice, there are at least 7 different codes in W and at least 5 codes repeated. But considering Claim 2 and its analogous results, it is a contradiction. So $px_3(K_{3,t}) = 3$ when $t \geq 13$. \square

Theorem 2.3. *For a complete bipartite graph $K_{s,t}$ with $t \geq s \geq 4$, we have $px_3(K_{s,t}) = 2$.*

Proof. Let U, W be the two partite sets of $K_{s,t}$, where $U = \{u_1, u_2, \dots, u_s\}$ and $W = \{w_1, w_2, \dots, w_t\}$. And denote a cycle $C_s = u_1 w_1 u_2 w_2 \dots u_s w_s u_1$. Moreover, if $u, v \in V(C_s)$, then we use $u C_s v$ to denote the segment of C_s from u to v in the clockwise direction, and we denote the opposite direction by $u C'_s v$. Then we demonstrate a 3-proper coloring of $K_{s,t}$ with two colors 0 and 1. Let $c(u_i w_i) = 0$ ($1 \leq i \leq s$) and $c(u_i w_j) = 1$ ($1 \leq i \neq j \leq s$). And assign $c(w_r u_i) = i \pmod{2}$ ($1 \leq i \leq s, s < r \leq t$). Now we prove that this coloring is a 3-proper coloring of $K_{s,t}$. Consider a 3-subset S .

i) $S \subseteq V(C_s)$. The proper path is a segment of C_s .

ii) $S = \{w_l, w_m, w_n\}$ where $l, m, n > s$. Then the path $P = w_l u_1 w_1 u_2 w_m u_3 w_3 u_4 w_n$ is a proper S -tree.

iii) $S = \{w_l, w_m, w_n\}$ where $l \leq s, m, n > s$. If $c(w_m u_l) = 1$, then the path $P = w_m u_l w_l C_s u_2 w_n$ is a proper S -tree. If $c(w_m u_l) = 0$, then the proper S -tree is the path $P = w_m u_l w_{l-1} u_{l-1} w_n u_{l-2} C'_s w_l$, where $u_0 = u_s, u_{-1} = u_{s-1}$ if $l_1 = 2$.

iv) $S = \{u_j, w_l, w_m\}$ where $l, m > s$. The way to find a proper S -tree is similar to that in iii).

v) $S = \{u_j, w_l, w_m\}$ where $l \leq s, m > s$. If $c(w_m u_j) = 1$, then the proper S -tree is the path $P = w_m u_j w_j C_s w_l$. If $c(w_m u_j) = 0$, then the path $P = w_m u_j C'_s w_l$ is a proper S -tree.

vi) $S = \{u_{j_1}, u_{j_2}, w_i\}$ where $i > s$. The way to find a proper S -tree is similar to that in v). \square

Remarks. Here, we introduce a generalization of k -proper index which was recently proposed by Chang et al. in [5]. Let G be a nontrivial κ -connected graph of order n , and let k and ℓ be two integers with $2 \leq k \leq n$ and $1 \leq \ell \leq \kappa$. For $S \subseteq V(G)$, let $\{T_1, T_2, \dots, T_\ell\}$ be a set of S -trees. They are *internally disjoint* if $E(T_i) \cap E(T_j) = \emptyset$ and $V(T_i) \cap V(T_j) = S$ for every pair of distinct integers i, j with $1 \leq i, j \leq \ell$. The (k, ℓ) -proper index of G , denoted by $px_{k,\ell}(G)$, is the minimum number of colors that are required in an edge-coloring of G such that for every k -subset S of $V(G)$,

there exist ℓ internally disjoint proper S -trees connecting them. In their paper, they investigated the complete bipartite graphs and obtained the following.

Theorem 2.4. [5] *Let s and t be two positive integers with $t = O(s^r)$, $r \in \mathbb{R}$ and $r \geq 1$. For every pair of integers k, ℓ with $k \geq 3$, there exists a positive integer $N_3 = N_3(k, \ell)$ such that $px_{k,\ell}(K_{s,t}) = 2$ for every integer $s \geq N_3$.*

Obviously, they did not give the exact value of $px_{k,\ell}(K_{s,t})$, even for $k = 3$ and $\ell = 1$. Our Theorem 2.3 completely determines the value of $px_{k,\ell}(K_{s,t})$ for $k = 3$ and $\ell = 1$, without using the condition that $t = O(s^r)$, $r \in \mathbb{R}$ and $r \geq 1$.

3 The 3-proper index of a complete multipartite graph

With the aids of Theorems 2.1, 2.2 and 2.3, we are now able to determine the 3-proper index of all complete multipartite graphs. First of all, we give a useful theorem.

Theorem 3.1. [8] *Let G be a graph with n vertices. If $\delta(G) \geq \frac{n-1}{2}$, then G has a Hamiltonian path (i.e. G is traceable).*

Theorem 3.2. *Let $G = K_{n_1, n_2, \dots, n_r}$ be a complete multipartite graph, where $r \geq 3$ and $n_1 \leq n_2 \leq \dots \leq n_r$. Set $s = \sum_{i=1}^{r-1} n_i$ and $t = n_r$. Then we have*

$$px_3(G) = \begin{cases} 3 & \text{if } G = K_{1,1,t}, 5 \leq t \leq 18 \\ & \text{or } G = K_{1,2,t}, t \geq 13 \\ & \text{or } G = K_{1,1,1,t}, t \geq 15; \\ \lceil \sqrt{\frac{t}{2}} \rceil & \text{if } G = K_{1,1,t}, t \geq 19; \\ 2 & \text{otherwise.} \end{cases}$$

Proof. The graph G has a $K_{s,t}$ as its spanning subgraph, so it follows from Propositions 1.1 and 1.2 that $2 \leq px_3(G) \leq px_3(K_{s,t})$. In the following, we discuss two cases according to the relationship between s and t .

Case 1: $s \leq t$. Let U_1, U_2, \dots, U_r denote the different r -partite sets of G , where $|U_i| = n_i$ for each integer $1 \leq i \leq r$.

When $s \geq 4$, then by Theorem 2.3, we have $px_3(G) = px_3(K_{s,t}) = 2$. When $s \leq 3$, there are only three possible values of $(n_1, n_2, \dots, n_{r-1})$.

Subcase 1: $(n_1, n_2, \dots, n_{r-1}) = (1, 1)$. Set $U_1 = \{u_1\}, U_2 = \{u_2\}$. Under this condition, giving the edge u_1u_2 an arbitrary color, the proof is exactly the same as that of Theorem 2.1. So it holds that $px_3(G) = px_3(K_{2,t})$.

Subcase 2: $(n_1, n_2, \dots, n_{r-1}) = (1, 2)$. Set $U_1 = \{u_1\}, U_2 = \{u_2, u_3\}$ and $W = U_r$. By Theorem 2.2, we have $px_3(G) = px_3(K_{3,t}) = 2$ if $t \leq 12$; $px_3(G) \leq px_3(K_{3,t}) = 3$ if $t > 12$. We claim that $px_3(G) = 3$ if $t > 12$. Assume, to the contrary, that G has a 3-proper coloring with two colors 0 and 1. By symmetry, we assume that $c(u_1u_2) = 0$. With the similar reason in Case 3 of the proof of Theorem 2.2, no code can appear more than twice. And recall the bijection f defined in that proof. To

label the vertices in W , we use its inverse $f^{-1} : (a_1, a_2, a_3) \mapsto w_{4a_1+2a_2+a_3+1}$, and denote by w'_i the copy of the vertex w_i with $1 \leq i \leq 8$. Then we prove the following results by contradiction.

Claim 1: $\{w_1, w'_1, w_2\} \not\subseteq W$ and $\{w_2, w'_2, w_1\} \not\subseteq W$.

Proof. Set $S = \{w_1, w'_1, w_2\}$. We know from the proof of Theorem 2.2 that there is no proper path of type I or II. So the proper path P connecting S is type III, defined as $w_a u_{a'} w_b u_{b'} u_{c'} w_c$. Then w_1, w'_1 must be the end vertices of P , and so $w_b = w_2$ and $u_{a'} = u_3$. Since $c(w_a u_{a'}) = 0$, $c(u_{b'} u_{c'}) = 1$, contradicting with $c(u_1 u_2) = 0$. Hence, we obtain $\{w_1, w'_1, w_2\} \not\subseteq W$. Similarly, we have $\{w_2, w'_2, w_1\} \not\subseteq W$. \square

Claim 2: $\{w_4, w'_4, w_8\} \not\subseteq W$ and $\{w_8, w'_8, w_4\} \not\subseteq W$.

Proof. Set $S = \{w_4, w'_4, w_8\}$. Similar to Claim 1, any proper path P connecting S should be type III: $w_a u_{a'} w_b u_{b'} u_{c'} w_c$. Then w_8 must be an end vertex of P , and so both of the end edges of P are colored with 1. Thus $u_{a'} = u_1$. Then $\{u_{b'}, u_{c'}\} = \{u_2, u_3\}$ and $c(u_2 u_3) = 0$, contradicting with the fact that $u_2 u_3 \notin E(G)$. Similarly, we have $\{w_8, w'_8, w_4\} \not\subseteq W$. \square

So there are four cases that some vertices can not exist in W at the same time, and each code appears at most twice. However, there are more than 12 vertices in W , a contradiction. So $px_3(G) = px_3(K_{3,t}) = 3$ when $t > 12$.

Subcase 3: $(n_1, n_2, \dots, n_{r-1}) = (1, 1, 1)$. Set $U = \cup_{j=1}^{r-1} U_j = \{u_1, u_2, u_3\}$ and $W = U_r$.

Claim 3: $px_3(G) = 2$ if $t \leq 14$.

Proof. By Theorem 2.2, we have $px_3(G) = px_3(K_{3,t}) = 2$ if $t \leq 12$; $px_3(G) \leq px_3(K_{3,t}) = 3$ if $t > 12$. When $t = 13$ or 14 , we recall $code(w)$ defined in Case 2 of Theorem 2.2. Set

$$code(w_1) = (0, 0, 1), code(w_2) = (0, 1, 0), code(w_3) = (0, 1, 1), code(w_4) = (1, 0, 0), \\ code(w_5) = (1, 0, 1), code(w_6) = (1, 1, 0), code(w_7) = (1, 1, 1).$$

And let $code(w_{7+i}) = code(w_i)$ for $1 \leq i \leq 7$ (if each of these vertices exists) and $c(u_i u_j) = 0$ for $1 \leq i \neq j \leq 3$. For convenience, we denote $w_{7+i} = w'_i$. Now, we claim that this induces a 3-proper coloring of G . Let S be an arbitrary 3-subset of G . Based on Theorem 2.2, we only consider about the case that $w_7(w'_7) \in S$. When $S = \{w_1, w_7, w'_7\}$, then the path $P = w_7 u_1 w_1 u_3 u_2 w'_7$ is a proper path connecting S . Similarly, we can find a proper path in type III connecting S whichever the two other vertices of S are. \square

Claim 4: $px_3(G) = 3$ if $t > 14$.

Proof. Assume, to the contrary, that G has a 3-proper coloring with two colors 0 and 1. If the edges of $G[U]$ are colored with two different colors, then we set u_2

the common vertex of two edges with two different colors. Moreover, without loss of generality, we suppose that $c(u_1u_2) = 0$. Similar to Subcase 2, we have $px_3(G) = 3$ if $t > 12$. If all the edges of $G[U]$ are colored with one color, say 0. Repeat the discussion in Subcase 2, then we know Claim 1 is also true under this condition. As $t \geq 15$ and no code could appear more than twice, there are at least 8 different codes in W and at least 7 codes repeated. But from Claim 1, we know $\{w_1, w'_1, w_2\} \not\subseteq W$ and $\{w_2, w'_2, w_1\} \not\subseteq W$. So $px_3(G) = 3$ when $t \geq 15$. \square

Case 2: $s \geq t$. Under this condition, we have $\delta(G) \geq \frac{n-1}{2}$. By Theorem 3.1, we know G is traceable. Thus, it follows from Proposition 1.3 that $px_3(G) = 2$. \square

4 The k -proper index

Now, we turn to the k -proper index of a complete bipartite graph and a complete multipartite graph for general k . Throughout this section, let k be a fixed integer with $k \geq 3$. Firstly, we generalize Theorem 1.1 to the k -proper index.

Theorem 4.1. *If D is a connected k -dominating set of a connected graph G with minimum degree $\delta(G) \geq k$, then $px_k(G) \leq px_k(G[D]) + 1$.*

Proof. Since D is a connected k -dominating set, every vertex v in \overline{D} has at least k neighbors in D . Let $x = px_k(G[D])$. We first color the edges in $G[D]$ with x different colors from $\{2, 3, \dots, x + 1\}$ such that for every k vertices in D , there exists a proper tree in $G[D]$ connecting them. Then we color the remaining edges with color 1.

Next, we will show that this coloring makes G k -proper connected. Let $S = \{v_1, v_2, \dots, v_k\}$ be any set of k vertices in G . Without loss of generality, we assume that $\{v_1, \dots, v_p\} \subseteq D$ and $\{v_{p+1}, \dots, v_k\} \subseteq \overline{D}$ for some p ($0 \leq p \leq k$). For each $v_i \in \overline{D}$ ($p + 1 \leq i \leq k$), let u_i be the neighbour of v_i in D such that $\{u_{p+1}, \dots, u_k\}$ is a $(k - p)$ -set. It is possible since D is a k -dominating set. Then the edges $\{u_{p+1}v_{p+1}, \dots, u_kv_k\}$ together with the proper tree connecting the vertices $\{v_1, \dots, v_p, u_{p+1}, \dots, u_k\}$ in $G[D]$ induces a proper S -tree. Thus, we have $px_k(G) \leq px_k(G[D]) + 1$. \square

Based on this theorem, we can give a lower bound and an upper bound on the k -proper index of a complete bipartite graph, whose proof is similar to Theorem 1.2.

Theorem 4.2. *For a complete bipartite graph $K_{s,t}$ with $t \geq s \geq k$, we have $2 \leq px_k(K_{s,t}) \leq 3$.*

Let G be a complete bipartite graph. Using the techniques in Theorem 2.3, we can obtain the sufficient condition such that $px_k(G) = 2$.

Theorem 4.3. *For a complete bipartite graph $K_{s,t}$ with $t \geq s \geq 2(k - 1)$, we have $px_k(K_{s,t}) = 2$.*

Proof. We demonstrate a k -proper coloring of $K_{s,t}$ with two colors 0 and 1, the same as Theorem 2.3. For completeness, we restate the coloring. Let U, W be the two partite sets of $K_{s,t}$, where $U = \{u_1, u_2, \dots, u_s\}$ and $W = \{w_1, w_2, \dots, w_t\}$, $t \geq s \geq 2(k-1)$. Denote a cycle $C_s = u_1 w_1 u_2 w_2 \dots u_s w_s u_1$. Let $c(u_i w_i) = 0$ ($1 \leq i \leq s$) and $c(u_i w_j) = 1$ ($1 \leq i \neq j \leq s$). And assign $c(w_r u_i) = i \pmod{2}$ ($1 \leq i \leq s, s < r \leq t$). Now, we show that for any k -subset $S \subseteq V(K_{s,t})$, there is a proper path P_S connecting all the vertices in S . Set $W_1 = \{w_1, w_2, \dots, w_s\}$ and $W_2 = \{w_{s+1}, \dots, w_t\}$ (if $t > s$). Then S can be divided into three parts, i.e., $S = S_1 \cup S_2 \cup S_3$, where $S_1 = S \cap W_1$, $S_2 = S \cap W_2$ and $S_3 = S \cap U$. Suppose $|S_1| = p$, $|S_2| = q$, then $p + q \leq k$. If $q = 0$, the path $P = u_1 w_1 u_2 w_2 \dots u_s w_s$ is a proper path connecting S . If $q \geq 1$, set $S_2 = \{w_{\alpha_1}, w_{\alpha_2}, \dots, w_{\alpha_q}\}$, where $s < \alpha_1, \alpha_2, \dots, \alpha_q \leq t$. Let $P = w_{\alpha_q} u_1 w_1 u_2 w_2 \dots u_s w_s$. Then consider the vertex set $W'_S = \{w_{2i} : w_{2i} \in W_1 \setminus S_1\}$. We have $|W'_S| \geq s/2 - p \geq k - p - 1 \geq q - 1$. So set $|W'_S| = \ell$ and $W'_S = \{w_{\beta_1}, w_{\beta_2}, \dots, w_{\beta_{q-1}}, \dots, w_{\beta_\ell}\}$, where $2 \leq \beta_1, \beta_2, \dots, \beta_\ell \leq s$ are even. Then we construct a path P_S by replacing the subpath $u_{\beta_j} w_{\beta_j} u_{\beta_j+1}$ of P with $u_{\beta_j} w_{\alpha_j} u_{\beta_j+1}$ (and $u_s w_s$ with $u_s w_{\alpha_j}$ if $\beta_j = s$) for $1 \leq j \leq q-1$. Hence, the new path P_S is a proper path contains all the vertices of U so that P_S connects S_3 . By the replacement we know that P_S also connects S_1 as well as S_2 . Thus we complete the proof. \square

With the aid of Theorems 4.3 and 3.1, we can easily obtain the following, the proof of which is similar to that of Theorem 3.2.

Theorem 4.4. *Let $G = K_{n_1, n_2, \dots, n_r}$ be a complete multipartite graph, where $r \geq 3$ and $n_1 \leq n_2 \leq \dots \leq n_r$. Set $s = \sum_{i=1}^{r-1} n_i$ and $t = n_r$. If $t \geq s \geq 2(k-1)$ or $t \leq s$, then we have $px_k(G) = 2$.*

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