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Roman domination in complementary prisms

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Abstract

The complementary prism $G\overline{G}$ of a graph G is formed from the disjoint union of G and its complement $\overline{G}$ by adding the edges of a perfect matching between the corresponding vertices of G and $\overline{G}$. A Roman dominating function on a graph $G = (V, E)$ is a labeling $f : V(G) \mapsto \{0, 1, 2\}$ such that every vertex with label 0 is adjacent to a vertex with label 2. The Roman domination number $\gamma_R(G)$ of G is the minimum $f(V) = \Sigma_{v\in V} f(v)$ over all such functions of G. We study the Roman domination number of complementary prisms. Our main results show that $\gamma_R(G\overline{G})$ takes on a limited number of values in terms of the domination number of $G\overline{G}$ and the Roman domination numbers of G and $\overline{G}$.

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1 Introduction

Complementary products were introduced in [6] as a generalization of Cartesian products. Problems involving domination invariants of Cartesian products [9, 11] are among the most interesting and well-studied problems in graph theory. In this paper, we consider Roman domination in a sub-family of complementary products called complementary prisms.

For a graph $G = (V, E)$, the *complementary prism*, denoted $G\overline{G}$, is formed from the disjoint union of G and its complement $\overline{G}$ by adding a perfect matching between corresponding vertices of G and $\overline{G}$. For each $v \in V(G)$, let $\overline{v}$ denote the vertex corresponding to v in $\overline{G}$. Formally, the graph $G\overline{G}$ is formed from $G \cup \overline{G}$ by adding the edge $v\overline{v}$ for every $v \in V(G)$. We note that complementary prisms are a generalization of the Petersen graph. In particular, the Petersen graph is the complementary prism $C_5\overline{C}_5$. For another example of a complementary prism, consider the following. The *corona* of a graph G, denoted $G \circ K_1$, is formed from G by adding, for each $v \in V(G)$, a new vertex v' and the pendant edge vv'. Thus, the corona $K_n \circ K_1$ is the complementary prism $K_n\overline{K}_n$.

The hamiltonicity of complementary prisms is studied in [10], and domination parameters of complementary prisms have been studied in [5, 7] and elsewhere. As previously mentioned, our focus is on Roman domination in these graphs.

A *Roman dominating function (RDF)* on a graph G is a vertex labeling $f : V(G) \mapsto \{0, 1, 2\}$ such that every vertex with label 0 is adjacent to at least one vertex with label 2. For any Roman dominating function f of G, and $i \in \{0, 1, 2\}$, let $V_i = \{v \in V(G) \mid f(v) = i\}$. Since this partition determines f, we write $f = (V_0, V_1, V_2)$. The *weight* of a Roman dominating function f is defined as $w(f) = \Sigma_{v \in V} f(v)$, equivalently $w(f) = |V_1| + 2|V_2|$. The *Roman domination number* $\gamma_R(G)$ of G is the minimum weight of a Roman dominating function on the graph G. If a Roman dominating function of G has weight $\gamma_R(G)$, then it is referred to as a γ_R-function of G. Roman domination was introduced by Cockayne et al. [4] in 2004 and has received much attention in the literature, see for example [1, 2, 3, 8].

To aid in our discussion, we will need some more terminology. For a vertex $v \in V(G)$, the *open neighborhood* of v is $N(v) = \{u \in V(G) \mid uv \in E(G)\}$, and the *closed neighborhood* $N[v] = N(v) \cup \{v\}$. The degree of a vertex v is $deg_G(v) = |N(v)|$. A vertex of degree 0 is an *isolated vertex*. For two vertices u and v in a connected graph G, the *distance* $d_G(u, v)$ between u and v is the length of a shortest u–v path in G. The maximum distance among all pairs of vertices of G is its *diameter*, which is denoted by $\mathrm{diam}(G)$. We say that G is a diameter-k graph if $\mathrm{diam}(G) = k$. If G is disconnected, then $\mathrm{diam}(G) = \infty$. A set $S \subseteq V(G)$ is a *dominating set* of G if every vertex of $V(G) \setminus S$ is adjacent to a vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of any dominating set of G, and a dominating set of cardinality $\gamma(G)$ is called a γ-*set* of G.

We say that a vertex $v \in V_2$ Roman dominates the vertices in $N[v]$, and for a function $f = (V_0, V_1, V_2)$, we say that f Roman dominates the vertices in $V_1 \cup V_2$ as well as the vertices in V_0 that have a neighbor in V_2. We refer to the complementary prism $G\overline{G}$ as a copy of G and a copy of $\overline{G}$ with a perfect matching between corresponding vertices. For a set $P \subseteq V(G)$, let $\overline{P}$ denote the corresponding set of vertices in $V(\overline{G})$. We also shorten $V(G)$ to V and $V(\overline{G})$ to $\overline{V}$. Further, for any function f on $G\overline{G}$, we let $w(f_V)$ denote the weight of f on G, and $w(f_{\overline{V}})$ denote the weight of f on $\overline{G}$. We note that $G\overline{G}$ is isomorphic to $\overline{G}G$, so our results stated in terms of G also apply to $\overline{G}$ unless otherwise stated.

In this paper, we show that Roman domination numbers of complementary prisms $G\overline{G}$ take on a limited number of values in terms of the domination number of $G\overline{G}$ and the Roman domination numbers of G and $\overline{G}$. These values are summarized in Table 1 in Section 4. Among other results, we prove the lower bounds of Table 1 in Section 2 and the upper bounds in Section 3.

2 Small Values and Lower Bounds

Observe that $\gamma_R(G\overline{G}) \geq 2$ for any graph G. As examples, we determine the complementary prisms $G\overline{G}$ having small Roman domination numbers, namely, those with $\gamma_R(G\overline{G}) \in \{2, 3, 4\}$.

Theorem 2.1 *Let G be a graph of order n. Then*

1. $\gamma_R(G\overline{G}) = 2$ *if and only if* $G = K_1$.
2. $\gamma_R(G\overline{G}) = 3$ *if and only if* $G = K_2$ *or* $\overline{G} = K_2$.
3. $\gamma_R(G\overline{G}) = 4$ *if and only if* $\gamma_R(G) = 3$ *and G has an isolated vertex or* $\gamma_R(\overline{G}) = 3$ *and $\overline{G}$ has an isolated vertex.*

Proof. (1) If $G = K_1$, then $G\overline{G} = K_2$ and $\gamma_R(K_2) = 2$.

Assume that $\gamma_R(G\overline{G}) = 2$. Since a vertex in G (respectively, $\overline{G}$) can Roman dominate at most one vertex in $\overline{G}$ (respectively, G), it follows that any function of weight 2 can Roman dominate at most one vertex in G or at most one vertex in $\overline{G}$. Hence, $G = K_1$.

(2) If $G = K_2$, then $G\overline{G}$ is isomorphic to the path P_4 and $\gamma_R(P_4) = 3$.

Assume that $\gamma_R(G\overline{G}) = 3$. Then at most one vertex of $G\overline{G}$, say v, is assigned a 2 under any γ_R-function of $G\overline{G}$. It follows that $\overline{G} - \overline{v}$ must be Roman dominated with a weight of 1. Thus, $\overline{G} - \overline{v}$ consists of exactly one vertex, that is, $\overline{G}$, and hence, G has order 2. Thus, $\{G, \overline{G}\} = \{K_2, \overline{K}_2\}$.

(3) Without loss of generality, assume that $\gamma_R(G) = 3$ and G has an isolated vertex v. Then assigning a 2 to $\overline{v}$ Roman dominates $\overline{V} \cup \{v\}$. Further, since v is an isolate of G and $\gamma_R(G) = 3$, it follows that assigning a total weight of 2 on the vertices of $G - v$ yields an RDF of $G\overline{G}$. Thus, $\gamma_R(G\overline{G}) \leq 2 + 2 = 4$. Equality follows from (1) and (2).

Finally, assume that $\gamma_R(G\overline{G}) = 4$, let f be a γ_R-function of $G\overline{G}$. Note that (1) and (2) imply that G has order at least 3. If no vertex of $G\overline{G}$ is assigned 2, then the order of $G\overline{G}$ is 4, implying that $G = K_2$ or $\overline{G} = K_2$, a contradiction. Thus, we may assume, without loss of generality, that $f(v) = 2$. Moreover, if $w(f_{\overline{V}}) = 0$, then $\overline{G}$ has order at most 2, a contradiction. Hence, we have that $w(f_V) \geq 2$, $w(f_{\overline{V}}) \geq 1$, and $w(f_V) + w(f_{\overline{V}}) = 4$. Further, if $w(f_{\overline{V}}) = 1$, then $w(f_V) = 3$ implying that at most two vertices of $\overline{G}$ are Roman dominated by f, a contradiction since $\overline{G}$ has order at least 3. Hence, it must be the case that $w(f_V) = w(f_{\overline{V}}) = 2$. If two vertices of $\overline{G}$ are labeled 1, then v dominates G implying that $\overline{v}$ is an isolate in $\overline{G}$ and G has order exactly 3. It follows that $\overline{v}$ is assigned 0 under f. Assigning 1 to every vertex of $\overline{G}$ gives a RDF of $\overline{G}$, and so, $\gamma_R(\overline{G}) = 3$ and $\overline{G}$ has an isolated vertex. Hence, we may assume that there is a vertex $\overline{u} \in \overline{G}$ for which $f(\overline{u}) = 2$. Thus, v Roman dominates $G - u$ and $\overline{u}$ Roman dominates $\overline{G} - \overline{v}$. Now u and v are adjacent in G or $\overline{u}$ and $\overline{v}$ are adjacent in $\overline{G}$. Hence, either u is an isolate in G or $\overline{v}$ is an isolate in $\overline{G}$. Without loss of generality, let u be an isolate in G. As before, $\gamma_R(G) = 3$, and the result holds. □

Corollary 2.1 *If G and its complement $\overline{G}$ are isolate-free graphs, then $\gamma_R(G\overline{G}) \geq 5$.*

For our next example, we determine the Roman domination number of the complementary prism of a complete graph K_n.

Proposition 2.1 *If $G = K_n$, then $\gamma_R(G\overline{G}) = n + 1$.*

Proof. Let v be a vertex in G. First note that the function assigning 2 to v, 1 to each vertex in $\overline{V} \setminus \{\overline{v}\}$, and 0 otherwise is an RDF of $G\overline{G}$. Hence, $\gamma_R(G\overline{G}) \leq n + 1$.

To see that $\gamma_R(G\overline{G}) \geq n + 1$, let f be a γ_R-function of $G\overline{G}$. Note that for every vertex $\overline{v} \in \overline{V}$, either $f(\overline{v}) \geq 1$ or $f(v) = 2$, implying that $\gamma_R(G\overline{G}) \geq n$. Further note that if f has weight n, then every vertex of V is assigned 0 under f and every vertex of $\overline{V}$ is assigned 1. But then the vertices of G are not Roman dominated by f, a contradiction. Hence, $\gamma_R(G\overline{G}) \geq n + 1$. □

Notice that from Proposition 2.1,

$$\gamma_R(G\overline{G}) = n + 1 = \gamma_R(\overline{G}) + 1 = \max\{\gamma_{\mathrm{R}}(\mathrm{G}), \gamma_{\mathrm{R}}(\overline{\mathrm{G}})\} + 1$$

for $G = K_n$. Next we show that for any graph G, $\gamma_R(G\overline{G}) \geq \max\{\gamma_{\mathrm{R}}(\mathrm{G}), \gamma_{\mathrm{R}}(\overline{\mathrm{G}})\} + 1$.

Theorem 2.2 *For any graph G of order n, $\gamma_R(G\overline{G}) \geq \max\{\gamma_{\mathrm{R}}(\mathrm{G}), \gamma_{\mathrm{R}}(\overline{\mathrm{G}})\} + 1$ with equality if and only if G or $\overline{G}$ has an isolated vertex.*

Proof. If $\{G, \overline{G}\} = \{K_n, \overline{K}_n\}$, then $G\overline{G}$ is the corona $K_n \circ K_1$. By Proposition 2.1, $\gamma_R(G\overline{G}) = n + 1 = \max\{\gamma_{\mathrm{R}}(\mathrm{G}), \gamma_{\mathrm{R}}(\overline{\mathrm{G}})\} + 1$. Hence, we may assume that neither G nor $\overline{G}$ is complete (empty), and so, G has order at least three. If either G or $\overline{G}$ is a

P_3, then it is a simple exercise to see that $\gamma_R(G\overline{G}) = \max\{\gamma_R(G), \gamma_R(\overline{G})\} + 1$. Hence we may assume that $n \geq 4$.

Without loss of generality, assume that $\gamma_R(G) \geq \gamma_R(\overline{G})$. Let f be a γ_R-function of $G\overline{G}$.

If $f(\overline{v}) = 0$ for every $\overline{v} \in \overline{V}$, then we note that $f(v) = 2$ for all $v \in V$. Let uv be an edge in G. Then the function $f'(u) = 0$ and $f'(w) = f(w) = 2$ for every $w \in V \setminus \{u\}$ is an RDF of G such that $w(f) \geq w(f') + 2 \geq \gamma_R(G) + 2$.

Thus, we may assume that at least one vertex of $\overline{V}$ has a nonzero weight under f. If no vertex of $\overline{V}$ is assigned a 2, then G is Roman dominated by the vertices of G, implying that $w(f) \geq \gamma_R(G) + 1$. If some vertex of $\overline{V}$ is assigned a 2, then the function $f'(v) = 1$ if $f(\overline{v}) = 2$ and $f'(u) = f(u)$ otherwise, is a RDF of G such that $w(f) \geq w(f') + 1 \geq \gamma_R(G) + 1$.

Now, by the above explanation, $w(f) = \gamma_R(G) + 1$ if and only if $f(\overline{v}) = 2$ and $f(\overline{u}) = 0$ for all $\overline{u} \in \overline{V} \setminus \{\overline{v}\}$, or $f(\overline{v}) = 1$ and $f(\overline{u}) = 0$ for all $\overline{u} \in \overline{V} \setminus \{\overline{v}\}$. First suppose that $f(\overline{v}) = 2$ and $f(\overline{u}) = 0$ for all $\overline{u} \in \overline{V} \setminus \{\overline{v}\}$, and so, $f(v) = 0$. If v has a neighbor with weight 2 in G, then $\gamma_R(G) \leq w(f) - 2$, a contradiction. If there exists a vertex u adjacent to v with $f(u) = 0$ or $f(u) = 1$, then $f(\overline{u}) = 0$ and $\overline{u}$ is not adjacent to a vertex with weight 2, a contradiction. Thus, v is an isolated vertex in G. Second, suppose that $f(\overline{v}) = 1$ and $f(\overline{u}) = 0$ for all $\overline{u} \in \overline{V} \setminus \{\overline{v}\}$. In this case, $f(u) = 2$ for every $u \neq v$ in V. Recall that $n \geq 4$. If v is not an isolated vertex in G, then there exists a vertex w adjacent to v, and the function $f'(w) = 2 = f(w)$, $f'(v) = f(v)$ and $f'(u) = 1$ for any $u \neq w, v$ is an RDF of G such that $w(f) \geq w(f') + 2 \geq \gamma_R(G) + 2$. This proves the necessity.

For the sufficiency, assume without loss of generality that G has an isolated vertex v. Let f be a γ_R-function of G. Note that $f(v) = 1$. Moreover, the function assigning 0 to v, $f(u)$ to every $u \in V \setminus \{v\}$, 2 to $\overline{v}$, and 0 to every other vertex in $\overline{V}$ is a RDF of $G\overline{G}$ having weight $\gamma_R(G) - 1 + 2 = \gamma_R(G) + 1$. Hence, $\gamma_R(G\overline{G}) = \gamma_R(G) + 1$. □

Corollary 2.2 *If neither graph G nor its complement $\overline{G}$ has an isolated vertex, then* $\gamma_R(G\overline{G}) \geq \max\{\gamma_R(G), \gamma_R(\overline{G})\} + 2$.

Next we show that the bound of Corollary 2.2 is sharp for graphs with minimum degree one.

Theorem 2.3 *If neither graph G nor its complement $\overline{G}$ has an isolated vertex and one of them has a vertex of degree one, then* $\gamma_R(G\overline{G}) = \max\{\gamma_R(G), \gamma_R(\overline{G})\} + 2$.

Proof. Let G and $\overline{G}$ be isolate-free graphs. Assume that v is a vertex of degree one in G and that u is the neighbor of v. Two possible scenarios can occur. For the first case, there exists a γ_R-function of G such that $f(v) = 0$ and $f(u) = 2$. In this case, f can be extended to an RDF of $G\overline{G}$ by assigning 2 to $\overline{v}$ and 0 to every other vertex of $\overline{G}$. In the second case, every $\gamma_R(G)$-function f assigns $f(v) = 1$. In this case, f can be extended to an RDF of $G\overline{G}$ by reassigning 0 to v and assigning 2 to $\overline{v}$, 1 to

$\overline{u}$, and 0 to every remaining vertex in $\overline{V}$. In either case, $\gamma_R(G\overline{G}) \leq \gamma_R(G) + 2$, and by Corollary 2.2, $\gamma_R(G\overline{G}) = \gamma_R(G) + 2$. $\square$

As a corollary to Theorem 2.3, we obtain the exact value for the Roman domination number of the complementary prisms of paths. We use the following result from [4].

Proposition 2.2 [4] *For paths P_n, $\gamma_R(P_n) = \lceil 2n/3 \rceil$.*

Corollary 2.3 *For paths $G = P_n$ where $n \geq 3$, $\gamma_R(G\overline{G}) = \left\lceil \frac{2n}{3} \right\rceil + 2$.*

Proof. For $G = P_3$, $\gamma_R(G\overline{G}) = 4 = \gamma_R(P_3) + 2$, and so the result holds. Thus, we may assume that $n \geq 4$. Since neither P_n nor $\overline{P}_n$ for $n \geq 4$ has an isolated vertex, by Theorem 2.3 and Proposition 2.2, it follows that $\gamma_R(G\overline{G}) = \gamma_R(G) + 2 = \left\lceil \frac{2n}{3} \right\rceil + 2$. $\square$

We note that the converse of Theorem 2.3 is not necessarily true. For example, let $G = C_5$. Then $G\overline{G}$ is the Petersen graph and $\gamma_R(G\overline{G}) = 6 = \gamma_R(G) + 2$.

3 Upper Bounds

We begin with some results involving general graphs G. Note that assigning a weight of 2 to every vertex of a γ-set S of G and a weight of 0 to the vertices in $V \setminus S$ is an RDF of G. This useful observation was first made in [4] as follows.

Observation 3.1 [4] *For any graph G, $\gamma_R(G) \leq 2\gamma(G)$.*

In [4], a graph G is called *Roman* if $\gamma_R(G) = 2\gamma(G)$. We say that a graph G is *almost Roman* if $\gamma_R(G) = 2\gamma(G) - 1$. Using the following results from [4], we observe that every diameter-2 graph is either Roman or almost Roman.

Proposition 3.1 [4] *For any graph G with no isolated vertices, there exists a γ_R-function $f = (V_0, V_1, V_2)$ of G such that if $V_1 \neq \emptyset$, then V_1 is a 2-packing.*

Theorem 3.1 [4] *For any non-trivial connected graph G, $\gamma_R(G) = \min\{2\gamma(G - S) + |S| : S$ is a 2-packing$\}$.*

Note that if $\text{diam}(G) = 2$, then any maximal 2-packing of G contains exactly one vertex. Thus, for diameter-2 graphs, if S is the set in Theorem 3.1, then either $S = \emptyset$ or $|S| = 1$. Since removing a vertex can decrease the domination number of any graph by at most one, we have the following corollaries to Theorem 3.1.

Corollary 3.1 *If diam$(G) = 2$, then $\gamma_R(G) \in \{2\gamma(G), 2\gamma(G) - 1\}$.*

Corollary 3.2 *If G is a graph of diameter 2, then $\gamma_R(G) = 2\gamma(G) - 1$ if and only if G has a vertex v such that $\gamma(G - v) = \gamma(G) - 1$.*

Now turning our attention back to complementary prisms, we consider the following result from [6].

Theorem 3.2 [6] *For the complementary prism $G\overline{G}$, if diam$(G) =$ diam$(\overline{G}) = 2$ then diam$(G\overline{G}) = 2$, else diam$(G\overline{G}) = 3$.*

Corollary 3.1 and Theorem 3.2 now yield the following corollary.

Corollary 3.3 *For any graph G, if diam$(G) =$ diam$(\overline{G}) = 2$, then $\gamma_R(G\overline{G}) \in \{2\gamma(G\overline{G}), 2\gamma(G\overline{G}) - 1\}$.*

In other words, if diam$(G\overline{G}) = 2$, then $G\overline{G}$ is Roman or almost Roman. Now we consider complementary prisms with diameter 3. Clearly, an RDF of G combined with an RDF of $\overline{G}$ forms an RDF of $G\overline{G}$, so we make the following straightforward observation.

Observation 3.2 *For any graph G, $\gamma_R(G\overline{G}) \leq \gamma_R(G) + \gamma_R(\overline{G})$.*

Theorem 3.3 *Let G be a graph with diam$(G) \geq 3$ such that neither G nor $\overline{G}$ has an isolated vertex. Then $\gamma_R(G) + 2 \leq \gamma_R(G\overline{G}) \leq \gamma_R(G) + 4$.*

Proof. The lower bound follows directly from Theorem 2.2. For the upper bound, let u and v be peripheral vertices of G such that the distance between u and v equals diam$(G) \geq 3$. Since $\{\overline{u}, \overline{v}\}$ dominates $\overline{G}$, it follows that $\gamma(\overline{G}) \leq 2$. Observations 3.1 and 3.2 imply that $\gamma_R(G\overline{G}) \leq \gamma_R(G) + \gamma_R(\overline{G}) \leq \gamma_R(G) + 4$. $\square$

We note that the upper bound of Theorem 3.3 is tight. To see this we consider a family of strong product graphs.The *strong product* $G \boxtimes H$ of two graphs G and H has vertex set $V(G) \times V(H)$ and any two distinct vertices (u, u') and (v, v') are adjacent in $G \boxtimes H$ if and only if one of the following holds: $uv \in E(G)$ and $u' = v'$, or $u = v$ and $u'v' \in E(H)$, or $uv \in E(G)$ and $u'v' \in E(H)$. For $k \geq 2$, let $G_k = C_{3k} \boxtimes K_2$. For ease of discussion, we label the vertices of one copy of C_{3k} as u_i for $1 \leq i \leq 3k$ and the vertices of the other copy of C_{3k} as v_i for $1 \leq i \leq 3k$. The graph G_2 is illustrated in Figure 1. In our next result, we show that the complementary prisms $G_k\overline{G}_k$ are extremal graphs for the upper bound of Theorem 3.3.

Proposition 3.2 *For the graph G_k with $k \geq 2$, $\gamma_R(G_k\overline{G}_k) = \gamma_R(G_k) + 4$.*

Proof. Let $G_k = C_{3k} \boxtimes K_2$ with the vertex set described above. Let $A = \{u_i \mid i \equiv 2 \,(\mathrm{mod}\, 3)\}$. A function assigning a label of 2 to each vertex in A and 0 to each vertex of $V(G_k) \setminus A$ is an RDF of G_k. Hence, $\gamma_R(G_k) \leq 2k$. Any RDF of G_k that

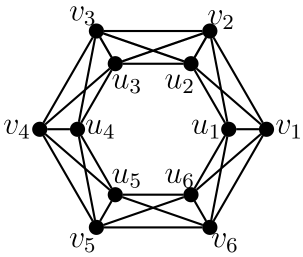

Figure 1: The graph G_k when $k = 2$.

assigns a value of 2 to $s < k$ vertices of G_k, must of necessity assign a value of 1 to at least $6(k - s)$ vertices of G_k. Thus, any such function f will have a weight $w(f) = 2s + 6(k - s) = 6k - 4s > 2k$. Hence, $\gamma_R(G_k) = 2k$. Note that $\{u_2, u_5\}$ is a dominating set for $\overline{G}_k$. Therefore, $\gamma_R(\overline{G}_k) \leq 4$. Any RDF of $\overline{G}_k$ that assigns no 2 will have a weight of $n = 6k$ and if it labels exactly one vertex with a 2, it will have a weight of at least 7. Thus, $\gamma_R(G_k) \geq \gamma_R(\overline{G}_k) = 4$.

We note that by Observation 3.2, $\gamma_R(G_k\overline{G}_k) \leq \gamma_R(G_k) + \gamma_R(\overline{G}_k) = 2k + 4$. Let f be a γ_R-function of $G_k\overline{G}_k$. We aim to show that $w(f) \geq \gamma_R(G_k) + \gamma_R(\overline{G}_k) = 2k + 4$. If f assigns a value of 2 to $s < k$ vertices of $V(G_k)$, then it must either assign a value of 1 to at least $6(k - s)$ vertices of $V(G_k)$ or a value of 2 to their counterparts in $V(\overline{G}_k)$. In either case, $w(f) \geq 2s + 6(k - s) = 6k - 4s \geq 2k + 4$. If f assigns a value of 2 to at least $k + 2$ vertices of $V(G_k\overline{G}_k)$, then $w(f) \geq 2k + 4$. If f assigns a value of 2 to exactly $k + 1$ vertices of $V(G_k)$, then in order to Roman dominate the $6k - (k + 1) = 5k - 1$ vertices of $V(\overline{G}_k)$ not Roman dominated by the vertices of $V(G_k)$, it will also be necessary for $w(f) \geq 2k + 2 + 5k - 1 > 2k + 4$. Thus, we may assume that exactly k vertices of $V(G_k)$ are assigned a label of 2 by f. If f does not assign a label of 2 to any vertex of $V(\overline{G}_k)$, then $w(f) \geq 2k + 5k = 7k > 2k + 4$. Hence, we may assume that f assigns a value of 2 to exactly one vertex of $V(\overline{G}_k)$ (if not, then $w(f) \geq 2k + 4$ and we would be finished). Without loss of generality, assume that $f(\overline{u}_1) = 2$.

Let S be the set of k vertices of $V(G_k)$ assigned a label of 2 by f. Now $\overline{V} \setminus N[\overline{u}_1] = \{\overline{u}_2, \overline{u}_{3k}, \overline{v}_2, \overline{v}_{3k}, \overline{v}_1\}$. Moreover, if more than one of these vertices is assigned a 1 under f, then we have the desired result. This implies that at least four of these vertices are dominated by vertices in S. Hence, at least four of the vertices of $\{u_2, u_{3k}, v_2, v_{3k}, v_1\}$ are in S. But then the k vertices of S do not dominate all the vertices $V(G_k) \setminus S$, a contradiction. It follows that $\gamma_R(G_k\overline{G}_k) = \gamma_R(G_k) + 4$. □

As we have seen, the complementary prisms of paths attain the lower bound of Theorem 3.3. Next we determine two additional families of complementary prisms attaining this lower bound. Note that since $\gamma_R(G) \leq 2\gamma(G)$, it follows that if a graph G is neither Roman nor almost Roman, then $\gamma_R(G) \leq 2\gamma(G) - 2$. Also, we have the following from [4].

Proposition 3.3 [4] *A graph G is Roman if and only if it has a γ_R-function $f = (V_0, V_1, V_2)$ where $V_1 = \emptyset$.*

Theorem 3.4 *If G is a graph that is neither Roman nor almost Roman and diam$(G) \geq 3$, then $\gamma_R(G\overline{G}) \leq \gamma_R(G) + 2 \leq 2\gamma(G)$.*

Proof. Select a γ_R-function $f = (V_0, V_1, V_2)$ of G such that $|V_2|$ is maximized. By Proposition 3.1 and Theorem 3.1, $V_1 = \emptyset$ or V_1 is a 2-packing of G. Since G is not Roman, it follows from Proposition 3.3 that $V_1 \neq \emptyset$.

Assume that $|V_1| = 1$, and let $V_1 = \{v\}$. In this case, $\gamma_R(G) = 2|V_2| + 1$ and V_2 dominates $V \setminus \{v\}$. Since $\gamma_R(G) \leq 2\gamma(G)$, it follows that $2|V_2| + 1 \leq 2\gamma(G)$. Hence, $|V_2| \leq \lfloor \gamma(G) - 1/2 \rfloor = \gamma(G) - 1$. If $|V_2| \leq \gamma(G) - 2$, then $V_2 \cup \{v\}$ is a dominating set of G with cardinality at most $\gamma(G) - 1$, a contradiction. Hence, $|V_2| = \gamma(G) - 1$ and $\gamma_R(G) = 2|V_2| + |V_1| = 2\gamma(G) - 1$, contrary to our assumption that G is not an almost Roman graph.

Thus, we may assume that $|V_1| \geq 2$. Since, V_1 is a 2-packing, there exists vertices u and v in V_1 such that $d(u, v) \geq 3$. Define the function f^* on $G\overline{G}$ as follows. If $x \in V \setminus \{u, v\}$, let $f^*(x) = f(x)$. Let $f^*(u) = f^*(v) = 0$ and $f^*(\overline{u}) = f^*(\overline{v}) = 2$. For all $\overline{x} \in \overline{V} \setminus \{\overline{u}, \overline{v}\}$, let $f^*(\overline{x}) = 0$. We note that $\{\overline{u}, \overline{v}\}$, dominates $\overline{V}$. Thus, f^* is an RDF of $G\overline{G}$, implying that $\gamma_R(G\overline{G}) \leq w(f) = \gamma_R(G) - 2 + 4 = \gamma_R(G) + 2$. Furthermore, since G is not Roman or almost Roman, $\gamma_R(G) \leq 2\gamma(G) - 2$ and the result follows. □

Theorem 2.2 and Theorem 3.4 yield the following corollary.

Corollary 3.4 *Let G be a graph such that both G and $\overline{G}$ are isolate-free. If G is neither Roman nor almost Roman and diam$(G) \geq 3$, then $\gamma_R(G\overline{G}) = \gamma_R(G) + 2$.*

We need the following definition before proceeding. A set $S \subseteq V(G)$ is a *restrained dominating set* if S is a dominating set of G and every vertex $v \in V(G) \setminus S$ has a neighbor in $V(G) \setminus S$. The minimum cardinality of a restrained dominating set of G is called the *restrained domination number* of G and is denoted by $\gamma_r(G)$ (not to be confused with $\gamma_R(G)$).

Theorem 3.5 *If G is a Roman graph such that $\gamma_r(G) > \gamma(G)$ and $\overline{G}$ has no isolated vertices, then $\gamma_R(G\overline{G}) = \gamma_R(G) + 2$.*

Proof. Let S be a γ-set of G. Since $\gamma_r(G) \neq \gamma(G)$, it follows that there exists a vertex $v \in V \setminus S$ such that $N(v) \subseteq S$. Let f be a function $f : V(G\overline{G}) \mapsto \{0, 1, 2\}$ such that $f(u) = 2$ if $u \in S \cup \{\overline{v}\}$ and $f(u) = 0$ otherwise. The function f is an RDF on $G\overline{G}$ with weight $2|S| + 2 = 2\gamma(G) + 2 = \gamma_R(G) + 2$. Hence, $\gamma_R(G\overline{G}) \leq \gamma_R(G) + 2$. Note that an isolated vertex of G would be in V_1. Since G is a Roman graph, Proposition 3.3 implies that G has no isolated vertices. Further, since $\overline{G}$ has no isolated vertices, the result follows from Theorem 2.2. □

We conclude this section by noting that the middle value of $\gamma_R(G) + 3$ is also attainable. Let $G = C_n$ for $n \geq 6$ such that n is congruent to 0 or 1 modulo 3. Then $\gamma_R(G\overline{G}) = \gamma_R(G) + 3 = \lceil 2n/3 \rceil + 3$. Furthermore, we note that if G is a graph with $\text{diam}(G) \geq 3$ and G is neither Roman nor almost Roman, by Theorem 3.4, $\gamma_R(G\overline{G}) = \gamma_R(G) + 2$. Hence, for any graph $G\overline{G}$ attaining the upper bound of Theorem 3.3, each of G and $\overline{G}$ must be Roman or almost Roman. However, this is not sufficient for a characterization of such graphs. For example, if $G = C_n$ where $n \geq 6$ and n is congruent to 0 modulo 3, then both G and $\overline{G}$ are Roman but $\gamma_R(G\overline{G}) < \gamma_R(G) + 4$.

4 Summary

Let G be a graph such that G and $\overline{G}$ are isolate-free graphs. Hence, $\text{diam}(G) \geq 2$ and $\text{diam}(\overline{G}) \geq 2$. Then Table 1 summarizes the results from Sections 2 and 3 for such graphs. The values for $\text{diam}(G)$ and $\text{diam}(\overline{G})$ are given in the first two columns and the last column lists the possible values of $\gamma_R(G\overline{G})$ for graphs $G\overline{G}$ where G and $\overline{G}$ have these given diameters. The first row in the table is directly from Corollary 3.3. It is well known that if $\text{diam}(G) \geq 3$, then $\text{diam}(\overline{G}) \leq 3$. Thus, the second row follows directly from Theorem 3.3 . If $\text{diam}(G) = \text{diam}(\overline{G}) = 3$, then by Observation 3.1, $\gamma_R(G\overline{G}) \leq \gamma_R(G) + \gamma_R(\overline{G}) \leq 4 + 4 = 8$. Given this small upper bound, we treat this subcase of the second row result separately in Row 3. Note that in Table 1, $k = \max\{\gamma_R(G), \gamma_R(\overline{G})\}$.

Table 1: Roman Domination Numbers of Complementary Prisms

$\text{diam}(G)$	$\text{diam}(\overline{G})$	$\gamma_R(G\overline{G})$
2	2	$\{2\gamma(G\overline{G}), 2\gamma(G\overline{G}) - 1\}$
≥ 3	$\{2, 3\}$	$\{k+2, k+3, k+4\}$
3	3	≤ 8

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References

[1] S. Bermudo, H. Fernau and J. M. Sigarreta, The differential and the Roman domination number of a graph, *Appl. Anal. Discrete Math.* 8 (2014), 155–171.

[2] E. W. Chambers, B. Kinnersley, N. Prince and D. B. West, Extremal problems for Roman domination, *SIAM J. Discrete Math.* 23 (2009), 1575–1586.

[3] M. Chellali and N. Jafari Rad, Double equality between the Roman domination and independent Roman domination numbers in trees, *Discuss. Math. Graph Theory* 33 (2013), 337–346.

[4] E. J. Cockayne, P. M. Dreyer Sr., S. M. Hedetniemi and S. T. Hedetniemi, Roman domination in graphs, *Discrete Math.* 278 (2004), 11–22.

[5] W. J. Desormeaux and T. W. Haynes, Restrained domination in complementary prisms, *Util. Math.* 86 (2011), 267–278.

[6] T. W. Haynes, M. A. Henning, P. J. Slater and L. C. van der Merwe, The complementary product of two graphs, *Bull. Instit. Combin. Appl.* 51 (2007), 21–30.

[7] T. W. Haynes, M. A. Henning and L. C. van der Merwe, Domination and total domination in complementary prisms, *J. Combin. Optim.* 18 (2009), 23–37.

[8] M. A. Henning and S. T. Hedetniemi, Defending the Roman Empire — A new strategy, *Discrete Math.* 266 (2003), 239–251.

[9] W. Imrich and S. Klavzar, *Product Graphs, Structure and Recognition*, Wiley (2000).

[10] D. Meierling, F. Protti, D. Rautenbach and A. R. de Almeida, Cycles in complementary prisms, *Discrete Appl. Math.* 193 (2015), 180–186.

[11] R. J. Nowakowski and D. F. Rall, Associative graph products and their independence, domination, and coloring numbers, *Discuss. Math. Graph Theory* 16 (1996), 53–79.

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