AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **68(2)** (2017), Pages 265–275

On the complexity of some bondage problems in graphs

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Abstract

The paired bondage number (total restrained bondage number, independent bondage number, k-rainbow bondage number) of a graph G, is the minimum number of edges whose removal from G results in a graph with larger paired domination number (respectively, total restrained domination number, independent domination number, k-rainbow domination number). In this paper we show that the decision problems for these variants are NP-hard, even when restricted to bipartite graphs.

1 Introduction

Let G be a graph with vertex set $V(G) = V$ of order $|V| = n$ and size $|E(G)| = m$, and let v be a vertex in V. The *open neighborhood* of v is $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. If the graph G is clear from the context, we simply write $N(v)$ rather than $N_G(v)$. The *degree* of a vertex v, is $\deg(v) = |N(v)|$. A vertex of degree one is called a *leaf* and its neighbor a *support vertex*. A *pendant edge* is an edge that one of its endpoints is a leaf. We denote the set of leaves and support vertices of a graph G by $L(G)$ and $S(G)$, respectively. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v\in S} N(v)$, and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. A *matching* in a graph G is a set of independent edges in G. A *perfect matching* M in G is a matching such that every vertex of G is incident to an element of M. For a subset S of vertices of G we refer to $G[S]$ as the subgraph of G induced by S. For notation and graph theory terminology, we in general follow [8, 10].

A subset $S \subseteq V$ is a *dominating set* of G if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set S in a graph with no isolated vertex is a *total dominating set* if the induced subgraph $G[S]$ has no isolated vertex. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. A dominating set S in a graph G with no isolated vertex is called a *paired dominating set* if the induced subgraph $G[S]$ contains a perfect matching. The *paired domination number* of G, denoted by $\gamma_{pr}(G)$, is the minimum cardinality of a paired dominating set of G. A dominating set S is called an *independent dominating set* if the induced graph $G[S]$ has no edge. The *independent domination number* of G, denoted by $i(G)$, is the minimum cardinality of an independent dominating set of G. A total dominating set S is called a *total restrained dominating set* if every vertex of $V - S$ is adjacent to another vertex in $V - S$. The *total restrained domination number* of G, denoted by $\gamma_{tr}(G)$, is the minimum cardinality of a total restrained dominating set of G. A dominating set S is called an $\gamma(G)$-set of G if $|S| = \gamma(G)$. Similarly a $\gamma_t(G)$-set, an $i(G)$-set, a $\gamma_{pr}(G)$-set, and a $\gamma_{tr}(G)$-set are defined. For references on domination and total domination in graphs see for example [8, 10].

For a graph G and an integer $k \geq 2$, let $f : V(G) \to \mathcal{P}(\{1, 2, ..., k\})$ be a function. If for each vertex $v \in V(G)$ such that $f(v) = \emptyset$ we have $\cup_{u \in N(v)} f(u) = \{1, 2, ..., k\}$, then f is called a *k-rainbow dominating function* (or simply kRDF) of G. The *weight*, $w(f)$, of f is defined as $w(f) = \sum_{v \in V(G)} |f(v)|$. The minimum weight of a kRDF of G is called the *k-rainbow domination number* of G, and is denoted by $\gamma_{rk}(G)$. If f is a kRDF of G, then we denote by $V^f_{12...k}$ the set of all vertices u with $|f(u)| = k$. We refer to a γ_{rk}-function in a graph G as a kRDF with minimum weight. If f is a kRDF of G, then we say that a vertex v is not k-rainbow dominated by f if $f(v) = \emptyset$ and $\cup_{u \in N(v)} f(u) \neq \{1, 2, ..., k\}$. For references in rainbow domination see for example [2, 3, 17].

The *bondage number* of G, denoted by $b(G)$, is the minimum number of edges whose removal from G results in a graph with larger domination number. The concept of bondage in graphs was introduced by Bauer, Harary, Nieminen and Suffel in [1], and has been further studied for example in [4, 5, 16]. Raczek [15] introduced the concept of paired bondage in graphs. The *paired bondage number* $b_{pr}(G)$ of a graph G with no isolated vertex is the cardinality of a smallest set of edges $E' \subseteq E(G)$ for which (1) $G - E'$ has no isolated vertex, and (2) $\gamma_{pr}(G - E') > \gamma_{pr}(G)$. Zhang, Liu and Sun [19] defined the *independent bondage number* $b_i(G)$ of G to be the minimum cardinality among all subsets $E' \subseteq E(G)$ for which $i(G - E') > i(G)$. The *total restrained bondage number* $b_{tr}(G)$ of a graph G with no isolated vertex is the cardinality of a smallest set of edges $E' \subseteq E(G)$ for which (1) $G - E'$ has no isolated vertex, and (2) $\gamma_{tr}(G - E') > \gamma_{tr}(G)$. The concept of total restrained bondage is studied in [13]. The *k-rainbow bondage number* $b_{rk}(G)$ of a graph G with maximum degree at least two is the minimum cardinality among all sets $E' \subseteq E(G)$ for which $\gamma_{rk}(G - E') > \gamma_{rk}(G)$. For a survey of results and recent developments on bondage we refer the reader to [18].

The complexity issue of several parameters in the theory of domination have been studied, see for example [6, 8]. The decision problem for some bondage problems has been proven to be NP-hard, see for example [7, 11, 12, 14, 18].

Conjecture 1.1 (Xu, [18])**.** *The paired bondage problem is NP-complete.*

In this paper, we consider the complexity issue for paired bondage problem, total restrained bondage problem, independent bondage problem, and k-rainbow bondage problem. We prove that the decision problem for these bondage problems is NP-hard, even when restricted to bipartite graphs. Our proofs are by a transformation from the 3-satisfiability problem (known as 3-SAT problem) that we describe it as follows. A *truth assignment* for a set U of Boolean variables is a mapping $t : U \to \{T, F\}$. A variable u is said to be *true* (or *false*) under t if $t(u) = T$ (or $t(u) = F$). If u is a variable in U, then u and $\overline{u}$ are *literals* over U. The literal u is true under t if and only if the variable u is true under t, and the literal $\overline{u}$ is true if and only if the variable u is false. A *clause* over U is a set of literals over U, and it is *satisfied* by a truth assignment if and only if at least one of its members is true under that assignment. A collection $\mathcal{C}$ of clauses over U is satisfiable if and only if there exists some truth assignment for U that simultaneously satisfies all the clauses in $\mathcal{C}$. Such a truth assignment is called a satisfying truth assignment for $\mathcal{C}$. The 3-SAT problem is specified as follows.

3-SAT problem:
Instance: A collection $\mathcal{C} = \{C_1, C_2, ..., C_m\}$ of clauses over a finite set U of variables such that $|C_j| = 3$ for $j = 1, 2, ..., m$.
Question: Is there a truth assignment for U that satisfies all the clauses in C?

Note that the 3-SAT problem was proven to be NP-complete in [6].

2 Main results

Consider the following decision problems.

Paired bondage problem (**PB**):
Instance: A graph G with no isolated vertex and a positive integer κ.
Question: Is $b_p(G) \leq \kappa$?

Total restrained bondage problem (**TRB**):
Instance: A graph G with no isolated vertex and a positive integer κ.
Question: Is $b_{tr}(G) \leq \kappa$?

Independent bondage problem (**IB**):
Instance: A nonempty graph G and a positive integer κ.

Question: Is $b_i(G) \leq \kappa$?

k-rainbow bondage problem (k**RB**):
Instance: A nonempty graph G and a positive integer κ.
Question: Is $b_{rk}(G) \leq \kappa$?

We will prove the following.

Theorem 2.1. **PB** *is NP-hard for bipartite graphs.*

Theorem 2.2. **TRB** *is NP-hard for general graphs.*

Theorem 2.3. **IB** *is NP-hard for bipartite graphs.*

Theorem 2.4. k**RB** *is NP-hard for bipartite graphs.*

3 Proofs

We show the NP-hardness of each problem by transforming the 3-SAT problem to it in polynomial time. Let $U = \{u_1, u_2, ..., u_n\}$ and $\mathcal{C} = \{C_1, C_2, ..., C_m\}$ be an arbitrary instance of the 3-SAT.

3.1 Proof of Theorem 2.1

We construct a bipartite graph G and an integer κ such that $\mathcal{C}$ is satisfiable if and only if $b_p(G) \leq \kappa$. Corresponding to each variable $u_i \in U$, we associate the graph H_i shown in Figure 1. Corresponding to each clause $C_j = \{x_j, y_j, z_j\} \in \mathcal{C}$, we associate a single vertex c_j and add the edge-set $E_j = \{c_jx_j, c_jy_j, c_jz_j\}$. Next add the graph J shown in Figure 1, and join s_2 to each vertex c_j with $1 \leq j \leq m$, to obtain a bipartite graph G. Set $\kappa = 1$.

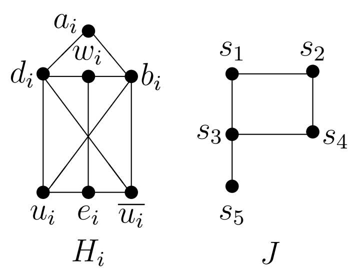

Figure 1. The graphs H_i and J.

Let S be a $\gamma_{pr}(G)$-set. Clearly $|S \cap V(H_i)| \geq 2$. Moreover, $|S \cap \{s_1, s_2, s_3, s_4, s_5\}| \geq 2$. Thus we have the following.

Lemma 3.1. $\gamma_{pr}(G) = |S| \geq 2n + 2$.

Lemma 3.2. $\gamma_{pr}(G) = 2n + 2$ *if and only if* $\mathcal{C}$ *is satisfiable.*

Proof. Assume that $\gamma_{pr}(G) = 2n+2$. Then $|S \cap V(H_i)| = 2$ for $i = 1, 2, ..., n$, $|S \cap \{s_1, s_2, s_3, s_4, s_5\}| = 2$ and $S \cap \{c_1, ..., c_m\} = \emptyset$. Clearly $s_3 \in S$, and thus we may assume, without loss of generality, that $S \cap \{s_1, s_2, s_3, s_4, s_5\} = \{s_3, s_4\}$. Moreover, S does not contain both u_i and $\overline{u_i}$ for $i = 1, 2, ..., n$. If $S \cap \{u_j, \overline{u_j}\} = \emptyset$ for some j, then we replace the vertices of $S \cap V(H_j)$ by u_i and b_i. We thus may assume that $|S \cap \{u_i, \overline{u_i}\}| = 1$ for $i = 1, 2, ..., n$. Let $t : U \longrightarrow \{T, F\}$ be a mapping defined by $t(u_i) = T$ if $u_i \in S$ and $t(u_i) = F$ if $\overline{u_i} \in S$. For each $j \in \{1, 2, .., m\}$, the corresponding vertex c_j in G is not dominated by s_3 or s_4, and thus there is an integer $i \in \{1, 2, ..., n\}$ such that c_j is dominated by $S \cap \{u_i, \overline{u_i}\}$. Assume that $u_i \in S$, and c_j is dominated by u_i. By the construction of G the literal u_i is in the clause C_j. Then $t(u_i) = T$, which implies that the clause C_j is satisfied by t. Next assume that $\overline{u_i} \in S$, and c_j is dominated by $\overline{u_i}$. By the construction of G the literal $\overline{u_i}$ is in the clause C_j. Then $t(u_i) = F$. Thus, t assigns $\overline{u_i}$ the truth value T, that is, t satisfies the clause C_j. Hence $\mathcal{C}$ is satisfiable.

Conversely, suppose that $\mathcal{C}$ is satisfiable. Let $t : U \longrightarrow \{T, F\}$ be a satisfying truth assignment for $\mathcal{C}$. We construct a paired dominating set S for G of cardinality $2n+2$. For this purpose, we construct a subset D of vertices of G as follows. If $t(u_i) = T$, then we put the vertices u_i and b_i in D; if $t(u_i) = F$, then put the vertices $\overline{u_i}$ and d_i in D. Clearly, $|D| = 2n$. Since t is a satisfying truth assignment for $\mathcal{C}$, for each $j = 1, 2, ..., m$, at least one of literals in C_j is true under the assignment t. It follows that the corresponding vertex c_j in G is adjacent to at least one vertex in D, since c_j is adjacent to each literal in C_j. Thus $S = D \cup \{s_3, s_4\}$ is a paired dominating set of G of cardinality $2n+2$, and so $\gamma_{pr}(G) \leq 2n+2$. By Lemma 3.1, $\gamma_{pr}(G) = 2n+2$. □

Lemma 3.3. *For any non-pendant edge $e \in E(G)$, $\gamma_{pr}(G-e) \leq 2n+4$.*

Proof. Let $e \in E(G)$ be a non-pendant edge. Assume that $e \notin E(H_i)$. If $e \notin \{s_2c_i : i = 1, 2, ..., m\}$, then $\{u_i, b_i : i = 1, 2, ..., n\} \cup \{s_1, s_2, s_3, s_4\}$ is a paired dominating set for $G-e$ of cardinality $2n+4$, and thus $\gamma_{pr}(G-e) \leq 2n+4$. Thus assume that $e = s_2c_i$, for some $i \in \{1, 2, ..., m\}$. There is an integer $j \in \{1, 2, ..., n\}$ such that $N(c_i) \cap \{u_j, \overline{u_j}\} \neq \emptyset$. Without loss of generality, assume that $u_j \in N(c_i)$. Then $\{u_i, b_i : i = 1, 2, ..., n, i \neq j\} \cup \{s_3, s_4\} \cup \{c_i, u_j, d_j, \overline{u_j}\}$ is a paired dominating set for $G-e$ of cardinality $2n+4$, and thus $\gamma_{pr}(G-e) \leq 2n+4$. Next assume that $e \in E(H_i)$ for some $i \in \{1, 2, ..., n\}$. If $e \in \{e_iw_i, e_i\overline{u_i}, d_i\overline{u_i}, d_iw_i, d_ia_i\}$, then $\{u_j, b_j : j = 1, 2, ..., n\} \cup \{s_1, s_2, s_3, s_4\}$ is a paired dominating set for $G-e$ of cardinality $2n+4$. If $e \in \{b_ia_i, e_iu_i\}$, then $\{\overline{u_j}, d_j : j = 1, 2, ..., n\} \cup \{s_1, s_2, s_3, s_4\}$ is a paired dominating set for $G-e$ of cardinality $2n+4$. If $e = u_id_i$, then $\{\overline{u_j}, b_j : j = 1, 2, ..., n\} \cup \{s_1, s_2, s_3, s_4\}$ is a paired dominating set for $G-e$ of cardinality $2n+4$. If $e = b_i\overline{u_i}$, then similarly $\gamma_{pr}(G-e) \leq 2n+4$. □

Lemma 3.4. *$\gamma_{pr}(G) = 2n+2$ if and only if $b_p(G) = 1$.*

Proof. Assume $\gamma_{pr}(G) = 2n+2$. Let D be a $\gamma_{pr}(G-e)$-set, where $e = s_3s_4$. Since s_2 and s_3 are a support vertices in $G-e$, we have $s_2, s_3 \in D$, and so $|D \cap \{s_1, s_2, s_3, s_4, s_5\}| \geq 3$. Since $|D \cap V(H_i)| \geq 2$, for $i = 1, 2, ..., n$, we deduce that $|D| > 2n+2$, and thus $b_p(G) = 1$. Conversely, assume that $b_p(G) = 1$. Let e be a non-pendant edge such that $\gamma_{pr}(G-e) > \gamma_{pr}(G)$. By Lemma 3.3, we have that $\gamma_{pr}(G-e) \leq 2n+4$. Since $\gamma_{pr}(G) \geq 2n+2$, and $\gamma_{pr}(G)$ is even, we conclude that $\gamma_{pr}(G) = 2n+2$. $\square$

From Lemmas 3.2, 3.3 and 3.4, it follows that $b_p(G) \leq 1$ if and only if $\mathcal{C}$ is satisfiable. Since the construction of the paired bondage instance is straightforward from a 3-SAT instance, the size of the paired bondage instance is bounded from above by a polynomial function of the size of the 3-SAT instance. It follows that this is a polynomial transformation, and the proof is complete.

3.2 Proof of Theorem 2.2

We construct a bipartite graph G and an integer κ such that $\mathcal{C}$ is satisfiable if and only if $b_{tr}(G) \leq \kappa$. Corresponding to each variable $u_i \in U$, we associate a graph G_i obtained from the graph H_i shown in Figure 1 by adding a vertex f_i and joining f_i to a_i. Figure 2 shows the graph G_i.

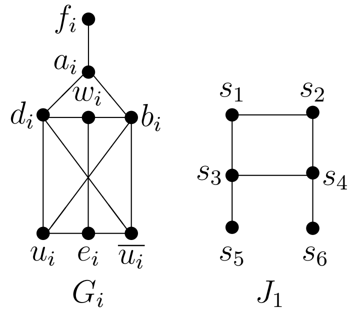

Figure 2. The graphs G_i and J_1.

Corresponding to each clause $C_j = \{x_j, y_j, z_j\} \in \mathcal{C}$, we associate a single vertex c_j and add the edge-set $E_j = \{c_jx_j, c_jy_j, c_jz_j\}$. Next we add the graph J_1 shown in the Figure 2, and join s_1 and s_2 to each vertex c_j with $1 \leq j \leq m$. Set $\kappa = 1$. Let S be a $\gamma_{tr}(G)$-set. For $i = 1, 2, ..., n$, clearly S contains f_i and a_i. Since e_i is dominated by S, we find that $|S \cap V(G_i)| \geq 4$, for $i = 1, 2, ..., n$. Since S contains s_3, s_4, s_5 and s_6, we obtain that $|S \cap V(J)| \geq 4$. Thus we have the following.

Lemma 3.5. $\gamma_{tr}(G) = |S| \geq 4n+4$.

Lemma 3.6. $\gamma_{tr}(G) = 4n+4$ *if and only if* $\mathcal{C}$ *is satisfiable.*

Proof. Assume that $\gamma_{tr}(G) = 4n+4$. Then $|S \cap V(G_i)| = 4$ for $i = 1, 2, ..., n$, $S \cap V(J) = \{s_3, s_4, s_5, s_6\}$ and $S \cap \{s_1, s_2\} = S \cap \{c_1, ..., c_m\} = \emptyset$. If $S \cap \{u_i, \overline{u_i}\} = \emptyset$ for some integer i, then $\{e_i, w_i\} \subseteq S$, since e_i and w_i are dominated by S. Then we replace w_i by b_i. Thus $S \cap \{u_i, \overline{u_i}\} \neq \emptyset$ for each $i = 1, 2, ..., n$. If $\{u_i, \overline{u_i}\} \subseteq S$ for some i, then $|S \cap V(G_i)| \geq 5$, a contradiction. Thus $|\{u_i, \overline{u_i}\} \cap S| = 1$ for $i = 1, 2, ..., n$. Let $t : U \longrightarrow \{T, F\}$ be a mapping defined by $t(u_i) = T$ if $u_i \in S$ and $t(u_i) = F$ if $\overline{u_i} \in S$. For each $j \in \{1, 2, .., m\}$, the corresponding vertex c_j in

G is not dominated by $\{s_3, s_4, s_5, s_6\}$, and thus there is an integer $i \in \{1, 2, ..., n\}$ such that c_j is dominated by $S \cap \{u_i, \overline{u_i}\}$. Assume that $u_i \in S$, and c_j is dominated by u_i. By the construction of G the literal u_i is in the clause C_j. Then $t(u_i) = T$, which implies that the clause C_j is satisfied by t. Next assume that $\overline{u_i} \in S$, and c_j is dominated by $\overline{u_i}$. Then by the construction of G the literal $\overline{u_i}$ is in the clause C_j. Then $t(u_i) = F$. Thus, t assigns $\overline{u_i}$ the truth value T, that is, t satisfies the clause C_j. Hence $\mathcal{C}$ is satisfiable.

Conversely, suppose that $\mathcal{C}$ is satisfiable. Let $t : U \longrightarrow \{T, F\}$ be a satisfying truth assignment for $\mathcal{C}$. We construct a total restrained dominating set S for G of cardinality $4n + 4$. For this purpose, we construct a subset D of vertices of G as follows. If $t(u_i) = T$, then we put the vertices u_i, e_i, a_i and f_i in D; if $t(u_i) = F$, then put the vertices $\overline{u_i}$, e_i, a_i and f_i in D. Clearly, $|D| = 4n$. Since t is a satisfying truth assignment for $\mathcal{C}$, for each $j = 1, 2, ..., m$, at least one of literals in C_j is true under the assignment t. It follows that the corresponding vertex c_j in G is adjacent to at least one vertex in D, since c_j is adjacent to each literal in C_j. Thus $S = D \cup \{s_3, s_4, s_5, s_6\}$ is a total restrained dominating set of G of cardinality $4n+4$, and so $\gamma_{tr}(G) \leq 4n+4$. By Lemma 3.5, $\gamma_{tr}(G) = 4n + 4$. □

Lemma 3.7. *For any non-pendant edge $e \in E(G)$, $\gamma_{tr}(G - e) \leq 4n + 5$.*

Proof. Let $e \in E(G)$ be a non-pendant edge. If $e = s_3s_4, s_1s_2$ or s_2s_4, then $\{s_1, s_3, s_4, s_5, s_6\} \cup \{u_i, e_i, a_i, f_i : i = 1, 2, ..., n\}$ is a total restrained dominating set for $G-e$ of cardinality $4n+5$, and thus $\gamma_{tr}(G-e) \leq 4n+5$. If $e = s_1s_3$, $e = s_1c_i$ for some $i \in \{1, 2, ..., m\}$, or $e = c_iu_j$ or $e = c_i\overline{u_j}$, for some $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., n\}$, then $\{s_2, s_3, s_4, s_5, s_6\} \cup \{u_i, e_i, a_i, f_i : i = 1, 2, ..., n\}$ is a total restrained dominating set for $G - e$ of cardinality $4n + 5$, and thus $\gamma_{tr}(G - e) \leq 4n + 5$. Similarly if $e = s_2c_i$ for some $i \in \{1, 2, ..., m\}$, then $\gamma_{tr}(G - e) \leq 4n + 5$. Thus assume that $e \in E(G_i)$ for some $i \in \{1, 2, ..., n\}$. If $e \in \{a_ib_i, a_id_i, w_ib_i, w_id_i, b_iu_i, b_i\overline{u_i}, d_iu_i, d_i\overline{u_i}\}$, then $\{s_2, s_3, s_4, s_5, s_6\} \cup \{u_j, e_j, a_j, f_j : j = 1, 2, ..., n\}$ is a total restrained dominating set for $G - e$ of cardinality $4n + 5$, and thus $\gamma_{tr}(G - e) \leq 4n + 5$. If $e = e_iw_i$, then $\{a_i, f_i, d_i, u_i\} \cup \{s_2, s_3, s_4, s_5, s_6\} \cup \{u_j, e_j, a_j, f_j : j = 1, 2, ..., n, j \neq i\}$ is a total restrained dominating set for $G-e$ of cardinality $4n+5$, and thus $\gamma_{tr}(G-e) \leq 4n+5$. If $e = u_ie_i$, then $\{a_i, f_i, \overline{u_i}, b_i\} \cup \{s_1, s_2, s_3, s_4, s_6, s_6\} \cup \{u_j, e_j, a_j, f_j : j = 1, 2, ..., n, j \neq i\}$ is a total restrained dominating set for $G - e$ of cardinality $4n + 5$, and thus $\gamma_{tr}(G - e) \leq 4n + 5$. □

Lemma 3.8. $\gamma_{tr}(G) = 4n + 4$ *if and only if* $b_{tr}(G) = 1$.

Proof. Assume $\gamma_{tr}(G) = 4n + 4$. Let D be a $\gamma_{tr}(G - e)$-set, where $e = s_1s_3$. Clearly $\{s_3, s_4, s_5, s_6\} \subseteq S$. Since s_1 is dominated by D, we obtain that $(N[s_1]-\{s_3\}) \cap D \neq \emptyset$. Since $|D \cap V(G_i)| \geq 4$, for $i = 1, 2, ..., n$, we deduce that $|D| > 4n + 4$, and thus $b_{tr}(G) = 1$. Conversely, assume that $b_{tr}(G) = 1$. Let e be an edge such that $\gamma_{tr}(G - e) > \gamma_{tr}(G)$. By Lemma 3.7, we have that $\gamma_{tr}(G - e) \leq 4n + 5$. Since $\gamma_{tr}(G) \geq 4n + 4$, we conclude that $\gamma_{tr}(G) = 4n + 4$. □

From Lemmas 3.6, 3.7 and 3.8, it follows that $b_{tr}(G) \leq 1$ if and only if $\mathcal{C}$ is satisfiable. Since the construction of the total restrained bondage instance is straightforward from a 3-SAT instance, the size of the total restrained bondage instance is bounded from above by a polynomial function of the size of the 3- SAT instance. It follows that this is a polynomial transformation, as desired.

3.3 Proof of Theorem 2.3

We construct a bipartite graph G and an integer κ such that $\mathcal{C}$ is satisfiable if and only if $b_i(G) \leq \kappa$. For $i = 1, 2, ..., n$, corresponding to each variable $u_i \in U$, associate a 6-cycle $H_i : u_i v_i \overline{u_i} a_i b_i d_i u_i$. Corresponding to each clause $C_j = \{x_j, y_j, z_j\} \in \mathcal{C}$, associate a single vertex c_j and add the edge-set $E_j = \{c_j x_j, c_j y_j, c_j z_j\}$ for $j = 1, 2, ..., m$. Finally add a path $P_3 : s_1 s_2 s_3$, and join s_1 and s_3 to each vertex c_j with $1 \leq j \leq m$ to obtain a bipartite graph G. Set $\kappa = 1$. Let S be an $i(G)$-set. Clearly $|S \cap V(H_i)| \geq 2$ for $i = 1, 2, ..., n$. Also $S \cap \{s_1, s_2, s_3\} \neq \emptyset$. Thus we have the following.

Lemma 3.9. $|S| = i(G) \geq 2n + 1$.

Lemma 3.10. *$i(G) = 2n + 1$ if and only if $\mathcal{C}$ is satisfiable.*

Proof. Assume that $i(G) = 2n + 1$. Then $|S \cap V(H_i)| = 2$ for $i = 1, 2, ..., n$, $s_2 \in S$, and $S \cap \{s_1, s_3\} = S \cap \{c_1, ..., c_m\} = \emptyset$. If $\{u_i, \overline{u_i}\} \subseteq S$ for some i, then b_i is not dominated by S, a contradiction. Thus $|S \cap \{u_i, \overline{u_i}\}| \leq 1$. If $S \cap \{u_i, \overline{u_i}\} = \emptyset$ for some i, then we can replace $S \cap V(H_i)$ by $\{u_i, a_i\}$. Thus we may assume that $|S \cap \{u_i, \overline{u_i}\}| = 1$ for $i = 1, 2, ..., n$. Let $t : U \longrightarrow \{T, F\}$ be a mapping defined by $t(u_i) = T$ if $u_i \in S$ and $t(u_i) = F$ if $\overline{u_i} \in S$. For each $j \in \{1, 2, .., m\}$, the corresponding vertex c_j in G is not dominated by $\{s_2\}$, and thus there is an integer $i \in \{1, 2, ..., n\}$ such that c_j is dominated by $S \cap \{u_i, \overline{u_i}\}$. Assume that $u_i \in S$, and c_j is dominated by u_i. By the construction of G the literal u_i is in the clause C_j. Then $t(u_i) = T$, which implies that the clause C_j is satisfied by t. Next assume that $\overline{u_i} \in S$, and c_j is dominated by $\overline{u_i}$. By the construction of G the literal $\overline{u_i}$ is in the clause C_j. Then $t(u_i) = F$. Thus, t assigns $\overline{u_i}$ the truth value T, that is, t satisfies the clause C_j. Hence $\mathcal{C}$ is satisfiable. Conversely, suppose that $\mathcal{C}$ is satisfiable. Let $t : U \longrightarrow \{T, F\}$ be a satisfying truth assignment for $\mathcal{C}$. We construct an independent dominating set S for G of cardinality $2n + 1$. For this purpose, we construct a subset D of vertices of G as follows. If $t(u_i) = T$, then we put u_i and a_i in D; if $t(u_i) = F$, then put $\overline{u_i}$ and d_i in D. Clearly, $|D| = 2n$. Since t is a satisfying truth assignment for $\mathcal{C}$, for each $j = 1, 2, ..., m$, at least one of literals in C_j is true under the assignment t. It follows that the corresponding vertex c_j in G is adjacent to at least one vertex in D, since c_j is adjacent to each literal in C_j. Thus $S = D \cup \{s_2\}$ is an independent dominating set of G of cardinality $2n + 1$, and so $i(G) \leq 2n + 1$. By Lemma 3.9, $i(G) = 2n + 1$. □

The proofs of the following lemmas are straightforward, and we omit them.

Lemma 3.11. *For any edge $e \in E(G)$, $i(G-e) \leq 2n+2$.*

Lemma 3.12. *$i(G) = 2n+1$ if and only if $b_i(G) = 1$.*

From Lemmas 3.10, 3.11 and 3.12 it follows that $b_i(G) \leq 1$ if and only if $\mathcal{C}$ is satisfiable. Since the construction of the independent bondage instance is straightforward from a 3-SAT instance, the size of the independent bondage instance is bounded from above by a polynomial function of the size of the 3-SAT instance. It follows that this is a polynomial transformation, as desired.

3.4 Proof of Theorem 2.4

We construct a bipartite graph G and an integer κ such that $\mathcal{C}$ is satisfiable if and only if $b_{rk}(G) \leq \kappa$. Corresponding to each variable $u_i \in U$, we associate a graph H_i with $V(H_i) = \{u_i, \overline{u_i}, b_i, d_i\} \cup \{c_{ij}, e_{ij} : j = 1, 2, ..., k+1\}$ and $E(H_i) = \{u_i d_i, \overline{u_i} b_i\} \cup \{c_{ij}e_{ij}, c_{ij}d_i, c_{ij}b_i, e_{ij}u_i, e_{ij}\overline{u_i} : j = 1, 2, ..., k+1\}$. Figure 3 shows the graph H_i for $k = 2$. Corresponding to each clause $C_j = \{x_j, y_j, z_j\} \in \mathcal{C}$, associate a single vertex c_j and add the edge-set $E_j = \{c_j x_j, c_j y_j, c_j z_j\}$. Finally, add a star $K_{1,k}$ with central vertex s and leaves $s_1, ..., s_k$, and join s_1 to each vertex c_j with $1 \leq j \leq m$, and set $\kappa = 1$.

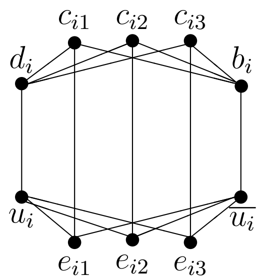

Figure 3. The graph H_i for $k = 2$.

Let f be a $\gamma_{rk}(G)$-function. It is straightforward to see that $\sum_{v \in V(H_i)} |f(v)| \geq 2k$ for $i = 1, 2, ..., n$. Since $|f(s)| + \sum_{j=1}^{k} |f(s_i)| + \sum_{j=1}^{m} |f(c_j)| \geq k$, we obtain that $\gamma_{rk}(G) = w(f) \geq 2kn + k$. Thus we obtain the following.

Lemma 3.13. $\gamma_{rk}(G) = w(f) \geq 2kn + k$.

Lemma 3.14. *$\gamma_{rk}(G) = 2kn + k$ if and only if $\mathcal{C}$ is satisfiable*

Proof. Assume that $\gamma_{rk}(G) = 2kn + k$. Let g be a $\gamma_{rk}(G)$-function. Clearly $\sum_{v \in V(H_i)} |g(v)| \geq 2k$ for $i = 1, 2, ..., n$. Also $|g(c_i)| = 0$ for $i = 1, 2, ..., m$. If $|g(s_1)| = k$, then s_2 is not k-rainbow dominated by g, a contradiction. Thus $|g(s_1)| < k$. This implies that for each $j \in \{1, 2, .., m\}$, there is an integer $i \in \{1, 2, ..., n\}$ such that c_j is dominated by $V^g_{12...k} \cap \{u_i, \overline{u_i}\}$. Assume that $u_i \in V^g_{12...k}$, and c_j is dominated by u_i. By the construction of G the literal u_i is in the clause C_j. Then $t(u_i) = T$, which implies that the clause C_j is satisfied by t. Next assume that $\overline{u_i} \in V^g_{12...k}$, and

c_j is dominated by $\overline{u_i}$. By the construction of G the literal $\overline{u_i}$ is in the clause C_j. Then $t(u_i) = F$. Thus, t assigns $\overline{u_i}$ the truth value T, that is, t satisfies the clause C_j. Hence $\mathcal{C}$ is satisfiable.

Conversely, assume that $\mathcal{C}$ is satisfiable. Let $t : U \longrightarrow \{T, F\}$ be a satisfying truth assignment for $\mathcal{C}$. We construct a subset D of vertices of G as follows. If $t(u_i) = T$, then we put the vertices u_i and b_i in D; if $t(u_i) = F$, then put the vertices $\overline{u_i}$ and d_i in D. Clearly, $|D| = 2n$. Now f defined on $V(G)$ by $f(u) = \{1, 2, ..., k\}$ if $u \in D$, $f(s) = \{1, 2, ..., k\}$, $f(u) = \emptyset$ otherwise, is a kRDF of weight $2kn + k$, and thus $\gamma_{rk}(G) \leq 2kn + k$. By Lemma 3.13, $\gamma_{rk}(G) = 2kn + k$. □

The following can be easily proved.

Lemma 3.15. *For any edge $e \in E(G)$, $\gamma_{rk}(G - e) \leq 2n + k + 1$.*

Lemma 3.16. *$\gamma_{rk}(G) = 2kn + k$ if and only if $b_{rk}(G) = 1$.*

Proof. Assume that $\gamma_{rk}(G) = 2kn + k$. Let h be a $\gamma_{rk}(G - ss_2)$-function. Then $\sum_{v \in V(H_i)} |h(v)| \geq 2k$ for $i = 1, 2, ..., n$, and $|h(s)| + \sum_{i=1}^{k} |h(s_i)| \geq k+1$. Consequently $b_{rk}(G) = 1$. Conversely assume that $b_{kr}(G) = 1$. Let e be an edge such that $\gamma_{rk}(G - e) > \gamma_{rk}(G)$. It is a routine matter to see that $\gamma_{rk}(G - e) \leq 2kn + k + 1$. Thus $2kn+k+1 \geq \gamma_{rk}(G-e) > \gamma_{rk}(G) \geq 2kn+k$ implying that $\gamma_{rk}(G) = 2kn+k$. □

Thus, from Lemmas 3.14, 3.15 and 3.16 it follows that $b_{rk}(G) \leq 1$ if and only if $\mathcal{C}$ is satisfiable. Since the construction of the k-rainbow bondage instance is straightforward from a 3-SAT instance, the size of the k-rainbow bondage instance is bounded from above by a polynomial function of the size of the 3-SAT instance. It follows that this is a polynomial transformation, as desired.

Acknowledgements

We would like to thank the referees for their careful review and some helpful comments.

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(Received 8 Sep 2016; revised 3 Apr 2017)