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# On the complexity of some bondage problems in graphs

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**Abstract**

The paired bondage number (total restrained bondage number, independent bondage number, $k$-rainbow bondage number) of a graph $G$, is the minimum number of edges whose removal from $G$ results in a graph with larger paired domination number (respectively, total restrained domination number, independent domination number, $k$-rainbow domination number). In this paper we show that the decision problems for these variants are NP-hard, even when restricted to bipartite graphs.

## 1 Introduction

Let $G$ be a graph with vertex set $V(G) = V$ of order $|V| = n$ and size $|E(G)| = m$, and let $v$ be a vertex in $V$. The *open neighborhood* of $v$ is $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and the *closed neighborhood of* $v$ is $N_G[v] = \{v\} \cup N_G(v)$. If the graph $G$ is clear from the context, we simply write $N(v)$ rather than $N_G(v)$. The *degree* of a vertex $v$, is $\deg(v) = |N(v)|$. A vertex of degree one is called a *leaf* and its neighbor a *support vertex*. A *pendant edge* is an edge that one of its endpoints is a leaf. We denote the set of leaves and support vertices of a graph $G$ by $L(G)$ and $S(G)$, respectively. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v\in S} N(v)$, and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. A *matching* in a graph $G$ is a set of independent edges in $G$. A *perfect matching* $M$ in $G$ is a matching such that every vertex of $G$ is incident to an element of $M$. For a subset $S$ of vertices of $G$ we refer to $G[S]$ as the subgraph of $G$ induced by $S$. For notation and graph theory terminology, we in general follow [8, 10].

A subset $S \subseteq V$ is a *dominating set* of $G$ if every vertex not in $S$ is adjacent to a vertex in $S$. The *domination number* of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. A dominating set $S$ in a graph with no isolated vertex is a *total dominating set* if the induced subgraph $G[S]$ has no isolated vertex. The *total domination number* of $G$, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of $G$. A dominating set $S$ in a graph $G$ with no isolated vertex is called a *paired dominating set* if the induced subgraph $G[S]$ contains a perfect matching. The *paired domination number* of $G$, denoted by $\gamma_{pr}(G)$, is the minimum cardinality of a paired dominating set of $G$. A dominating set $S$ is called an *independent dominating set* if the induced graph $G[S]$ has no edge. The *independent domination number* of $G$, denoted by $i(G)$, is the minimum cardinality of an independent dominating set of $G$. A total dominating set $S$ is called a *total restrained dominating set* if every vertex of $V - S$ is adjacent to another vertex in $V - S$. The *total restrained domination number* of $G$, denoted by $\gamma_{tr}(G)$, is the minimum cardinality of a total restrained dominating set of $G$. A dominating set $S$ is called an $\gamma(G)$-set of $G$ if $|S| = \gamma(G)$. Similarly a $\gamma_t(G)$-set, an $i(G)$-set, a $\gamma_{pr}(G)$-set, and a $\gamma_{tr}(G)$-set are defined. For references on domination and total domination in graphs see for example [8, 10].

For a graph $G$ and an integer $k \geq 2$, let $f : V(G) \to \mathcal{P}(\{1, 2, ..., k\})$ be a function. If for each vertex $v \in V(G)$ such that $f(v) = \emptyset$ we have $\cup_{u \in N(v)} f(u) = \{1, 2, ..., k\}$, then $f$ is called a *$k$-rainbow dominating function* (or simply $k$RDF) of $G$. The *weight*, $w(f)$, of $f$ is defined as $w(f) = \sum_{v \in V(G)} |f(v)|$. The minimum weight of a $k$RDF of $G$ is called the *$k$-rainbow domination number* of $G$, and is denoted by $\gamma_{rk}(G)$. If $f$ is a $k$RDF of $G$, then we denote by $V^f_{12...k}$ the set of all vertices $u$ with $|f(u)| = k$. We refer to a $\gamma_{rk}$-function in a graph $G$ as a $k$RDF with minimum weight. If $f$ is a $k$RDF of $G$, then we say that a vertex $v$ is not $k$-rainbow dominated by $f$ if $f(v) = \emptyset$ and $\cup_{u \in N(v)} f(u) \neq \{1, 2, ..., k\}$. For references in rainbow domination see for example [2, 3, 17].

The *bondage number* of $G$, denoted by $b(G)$, is the minimum number of edges whose removal from $G$ results in a graph with larger domination number. The concept of bondage in graphs was introduced by Bauer, Harary, Nieminen and Suffel in [1], and has been further studied for example in [4, 5, 16]. Raczek [15] introduced the concept of paired bondage in graphs. The *paired bondage number* $b_{pr}(G)$ of a graph $G$ with no isolated vertex is the cardinality of a smallest set of edges $E' \subseteq E(G)$ for which (1) $G - E'$ has no isolated vertex, and (2) $\gamma_{pr}(G - E') > \gamma_{pr}(G)$. Zhang, Liu and Sun [19] defined the *independent bondage number* $b_i(G)$ of $G$ to be the minimum cardinality among all subsets $E' \subseteq E(G)$ for which $i(G - E') > i(G)$. The *total restrained bondage number* $b_{tr}(G)$ of a graph $G$ with no isolated vertex is the cardinality of a smallest set of edges $E' \subseteq E(G)$ for which (1) $G - E'$ has no isolated vertex, and (2) $\gamma_{tr}(G - E') > \gamma_{tr}(G)$. The concept of total restrained bondage is studied in [13]. The *k-rainbow bondage number* $b_{rk}(G)$ of a graph $G$ with maximum degree at least two is the minimum cardinality among all sets $E' \subseteq E(G)$ for which $\gamma_{rk}(G - E') > \gamma_{rk}(G)$. For a survey of results and recent developments on bondage we refer the reader to [18].

The complexity issue of several parameters in the theory of domination have been studied, see for example [6, 8]. The decision problem for some bondage problems has been proven to be NP-hard, see for example [7, 11, 12, 14, 18].

**Conjecture 1.1** (Xu, [18])**.** *The paired bondage problem is NP-complete.*

In this paper, we consider the complexity issue for paired bondage problem, total restrained bondage problem, independent bondage problem, and $k$-rainbow bondage problem. We prove that the decision problem for these bondage problems is NP-hard, even when restricted to bipartite graphs. Our proofs are by a transformation from the 3-satisfiability problem (known as 3-SAT problem) that we describe it as follows. A *truth assignment* for a set $U$ of Boolean variables is a mapping $t : U \to \{T, F\}$. A variable $u$ is said to be *true* (or *false*) under $t$ if $t(u) = T$ (or $t(u) = F$). If $u$ is a variable in $U$, then $u$ and $\overline{u}$ are *literals* over $U$. The literal $u$ is true under $t$ if and only if the variable $u$ is true under $t$, and the literal $\overline{u}$ is true if and only if the variable $u$ is false. A *clause* over $U$ is a set of literals over $U$, and it is *satisfied* by a truth assignment if and only if at least one of its members is true under that assignment. A collection $\mathcal{C}$ of clauses over $U$ is satisfiable if and only if there exists some truth assignment for $U$ that simultaneously satisfies all the clauses in $\mathcal{C}$. Such a truth assignment is called a satisfying truth assignment for $\mathcal{C}$. The 3-SAT problem is specified as follows.

**3-SAT problem**:
**Instance:** A collection $\mathcal{C} = \{C_1, C_2, ..., C_m\}$ of clauses over a finite set $U$ of variables such that $|C_j| = 3$ for $j = 1, 2, ..., m$.
**Question:** Is there a truth assignment for $U$ that satisfies all the clauses in $C$?

Note that the 3-SAT problem was proven to be NP-complete in [6].

## 2 Main results

Consider the following decision problems.

Paired bondage problem (**PB**):
Instance: A graph $G$ with no isolated vertex and a positive integer $\kappa$.
Question: Is $b_p(G) \leq \kappa$?

Total restrained bondage problem (**TRB**):
Instance: A graph $G$ with no isolated vertex and a positive integer $\kappa$.
Question: Is $b_{tr}(G) \leq \kappa$?

Independent bondage problem (**IB**):
Instance: A nonempty graph $G$ and a positive integer $\kappa$.

Question: Is $b_i(G) \leq \kappa$?

$k$-rainbow bondage problem ($k$**RB**):
Instance: A nonempty graph $G$ and a positive integer $\kappa$.
Question: Is $b_{rk}(G) \leq \kappa$?

We will prove the following.

**Theorem 2.1.** **PB** *is NP-hard for bipartite graphs.*

**Theorem 2.2.** **TRB** *is NP-hard for general graphs.*

**Theorem 2.3.** **IB** *is NP-hard for bipartite graphs.*

**Theorem 2.4.** $k$**RB** *is NP-hard for bipartite graphs.*

## 3 Proofs

We show the NP-hardness of each problem by transforming the 3-SAT problem to it in polynomial time. Let $U = \{u_1, u_2, ..., u_n\}$ and $\mathcal{C} = \{C_1, C_2, ..., C_m\}$ be an arbitrary instance of the 3-SAT.

### 3.1 Proof of Theorem 2.1

We construct a bipartite graph $G$ and an integer $\kappa$ such that $\mathcal{C}$ is satisfiable if and only if $b_p(G) \leq \kappa$. Corresponding to each variable $u_i \in U$, we associate the graph $H_i$ shown in Figure 1. Corresponding to each clause $C_j = \{x_j, y_j, z_j\} \in \mathcal{C}$, we associate a single vertex $c_j$ and add the edge-set $E_j = \{c_jx_j, c_jy_j, c_jz_j\}$. Next add the graph $J$ shown in Figure 1, and join $s_2$ to each vertex $c_j$ with $1 \leq j \leq m$, to obtain a bipartite graph $G$. Set $\kappa = 1$.

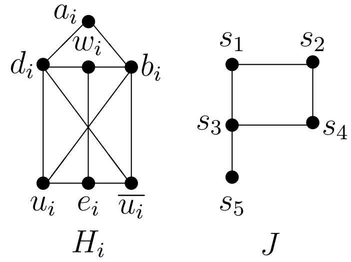


Figure 1. The graphs $H_i$ and $J$.

Let $S$ be a $\gamma_{pr}(G)$-set. Clearly $|S \cap V(H_i)| \geq 2$. Moreover, $|S \cap \{s_1, s_2, s_3, s_4, s_5\}| \geq 2$. Thus we have the following.

**Lemma 3.1.** $\gamma_{pr}(G) = |S| \geq 2n + 2$.

**Lemma 3.2.** $\gamma_{pr}(G) = 2n + 2$ *if and only if* $\mathcal{C}$ *is satisfiable.*

*Proof.* Assume that $\gamma_{pr}(G) = 2n+2$. Then $|S \cap V(H_i)| = 2$ for $i = 1,2,...,n$, $|S \cap \{s_1,s_2,s_3,s_4,s_5\}| = 2$ and $S \cap \{c_1,...,c_m\} = \emptyset$. Clearly $s_3 \in S$, and thus we may assume, without loss of generality, that $S \cap \{s_1,s_2,s_3,s_4,s_5\} = \{s_3,s_4\}$. Moreover, $S$ does not contain both $u_i$ and $\overline{u_i}$ for $i = 1,2,...,n$. If $S \cap \{u_j, \overline{u_j}\} = \emptyset$ for some $j$, then we replace the vertices of $S \cap V(H_j)$ by $u_i$ and $b_i$. We thus may assume that $|S \cap \{u_i, \overline{u_i}\}| = 1$ for $i = 1,2,...,n$. Let $t : U \longrightarrow \{T,F\}$ be a mapping defined by $t(u_i) = T$ if $u_i \in S$ and $t(u_i) = F$ if $\overline{u_i} \in S$. For each $j \in \{1,2,..,m\}$, the corresponding vertex $c_j$ in $G$ is not dominated by $s_3$ or $s_4$, and thus there is an integer $i \in \{1,2,...,n\}$ such that $c_j$ is dominated by $S \cap \{u_i, \overline{u_i}\}$. Assume that $u_i \in S$, and $c_j$ is dominated by $u_i$. By the construction of $G$ the literal $u_i$ is in the clause $C_j$. Then $t(u_i) = T$, which implies that the clause $C_j$ is satisfied by $t$. Next assume that $\overline{u_i} \in S$, and $c_j$ is dominated by $\overline{u_i}$. By the construction of $G$ the literal $\overline{u_i}$ is in the clause $C_j$. Then $t(u_i) = F$. Thus, $t$ assigns $\overline{u_i}$ the truth value $T$, that is, $t$ satisfies the clause $C_j$. Hence $\mathcal{C}$ is satisfiable.

Conversely, suppose that $\mathcal{C}$ is satisfiable. Let $t : U \longrightarrow \{T,F\}$ be a satisfying truth assignment for $\mathcal{C}$. We construct a paired dominating set $S$ for $G$ of cardinality $2n+2$. For this purpose, we construct a subset $D$ of vertices of $G$ as follows. If $t(u_i) = T$, then we put the vertices $u_i$ and $b_i$ in $D$; if $t(u_i) = F$, then put the vertices $\overline{u_i}$ and $d_i$ in $D$. Clearly, $|D| = 2n$. Since $t$ is a satisfying truth assignment for $\mathcal{C}$, for each $j = 1,2,...,m$, at least one of literals in $C_j$ is true under the assignment $t$. It follows that the corresponding vertex $c_j$ in $G$ is adjacent to at least one vertex in $D$, since $c_j$ is adjacent to each literal in $C_j$. Thus $S = D \cup \{s_3,s_4\}$ is a paired dominating set of $G$ of cardinality $2n+2$, and so $\gamma_{pr}(G) \leq 2n+2$. By Lemma 3.1, $\gamma_{pr}(G) = 2n+2$. □

**Lemma 3.3.** *For any non-pendant edge $e \in E(G)$, $\gamma_{pr}(G-e) \leq 2n+4$.*

*Proof.* Let $e \in E(G)$ be a non-pendant edge. Assume that $e \notin E(H_i)$. If $e \notin \{s_2c_i : i = 1,2,...,m\}$, then $\{u_i, b_i : i = 1,2,...,n\} \cup \{s_1,s_2,s_3,s_4\}$ is a paired dominating set for $G-e$ of cardinality $2n+4$, and thus $\gamma_{pr}(G-e) \leq 2n+4$. Thus assume that $e = s_2c_i$, for some $i \in \{1,2,...,m\}$. There is an integer $j \in \{1,2,...,n\}$ such that $N(c_i) \cap \{u_j, \overline{u_j}\} \neq \emptyset$. Without loss of generality, assume that $u_j \in N(c_i)$. Then $\{u_i, b_i : i = 1,2,...,n, i \neq j\} \cup \{s_3,s_4\} \cup \{c_i, u_j, d_j, \overline{u_j}\}$ is a paired dominating set for $G-e$ of cardinality $2n+4$, and thus $\gamma_{pr}(G-e) \leq 2n+4$. Next assume that $e \in E(H_i)$ for some $i \in \{1,2,...,n\}$. If $e \in \{e_iw_i, e_i\overline{u_i}, d_i\overline{u_i}, d_iw_i, d_ia_i\}$, then $\{u_j, b_j : j = 1,2,...,n\} \cup \{s_1,s_2,s_3,s_4\}$ is a paired dominating set for $G-e$ of cardinality $2n+4$. If $e \in \{b_ia_i, e_iu_i\}$, then $\{\overline{u_j}, d_j : j = 1,2,...,n\} \cup \{s_1,s_2,s_3,s_4\}$ is a paired dominating set for $G-e$ of cardinality $2n+4$. If $e = u_id_i$, then $\{\overline{u_j}, b_j : j = 1,2,...,n\} \cup \{s_1,s_2,s_3,s_4\}$ is a paired dominating set for $G-e$ of cardinality $2n+4$. If $e = b_i\overline{u_i}$, then similarly $\gamma_{pr}(G-e) \leq 2n+4$. □

**Lemma 3.4.** $\gamma_{pr}(G) = 2n+2$ *if and only if* $b_p(G) = 1$.

*Proof.* Assume $\gamma_{pr}(G) = 2n+2$. Let $D$ be a $\gamma_{pr}(G-e)$-set, where $e = s_3s_4$. Since $s_2$ and $s_3$ are a support vertices in $G-e$, we have $s_2, s_3 \in D$, and so $|D \cap \{s_1,s_2,s_3,s_4,s_5\}| \geq 3$. Since $|D \cap V(H_i)| \geq 2$, for $i = 1,2,...,n$, we deduce that $|D| > 2n+2$, and thus $b_p(G) = 1$. Conversely, assume that $b_p(G) = 1$. Let $e$ be a non-pendant edge such that $\gamma_{pr}(G-e) > \gamma_{pr}(G)$. By Lemma 3.3, we have that $\gamma_{pr}(G-e) \leq 2n+4$. Since $\gamma_{pr}(G) \geq 2n+2$, and $\gamma_{pr}(G)$ is even, we conclude that $\gamma_{pr}(G) = 2n+2$. □

From Lemmas 3.2, 3.3 and 3.4, it follows that $b_p(G) \leq 1$ if and only if $\mathcal{C}$ is satisfiable. Since the construction of the paired bondage instance is straightforward from a 3-SAT instance, the size of the paired bondage instance is bounded from above by a polynomial function of the size of the 3-SAT instance. It follows that this is a polynomial transformation, and the proof is complete.

## 3.2 Proof of Theorem 2.2

We construct a bipartite graph $G$ and an integer $\kappa$ such that $\mathcal{C}$ is satisfiable if and only if $b_{tr}(G) \leq \kappa$. Corresponding to each variable $u_i \in U$, we associate a graph $G_i$ obtained from the graph $H_i$ shown in Figure 1 by adding a vertex $f_i$ and joining $f_i$ to $a_i$. Figure 2 shows the graph $G_i$.

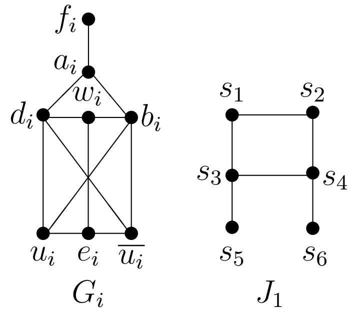


Figure 2. The graphs $G_i$ and $J_1$.

Corresponding to each clause $C_j = \{x_j, y_j, z_j\} \in \mathcal{C}$, we associate a single vertex $c_j$ and add the edge-set $E_j = \{c_jx_j, c_jy_j, c_jz_j\}$. Next we add the graph $J_1$ shown in the Figure 2, and join $s_1$ and $s_2$ to each vertex $c_j$ with $1 \leq j \leq m$. Set $\kappa = 1$. Let $S$ be a $\gamma_{tr}(G)$-set. For $i = 1, 2, ..., n$, clearly $S$ contains $f_i$ and $a_i$. Since $e_i$ is dominated by $S$, we find that $|S \cap V(G_i)| \geq 4$, for $i = 1, 2, ..., n$. Since $S$ contains $s_3$, $s_4$, $s_5$ and $s_6$, we obtain that $|S \cap V(J)| \geq 4$. Thus we have the following.

**Lemma 3.5.** $\gamma_{tr}(G) = |S| \geq 4n+4$.

**Lemma 3.6.** $\gamma_{tr}(G) = 4n+4$ *if and only if* $\mathcal{C}$ *is satisfiable.*

*Proof.* Assume that $\gamma_{tr}(G) = 4n+4$. Then $|S \cap V(G_i)| = 4$ for $i = 1, 2, ..., n$, $S \cap V(J) = \{s_3, s_4, s_5, s_6\}$ and $S \cap \{s_1, s_2\} = S \cap \{c_1, ..., c_m\} = \emptyset$. If $S \cap \{u_i, \overline{u_i}\} = \emptyset$ for some integer $i$, then $\{e_i, w_i\} \subseteq S$, since $e_i$ and $w_i$ are dominated by $S$. Then we replace $w_i$ by $b_i$. Thus $S \cap \{u_i, \overline{u_i}\} \neq \emptyset$ for each $i = 1, 2, ..., n$. If $\{u_i, \overline{u_i}\} \subseteq S$ for some $i$, then $|S \cap V(G_i)| \geq 5$, a contradiction. Thus $|\{u_i, \overline{u_i}\} \cap S| = 1$ for $i = 1, 2, ..., n$. Let $t : U \longrightarrow \{T, F\}$ be a mapping defined by $t(u_i) = T$ if $u_i \in S$ and $t(u_i) = F$ if $\overline{u_i} \in S$. For each $j \in \{1, 2, .., m\}$, the corresponding vertex $c_j$ in

$G$ is not dominated by $\{s_3, s_4, s_5, s_6\}$, and thus there is an integer $i \in \{1, 2, ..., n\}$ such that $c_j$ is dominated by $S \cap \{u_i, \overline{u_i}\}$. Assume that $u_i \in S$, and $c_j$ is dominated by $u_i$. By the construction of $G$ the literal $u_i$ is in the clause $C_j$. Then $t(u_i) = T$, which implies that the clause $C_j$ is satisfied by $t$. Next assume that $\overline{u_i} \in S$, and $c_j$ is dominated by $\overline{u_i}$. Then by the construction of $G$ the literal $\overline{u_i}$ is in the clause $C_j$. Then $t(u_i) = F$. Thus, $t$ assigns $\overline{u_i}$ the truth value $T$, that is, $t$ satisfies the clause $C_j$. Hence $\mathcal{C}$ is satisfiable.

Conversely, suppose that $\mathcal{C}$ is satisfiable. Let $t : U \longrightarrow \{T, F\}$ be a satisfying truth assignment for $\mathcal{C}$. We construct a total restrained dominating set $S$ for $G$ of cardinality $4n + 4$. For this purpose, we construct a subset $D$ of vertices of $G$ as follows. If $t(u_i) = T$, then we put the vertices $u_i$, $e_i$, $a_i$ and $f_i$ in $D$; if $t(u_i) = F$, then put the vertices $\overline{u_i}$, $e_i$, $a_i$ and $f_i$ in $D$. Clearly, $|D| = 4n$. Since $t$ is a satisfying truth assignment for $\mathcal{C}$, for each $j = 1, 2, ..., m$, at least one of literals in $C_j$ is true under the assignment $t$. It follows that the corresponding vertex $c_j$ in $G$ is adjacent to at least one vertex in $D$, since $c_j$ is adjacent to each literal in $C_j$. Thus $S = D \cup \{s_3, s_4, s_5, s_6\}$ is a total restrained dominating set of $G$ of cardinality $4n+4$, and so $\gamma_{tr}(G) \leq 4n+4$. By Lemma 3.5, $\gamma_{tr}(G) = 4n + 4$. □

**Lemma 3.7.** *For any non-pendant edge $e \in E(G)$, $\gamma_{tr}(G - e) \leq 4n + 5$.*

*Proof.* Let $e \in E(G)$ be a non-pendant edge. If $e = s_3s_4, s_1s_2$ or $s_2s_4$, then $\{s_1, s_3, s_4, s_5, s_6\} \cup \{u_i, e_i, a_i, f_i : i = 1, 2, ..., n\}$ is a total restrained dominating set for $G-e$ of cardinality $4n+5$, and thus $\gamma_{tr}(G-e) \leq 4n+5$. If $e = s_1s_3$, $e = s_1c_i$ for some $i \in \{1, 2, ..., m\}$, or $e = c_iu_j$ or $e = c_i\overline{u_j}$, for some $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., n\}$, then $\{s_2, s_3, s_4, s_5, s_6\} \cup \{u_i, e_i, a_i, f_i : i = 1, 2, ..., n\}$ is a total restrained dominating set for $G - e$ of cardinality $4n + 5$, and thus $\gamma_{tr}(G - e) \leq 4n + 5$. Similarly if $e = s_2c_i$ for some $i \in \{1, 2, ..., m\}$, then $\gamma_{tr}(G - e) \leq 4n + 5$. Thus assume that $e \in E(G_i)$ for some $i \in \{1, 2, ..., n\}$. If $e \in \{a_ib_i, a_id_i, w_ib_i, w_id_i, b_iu_i, b_i\overline{u_i}, d_iu_i, d_i\overline{u_i}\}$, then $\{s_2, s_3, s_4, s_5, s_6\} \cup \{u_j, e_j, a_j, f_j : j = 1, 2, ..., n\}$ is a total restrained dominating set for $G - e$ of cardinality $4n + 5$, and thus $\gamma_{tr}(G - e) \leq 4n + 5$. If $e = e_iw_i$, then $\{a_i, f_i, d_i, u_i\} \cup \{s_2, s_3, s_4, s_5, s_6\} \cup \{u_j, e_j, a_j, f_j : j = 1, 2, ..., n, j \neq i\}$ is a total restrained dominating set for $G-e$ of cardinality $4n+5$, and thus $\gamma_{tr}(G-e) \leq 4n+5$. If $e = u_ie_i$, then $\{a_i, f_i, \overline{u_i}, b_i\} \cup \{s_1, s_2, s_3, s_4, s_6, s_6\} \cup \{u_j, e_j, a_j, f_j : j = 1, 2, ..., n, j \neq i\}$ is a total restrained dominating set for $G - e$ of cardinality $4n + 5$, and thus $\gamma_{tr}(G - e) \leq 4n + 5$. □

**Lemma 3.8.** $\gamma_{tr}(G) = 4n + 4$ *if and only if* $b_{tr}(G) = 1$.

*Proof.* Assume $\gamma_{tr}(G) = 4n + 4$. Let $D$ be a $\gamma_{tr}(G - e)$-set, where $e = s_1s_3$. Clearly $\{s_3, s_4, s_5, s_6\} \subseteq S$. Since $s_1$ is dominated by $D$, we obtain that $(N[s_1]-\{s_3\}) \cap D \neq \emptyset$. Since $|D \cap V(G_i)| \geq 4$, for $i = 1, 2, ..., n$, we deduce that $|D| > 4n + 4$, and thus $b_{tr}(G) = 1$. Conversely, assume that $b_{tr}(G) = 1$. Let $e$ be an edge such that $\gamma_{tr}(G - e) > \gamma_{tr}(G)$. By Lemma 3.7, we have that $\gamma_{tr}(G - e) \leq 4n + 5$. Since $\gamma_{tr}(G) \geq 4n + 4$, we conclude that $\gamma_{tr}(G) = 4n + 4$. □

From Lemmas 3.6, 3.7 and 3.8, it follows that $b_{tr}(G) \leq 1$ if and only if $\mathcal{C}$ is satisfiable. Since the construction of the total restrained bondage instance is straightforward from a 3-SAT instance, the size of the total restrained bondage instance is bounded from above by a polynomial function of the size of the 3- SAT instance. It follows that this is a polynomial transformation, as desired.

### 3.3 Proof of Theorem 2.3

We construct a bipartite graph $G$ and an integer $\kappa$ such that $\mathcal{C}$ is satisfiable if and only if $b_i(G) \leq \kappa$. For $i = 1, 2, ..., n$, corresponding to each variable $u_i \in U$, associate a 6-cycle $H_i : u_i v_i \overline{u_i} a_i b_i d_i u_i$. Corresponding to each clause $C_j = \{x_j, y_j, z_j\} \in \mathcal{C}$, associate a single vertex $c_j$ and add the edge-set $E_j = \{c_j x_j, c_j y_j, c_j z_j\}$ for $j = 1, 2, ..., m$. Finally add a path $P_3 : s_1 s_2 s_3$, and join $s_1$ and $s_3$ to each vertex $c_j$ with $1 \leq j \leq m$ to obtain a bipartite graph $G$. Set $\kappa = 1$. Let $S$ be an $i(G)$-set. Clearly $|S \cap V(H_i)| \geq 2$ for $i = 1, 2, ..., n$. Also $S \cap \{s_1, s_2, s_3\} \neq \emptyset$. Thus we have the following.

**Lemma 3.9.** $|S| = i(G) \geq 2n + 1$.

**Lemma 3.10.** *$i(G) = 2n + 1$ if and only if $\mathcal{C}$ is satisfiable.*

*Proof.* Assume that $i(G) = 2n + 1$. Then $|S \cap V(H_i)| = 2$ for $i = 1, 2, ..., n$, $s_2 \in S$, and $S \cap \{s_1, s_3\} = S \cap \{c_1, ..., c_m\} = \emptyset$. If $\{u_i, \overline{u_i}\} \subseteq S$ for some $i$, then $b_i$ is not dominated by $S$, a contradiction. Thus $|S \cap \{u_i, \overline{u_i}\}| \leq 1$. If $S \cap \{u_i, \overline{u_i}\} = \emptyset$ for some $i$, then we can replace $S \cap V(H_i)$ by $\{u_i, a_i\}$. Thus we may assume that $|S \cap \{u_i, \overline{u_i}\}| = 1$ for $i = 1, 2, ..., n$. Let $t : U \longrightarrow \{T, F\}$ be a mapping defined by $t(u_i) = T$ if $u_i \in S$ and $t(u_i) = F$ if $\overline{u_i} \in S$. For each $j \in \{1, 2, .., m\}$, the corresponding vertex $c_j$ in $G$ is not dominated by $\{s_2\}$, and thus there is an integer $i \in \{1, 2, ..., n\}$ such that $c_j$ is dominated by $S \cap \{u_i, \overline{u_i}\}$. Assume that $u_i \in S$, and $c_j$ is dominated by $u_i$. By the construction of $G$ the literal $u_i$ is in the clause $C_j$. Then $t(u_i) = T$, which implies that the clause $C_j$ is satisfied by $t$. Next assume that $\overline{u_i} \in S$, and $c_j$ is dominated by $\overline{u_i}$. By the construction of $G$ the literal $\overline{u_i}$ is in the clause $C_j$. Then $t(u_i) = F$. Thus, $t$ assigns $\overline{u_i}$ the truth value $T$, that is, $t$ satisfies the clause $C_j$. Hence $\mathcal{C}$ is satisfiable. Conversely, suppose that $\mathcal{C}$ is satisfiable. Let $t : U \longrightarrow \{T, F\}$ be a satisfying truth assignment for $\mathcal{C}$. We construct an independent dominating set $S$ for $G$ of cardinality $2n + 1$. For this purpose, we construct a subset $D$ of vertices of $G$ as follows. If $t(u_i) = T$, then we put $u_i$ and $a_i$ in $D$; if $t(u_i) = F$, then put $\overline{u_i}$ and $d_i$ in $D$. Clearly, $|D| = 2n$. Since $t$ is a satisfying truth assignment for $\mathcal{C}$, for each $j = 1, 2, ..., m$, at least one of literals in $C_j$ is true under the assignment $t$. It follows that the corresponding vertex $c_j$ in $G$ is adjacent to at least one vertex in $D$, since $c_j$ is adjacent to each literal in $C_j$. Thus $S = D \cup \{s_2\}$ is an independent dominating set of $G$ of cardinality $2n + 1$, and so $i(G) \leq 2n + 1$. By Lemma 3.9, $i(G) = 2n + 1$. □

The proofs of the following lemmas are straightforward, and we omit them.

**Lemma 3.11.** *For any edge* $e \in E(G)$, $i(G - e) \leq 2n + 2$.

**Lemma 3.12.** $i(G) = 2n + 1$ *if and only if* $b_i(G) = 1$.

From Lemmas 3.10, 3.11 and 3.12 it follows that $b_i(G) \leq 1$ if and only if $\mathcal{C}$ is satisfiable. Since the construction of the independent bondage instance is straightforward from a 3-SAT instance, the size of the independent bondage instance is bounded from above by a polynomial function of the size of the 3-SAT instance. It follows that this is a polynomial transformation, as desired.

### 3.4 Proof of Theorem 2.4

We construct a bipartite graph $G$ and an integer $\kappa$ such that $\mathcal{C}$ is satisfiable if and only if $b_{rk}(G) \leq \kappa$. Corresponding to each variable $u_i \in U$, we associate a graph $H_i$ with $V(H_i) = \{u_i, \overline{u_i}, b_i, d_i\} \cup \{c_{ij}, e_{ij} : j = 1, 2, ..., k+1\}$ and $E(H_i) = \{u_i d_i, \overline{u_i} b_i\} \cup \{c_{ij}e_{ij}, c_{ij}d_i, c_{ij}b_i, e_{ij}u_i, e_{ij}\overline{u_i} : j = 1, 2, ..., k+1\}$. Figure 3 shows the graph $H_i$ for $k = 2$. Corresponding to each clause $C_j = \{x_j, y_j, z_j\} \in \mathcal{C}$, associate a single vertex $c_j$ and add the edge-set $E_j = \{c_j x_j, c_j y_j, c_j z_j\}$. Finally, add a star $K_{1,k}$ with central vertex $s$ and leaves $s_1, ..., s_k$, and join $s_1$ to each vertex $c_j$ with $1 \leq j \leq m$, and set $\kappa = 1$.

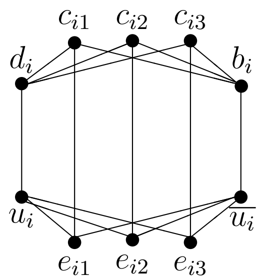


Figure 3. The graph $H_i$ for $k = 2$.

Let $f$ be a $\gamma_{rk}(G)$-function. It is straightforward to see that $\sum_{v \in V(H_i)} |f(v)| \geq 2k$ for $i = 1, 2, ..., n$. Since $|f(s)| + \sum_{j=1}^{k} |f(s_i)| + \sum_{j=1}^{m} |f(c_j)| \geq k$, we obtain that $\gamma_{rk}(G) = w(f) \geq 2kn + k$. Thus we obtain the following.

**Lemma 3.13.** $\gamma_{rk}(G) = w(f) \geq 2kn + k$.

**Lemma 3.14.** $\gamma_{rk}(G) = 2kn + k$ *if and only if* $\mathcal{C}$ *is satisfiable*

*Proof.* Assume that $\gamma_{rk}(G) = 2kn + k$. Let $g$ be a $\gamma_{rk}(G)$-function. Clearly $\sum_{v \in V(H_i)} |g(v)| \geq 2k$ for $i = 1, 2, ..., n$. Also $|g(c_i)| = 0$ for $i = 1, 2, ..., m$. If $|g(s_1)| = k$, then $s_2$ is not $k$-rainbow dominated by $g$, a contradiction. Thus $|g(s_1)| < k$. This implies that for each $j \in \{1, 2, .., m\}$, there is an integer $i \in \{1, 2, ..., n\}$ such that $c_j$ is dominated by $V^g_{12...k} \cap \{u_i, \overline{u_i}\}$. Assume that $u_i \in V^g_{12...k}$, and $c_j$ is dominated by $u_i$. By the construction of $G$ the literal $u_i$ is in the clause $C_j$. Then $t(u_i) = T$, which implies that the clause $C_j$ is satisfied by $t$. Next assume that $\overline{u_i} \in V^g_{12...k}$, and

$c_j$ is dominated by $\overline{u_i}$. By the construction of $G$ the literal $\overline{u_i}$ is in the clause $C_j$. Then $t(u_i) = F$. Thus, $t$ assigns $\overline{u_i}$ the truth value $T$, that is, $t$ satisfies the clause $C_j$. Hence $\mathcal{C}$ is satisfiable.

Conversely, assume that $\mathcal{C}$ is satisfiable. Let $t : U \longrightarrow \{T, F\}$ be a satisfying truth assignment for $\mathcal{C}$. We construct a subset $D$ of vertices of $G$ as follows. If $t(u_i) = T$, then we put the vertices $u_i$ and $b_i$ in $D$; if $t(u_i) = F$, then put the vertices $\overline{u_i}$ and $d_i$ in $D$. Clearly, $|D| = 2n$. Now $f$ defined on $V(G)$ by $f(u) = \{1, 2, ..., k\}$ if $u \in D$, $f(s) = \{1, 2, ..., k\}$, $f(u) = \emptyset$ otherwise, is a $k$RDF of weight $2kn + k$, and thus $\gamma_{rk}(G) \leq 2kn + k$. By Lemma 3.13, $\gamma_{rk}(G) = 2kn + k$. □

The following can be easily proved.

**Lemma 3.15.** *For any edge $e \in E(G)$, $\gamma_{rk}(G - e) \leq 2n + k + 1$.*

**Lemma 3.16.** *$\gamma_{rk}(G) = 2kn + k$ if and only if $b_{rk}(G) = 1$.*

*Proof.* Assume that $\gamma_{rk}(G) = 2kn + k$. Let $h$ be a $\gamma_{rk}(G - ss_2)$-function. Then $\sum_{v\in V(H_i)} |h(v)| \geq 2k$ for $i = 1, 2, ..., n$, and $|h(s)|+\sum_{i=1}^{k} |h(s_i)| \geq k+1$. Consequently $b_{rk}(G) = 1$. Conversely assume that $b_{kr}(G) = 1$. Let $e$ be an edge such that $\gamma_{rk}(G - e) > \gamma_{rk}(G)$. It is a routine matter to see that $\gamma_{rk}(G - e) \leq 2kn + k + 1$. Thus $2kn+k+1 \geq \gamma_{rk}(G-e) > \gamma_{rk}(G) \geq 2kn+k$ implying that $\gamma_{rk}(G) = 2kn+k$. □

Thus, from Lemmas 3.14, 3.15 and 3.16 it follows that $b_{rk}(G) \leq 1$ if and only if $\mathcal{C}$ is satisfiable. Since the construction of the $k$-rainbow bondage instance is straightforward from a 3-SAT instance, the size of the $k$-rainbow bondage instance is bounded from above by a polynomial function of the size of the 3-SAT instance. It follows that this is a polynomial transformation, as desired.

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