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Pairwise additive 1-rotational BIB designs with $\lambda = 1$

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Abstract

The existence of pairwise additive balanced incomplete block (BIB) designs and pairwise additive cyclic BIB designs with $\lambda = 1$ has been discussed through direct and recursive constructions in the literature. This paper takes BIB designs with 1-rotational automorphisms and then the existence of pairwise additive 1-rotational BIB designs is investigated for $\lambda = 1$. Finally, it is shown that there exists a 2-pairwise additive 1-rotational BIB design with parameters v, k and $\lambda = 1$ if and only if any $v \geq 4$ and $k = 2$.

1 Introduction

A *balanced incomplete block* (BIB) *design* is a system $(V, \mathcal{B})$, where V is a set of v points and $\mathcal{B}$ ($|\mathcal{B}| = b$) is a family of k-subsets (blocks) of V, such that each point of V appears in r different blocks of $\mathcal{B}$ and any two different points of V appear in exactly λ blocks in $\mathcal{B}$ [21]. This is denoted by BIBD(v, b, r, k, λ) or B(v, k, λ).

For a BIB design $(V, \mathcal{B})$, let σ be a permutation on V. For a block $B = \{v_1, \ldots, v_k\} \in \mathcal{B}$ and a permutation σ on V, let $B^\sigma = \{v_1^\sigma, \ldots, v_k^\sigma\}$. When $\mathcal{B} = \{B^\sigma \mid B \in \mathcal{B}\}$, σ is called an *automorphism* of the design $(V, \mathcal{B})$. If there exists an automorphism σ of order $v = |V|$, then the BIB design is said to be *cyclic*. On the other hand, when there exists an automorphism σ of order $v-1$ with one fixed point, the BIB design is said to be 1-*rotational with respect to the cyclic group of order* $v-1$ [2, 19]. Throughout the paper, the BIB design being 1-rotational with respect to the cyclic group of order $v-1$ is simply said to be 1-rotational. Note that 1-rotational BIB designs with respect to other algebraic groups are not said to be 1-rotational in this paper.

For a cyclic BIB design $(V, \mathcal{B})$, the set V of v points can be identified with $Z_v = \{0, 1, \ldots, v-1\}$. In this case, the design has an automorphism $\sigma : i \mapsto i+1 \pmod{v}$. The *block orbit* containing $B = \{v_1, v_2, \ldots, v_k\} \in \mathcal{B}$ is a set of distinct blocks $B + i = \{v_1 + i, v_2 + i, \ldots, v_k + i\} \pmod{v}$ for $i \in Z_v$. A block orbit is said to be *full* or *short* according as $|\{B + i \mid 0 \le i \le v-1\}| = v$ or not.

For a 1-rotational BIB design $(V, \mathcal{B})$, the set V of v points can be identified with $Z_{v-1} \cup \{\infty\}$ and the block orbit containing $B = \{v_1, v_2, \ldots, v_k\} \in \mathcal{B}$ is a set of distinct blocks $B + i = \{v_1 + i, v_2 + i, \ldots, v_k + i\} \pmod{v-1}$ for $i \in Z_{v-1}$. When $B = \{\infty, v_2, \ldots, v_k\}$, $B + i = \{\infty, v_2 + i, \ldots, v_k + i\}$. Moreover, if $\lambda = 1$, then the orbit of $B = \{\infty, v_2, \ldots, v_k\}$ has cardinality $(v-1)/(k-1)$. Similarly, a block orbit is said to be full or short according as $|\{B + i \mid 0 \le i \le v-2\}| = v-1$ or not.

Choose an arbitrary block from each block orbit and call it an *initial block*. The initial block in a full block orbit and a short block orbit is called a full initial block and a short initial block, respectively. It is clear that a cyclic $\mathrm{B}(2t, 2, 1)$ or a 1-rotational $\mathrm{B}(2t+1, 2, 1)$ has a short orbit given by the 2-subset $\{0, t\}$ for any $t \ge 1$. This short orbit is denoted by $\{0, t\}\mathrm{PC}(t)$, where $\mathrm{PC}(t)$ means a short cycle of order t, i.e., only $0, 1, \ldots, t-1$ are to be added to the initial block.

Let $s = v/k$, where s need not be an integer, unlike other design parameters. A set of ℓ $\mathrm{BIBD}(v, b, r, k, \lambda)$'s, namely, $(V, \mathcal{B}_1), (V, \mathcal{B}_2), \ldots, (V, \mathcal{B}_\ell)$, is called an ℓ-*pairwise additive BIB design*, denoted by ℓ-$\mathrm{PAB}(v, k, \lambda)$, if it is possible to pair the designs $(V, \mathcal{B}_1), (V, \mathcal{B}_2), \ldots, (V, \mathcal{B}_\ell)$, in such a way that every pair $(V, \mathcal{B}_{i_1}), (V, \mathcal{B}_{i_2})$, where $1 \le i_1, i_2 \le \ell$, $i_1 \ne i_2$, gives rise to a new design $(V, \mathcal{B}^*_{i_1 i_2})$ with parameters $v^* = v = sk$, $b^* = b$, $r^* = 2r$, $k^* = 2k$, $\lambda^* = 2r(2k-1)/(sk-1)$. The family $\mathcal{B}^*_{i_1 i_2}$ is defined by $\mathcal{B}^*_{i_1 i_2} = \{B_{i_1 j} \cup B_{i_2 j} \mid 1 \le j \le b\}$ with B_{ij} being the jth block of an ith block family $\mathcal{B}_i$. When $\ell = s$, this is called an *additive BIB design* [16, 23], denoted by $\mathrm{AB}(v, k, \lambda)$. An ℓ-$\mathrm{PAB}(v, k, \lambda)$ is said to be cyclic or 1-rotational, denoted by ℓ-$\mathrm{PACB}(v, k, \lambda)$ or ℓ-$\mathrm{PARB}(v, k, \lambda)$, if (i) every design $(V, \mathcal{B}_1)$, $(V, \mathcal{B}_2), \ldots, (V, \mathcal{B}_\ell)$ is cyclic or 1-rotational, respectively, and (ii) every design $(V, \mathcal{B}^*_{i_1 i_2})$ arising from the pair $(V, \mathcal{B}_{i_1}), (V, \mathcal{B}_{i_2})$ is cyclic or 1-rotational and its initial blocks are obtained by joining an initial block in $(V, \mathcal{B}_{i_1})$ to an initial block in $(V, \mathcal{B}_{i_2})$, where two orbits given by $B_{i_1 j}$ and $B_{i_2 j}$ have the same cardinality for each $1 \le j \le b$. Note that if we join an initial block of a $\mathrm{B}(v, k, \lambda)$ to an initial block of another $\mathrm{B}(v, k, \lambda)$, then the resulting block might not be an initial block of a $\mathrm{B}(v, 2k, \lambda')$. When $\ell = s$, this is called an *additive cyclic BIB design* or an *additive* 1-*rotational BIB design*, denoted by $\mathrm{ACB}(v, k, \lambda)$ or $\mathrm{ARB}(v, k, \lambda)$, respectively. For example, it is checked that the four block families

$$\begin{array}{lcll} \mathcal{B}_1 & : & \{0,1\},\{4,2\},\{3,6\},\{5,\infty\} & \text{mod } 7 \\ \mathcal{B}_2 & : & \{4,2\},\{0,1\},\{5,\infty\},\{3,6\} & \text{mod } 7 \\ \mathcal{B}_3 & : & \{5,\infty\},\{3,6\},\{0,1\},\{4,2\} & \text{mod } 7 \\ \mathcal{B}_4 & : & \{3,6\},\{5,\infty\},\{4,2\},\{0,1\} & \text{mod } 7 \end{array}$$

yield an ARB$(8,2,1)$. Note that we allow repeated blocks in $(V,\mathcal{B}^*_{i_1i_2})$.

Some results on existence are reviewed. In a PAB$(v,k,1)$, it is shown that there are an AB$(2^n,2,1)$ and an AB$(3^n,3,1)$ for any integer $n \geq 2$ [22, 23], and there are a 2-PAB$(v,2,1)$ for any $v \geq 4$ and a 3-PAB$(v,2,1)$ for any $v \geq 6$ [11, 14]. Furthermore, partial results on asymptotic existence of ℓ-PAB(v,k,λ)'s and the existence of 2-PACB$(v,k,1)$'s are also shown in [12, 13, 15]. However, for an ℓ-PACB$(v,k,1)$, its complete existence is not yet known in the literature, even if $\ell = 2$ and $k = 2$, as the following shows.

Theorem 1.1 [12] There exists a 2-PACB$(v,2,1)$ for any odd integer $v \geq 5$ such that $\gcd(v,9) \neq 3$.

Theorem 1.2 [15] There exists a 2-PACB$(2^m t,2,1)$ for any integer $m \geq 2$ and any odd integer $t(\geq 1)$ such that $\gcd(t,27) \neq 3,9$.

We now focus on 1-rotational BIB designs and the complete existence of a 2-PARB$(v,k,1)$ will be established in Section 5 as follows. This will be the main result of the present paper.

Theorem 1.3 There exists a 2-PARB$(v,k,1)$ if and only if any $v \geq 4$ and $k = 2$.

Note that the existence of ℓ-pairwise additive BIB designs is equivalent to the existence of some kind of decompositions of a λ-fold complete graph λK_v into edge-disjoint subgraphs isomorphic to a complete graph K_k, denoted by a (v,K_k,λ)-design, in terms of graph embeddings (cf. [3, 7]). In fact, Theorem 1.3 is equivalent to say that there are two 1-rotational $(v,K_2,1)$-designs simultaneously embedded into a 1-rotational $(v,K_4,6)$-design allowed the repeated blocks such that two K_2's simultaneously embedded into each K_4 are vertex-disjoint. However, as far as the authors know, any existence result on graphs which is equivalent to Theorem 1.3 has not been provided in literature.

On the other hand, [14] gives a construction of an ℓ-PAB(v,k,λ) by use of nested BIB designs defined in [20]. A survey of nested BIB designs is given in [18] and a more general class of nested BIB designs is further discussed in [10, 17] with wide applicability for other designs. Unfortunately, to the best of our knowledge, by utilizing any result on nested BIB designs we cannot show the complete existence of a 2-PARB$(v,2,1)$.

In particular, Z-cyclic whist tournament designs of order $4n$ in [1] coincide with a special class of nested BIB designs having both a 1-rotational automorphism and the property of resolvability. It is seen that the Z-cyclic whist tournament designs of order $4n$ can give the 2-PARB$(4n, 2, 1)$ with resolvability by use of the construction method in [14]. However, the investigation of existence of a 2-PARB$(v, 2, 1)$ with resolvability may be as difficult as showing the existence of Z-cyclic whist tournament designs. The resolvability of a 2-PARB$(v, k, 1)$ will be discussed in another paper.

In Section 2, fundamental results for PAB$(v, k, 1)$'s and 1-rotational BIB designs will be reviewed and the nonexistence of a 2-PARB$(v, k, 1)$ for any $k \geq 3$ will be shown. In Section 3, a pairwise additive cyclic relative difference family (PACDF) used in the proof of Theorem 1.3 will be defined and recursive constructions used in [4, 9, 24] will be developed for the PACDF. Section 4 shows some existence of PACDFs and Section 5 is devoted to the proof of Theorem 1.3. As the appendix, individual examples will be presented.

2 Fundamental results

It is known [23] that in a PAB(v, k, λ)

$$2\lambda \equiv 0 \pmod{k-1} \tag{2.1}$$

which implies $k = 2$ or 3 when $\lambda = 1$.

A B$(v, 3, 1)$ is known as a Steiner triple system (STS). The existence of 1-rotational STSs with respect to an arbitrary group is studied in [2]. Moreover, a characterization of 1-rotational STSs with respect to the cyclic group of order $v - 1$ is known as follows.

Lemma 2.1 [19] Any 1-rotational B$(v, 3, 1)$ $(V, \mathcal{B})$ with a point set $V = Z_{v-1} \cup \{\infty\}$ contains the short orbit of the block $\{0, (v-1)/2, \infty\}$PC$((v-1)/2)$ and full orbits in $\mathcal{B}$.

Now the nonexistence of an ℓ-PARB$(v, k, 1)$ can be shown.

Theorem 2.2 There exists no ℓ-PARB$(v, k, 1)$ for any integers $\ell \geq 2$, $v \geq \ell k$ and $k \geq 3$.

Proof. When $k \geq 4$, (2.1) shows the nonexistence of the design. When $k = 3$, on account of Lemma 2.1, let $\{a, a + (v-1)/2, \infty\}$ and $\{a', a' + (v-1)/2, \infty\}$, $a, a' \in Z_{v-1}$, can be short initial blocks of $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. By the definition of a 2-PARB$(v, 3, 1)$, $\mathcal{B}^*_{12}$ must contain a set-union of two short initial blocks. However, both of the two blocks contain the element ∞ in common. Hence there does not exist the required design. □

Remark 2.3 By taking account of an idea used in the proof of Theorem 2.2, a general result can be shown such that there exists no ℓ-PARB$(v, k, (k-1)/2)$ for any $\ell \geq 2$, any $v \geq \ell k$ and any odd integer $k \geq 3$. Hence, it follows from (2.1) that $\lambda \geq k - 1$ in a PARB(v, k, λ).

From now on, we will discuss the remaining case $k = 2$ for $\lambda = 1$ and any $v \geq 4$ to obtain the main result of this paper.

3 Some combinatorial structures

In this section, cyclic difference matrices (CDMs) and cyclic relative difference families (CDFs) are reviewed and pairwise additive cyclic relative difference families (PACDFs) are newly defined. In [4, 9, 24], CDFs are used to construct designs with cyclic (or 1-rotational) automorphisms, and useful recursive constructions of CDFs are given by use of CDMs. Similarly, some constructions of PACDFs are discussed here.

At first CDMs are reviewed. A *cyclic difference matrix* on Z_v, denoted by CDM(k, v), is defined as a $k \times v$ array $(a(m, n))$, $a(m, n) \in Z_v$, $1 \leq m \leq k$, $1 \leq n \leq v$, that satisfies

$$Z_v = \{a(i, n) - a(j, n) \pmod v \mid 1 \leq n \leq v\}$$

for each $1 \leq i < j \leq k$, that is, the differences of any two distinct rows contain every element of Z_v exactly once (see [8]).

Lemma 3.1 [8] There exists a CDM$(4, v)$ for any odd integer $v \geq 5$ such that $\gcd(v, 27) \neq 9$.

Let G be a group and N be a subgroup of G. Then a family $\mathcal{F} = \{F_i \mid i \in I\}$ of k-subsets of G is called a *relative difference family*, denoted by (G, N, k, λ)-DF, if the list of differences $(d - d' \mid d, d' \in D_i, d \neq d', i \in I)$ contains each element of $G - N$ exactly λ times and each element of N zero time. When G is the cyclic group Z_v and N is the subgroup of Z_v of order n, the relative difference family is said to be *cyclic*, denoted by (v, n, k, λ)-CDF (cf. [4, 24]).

Some results on the existence of $(vg, g, 4, 1)$-CDFs are known as follows.

Lemma 3.2 [5, 6] There exists a $(2^{s+4}, 2^s, 4, 1)$-CDF for any integer $s \geq 2$.

Lemma 3.3 [6] There exist a $(81, 9, 4, 1)$-CDF and a $(243, 27, 4, 1)$-CDF.

A set of two families $\mathcal{F}_1$ and $\mathcal{F}_2$ is called a 2-*pairwise additive* (vg, g, k, λ)-CDF, denoted by 2-(vg, g, k, λ)-PACDF, if both $\mathcal{F}_1$ and $\mathcal{F}_2$ are (vg, g, k, λ)-CDFs and the family of set-unions of the jth k-subsets $B_j^{(1)} \in \mathcal{F}_1$ and $B_j^{(2)} \in \mathcal{F}_2$, $1 \leq j \leq |\mathcal{F}_1| = |\mathcal{F}_2|$, is also a $(vg, g, 2k, \lambda')$-CDF with $\lambda' = 2\lambda(2k-1)/(k-1)$. Throughout the paper, the above "2-(vg, g, k, λ)-PACDF" is simply denoted by "(vg, g, k, λ)-PACDF".

Next, some constructions of $(vg, g, 2, 1)$-PACDFs are provided.

Lemma 3.4 The existence of a $(vg, g, 4, 1)$-CDF implies the existence of a $(vg, g, 2, 1)$-PACDF.

Proof. Let 4-subsets of the $(vg, g, 4, 1)$-CDF on Z_{vg} be

$$\{a_i, b_i, c_i, d_i\},\ 1 \le i \le \frac{g(v-1)}{12}.$$

Then it is seen that the following families on subsets of Z_{vg} yield the required $(vg, g, 2, 1)$-PACDF:

$$\begin{array}{rcl} \mathcal{F}_1 & : & \{a_i, b_i\}, \{a_i, c_i\}, \{a_i, d_i\}, \{c_i, b_i\}, \{b_i, d_i\}, \{d_i, c_i\} \\ \mathcal{F}_2 & : & \{c_i, d_i\}, \{d_i, b_i\}, \{b_i, c_i\}, \{d_i, a_i\}, \{c_i, a_i\}, \{b_i, a_i\} \end{array}$$

for $1 \le i \le g(v-1)/12$. □

Note that in the proof of Lemma 3.4 the construction of $\mathcal{F}_1$ and $\mathcal{F}_2$ is skillful, since an initial subset of the CDF might not arise from the union of initial subsets belonging to families with other parameters.

Lemma 3.5 Let m be a divisor of g. Then the existence of a $(vg, g, 2, 1)$-PACDF and a $(g, m, 2, 1)$-PACDF implies the existence of a $(vg, m, 2, 1)$-PACDF.

Proof. Let $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_1', \mathcal{F}_2'$ be families of a $(vg, g, 2, 1)$-PACDF and a $(g, m, 2, 1)$-PACDF, respectively. Then combined families $\mathcal{F}_h^* = \mathcal{F}_h \cup \{\{vx,\ vy\} \mid \{x, y\} \in \mathcal{F}_h'\}$ on Z_{vg}, $h = 1, 2$, can yield a $(vg, m, 2, 1)$-PACDF. □

Lemma 3.6 The existence of a $(vg, g, 2, 1)$-PACDF and a CDM$(4, v')$ implies the existence of a $(vv'g, v'g, 2, 1)$-PACDF.

Proof. Let two families of a $(vg, g, 2, 1)$-PACDF be

$$\mathcal{F}_h \ : \ \{x_{hi}, y_{hi}\}$$

for $1 \le i \le g(v-1)/2$ and $h = 1, 2$. Further let A be the CDM$(4, v')$ with $a(m, n)$ as the (m, n)-entry for $1 \le m \le 4$ and $1 \le n \le v'$. Then, it can be shown that the following two families yield the required $(vv'g, v'g, 2, 1)$-PACDF on $Z_{vv'g}$:

$$\mathcal{F}_h^* \ : \ \{x_{hi} + a(2h-1, n)vg,\ y_{hi} + a(2h, n)vg\}$$

for $1 \le i \le g(v-1)/2, 1 \le n \le v'$ and $h = 1, 2$. In fact, let $\{x_{hj}^*, y_{hj}^*\}$ be the jth subset of $\mathcal{F}_h^*$ for $1 \le j \le v'g(v-1)/2$ and $h = 1, 2$. Then, by the property of the CDM$(4, v')$, it can be checked that the multiset of differences arising from the subsets of $\mathcal{F}_h^*$, $h = 1, 2$, is composed of (i) $\cup_{j=1}^{v'g(v-1)/2}\{\pm(x_{hj}^* - y_{hj}^*)\} = \{\pm(x_{hi} - y_{hi}) + nvg \mid 1 \le i \le g(v-1)/2, 0 \le n \le v'-1\}$ containing every element of $Z_{vv'g} - vZ_{vv'g}$ exactly once for each $h = 1, 2$ and (ii) $\cup_{j=1}^{v'g(v-1)/2}\{\pm(x_{1j}^* - x_{2j}^*), \pm(y_{1j}^* - y_{2j}^*), \pm(x_{1j}^* - y_{2j}^*), \pm(y_{1j}^* - x_{2j}^*)\} = \{\pm(x_{1i} - x_{2i} + nvg), \pm(y_{1i} - y_{2i} + nvg), \pm(x_{1i} - y_{2i} + nvg), \pm(y_{1i} - x_{2i} + nvg) \mid 1 \le i \le g(v-1)/2, 0 \le n \le v'-1\}$ containing every element of $Z_{vv'g} - vZ_{vv'g}$ exactly four times. Thus it is seen that both $\mathcal{F}_1^*$ and $\mathcal{F}_2^*$ are $(vv'g, v'g, 2, 1)$-CDFs, and the family of set-unions $\{x_{1j}^*, y_{1j}^*\} \cup \{x_{2j}^*, y_{2j}^*\}$, $1 \le j \le v'g(v-1)/2$, yields a $(vv'g, v'g, 4, 6)$-CDF. The proof is complete. □

Note that full initial blocks of a 2-PACB$(v, 2, 1)$ with no short initial blocks can be considered as a $(v, 1, 2, 1)$-PACDF. Hence, it is clear that Lemma 3.6 provides a $(vg, g, 2, 1)$-PACDF, by use of the 2-PACB$(v, 2, 1)$ with no short initial blocks and a CDM$(4, g)$.

On the other hand, it is obvious that there does not exist a CDM$(4, 2)$. Hence, Lemma 3.6 cannot be utilized for the case of $v' = 2$. However, the following recursive construction can be presented.

Lemma 3.7 The existence of a $(vg, g, 2, 1)$-PACDF implies the existence of a $(2vg, 2g, 2, 1)$-PACDF.

Proof. Let two families of a $(vg, g, 2, 1)$-PACDF be

$$\mathcal{F}_h \ : \ \{x_{hi}, y_{hi}\}$$

for $1 \le i \le g(v-1)/2$ and $h = 1, 2$. Then, by choosing arbitrary blocks in each orbit of $\{x_{1i}, y_{1i}\} \cup \{x_{2i}, y_{2i}\}$, without loss of generality it can be assumed that $\{x_{1i}, y_{1i}\} = \{0, i\}$.

Now it can be shown that the following two families yield the required $(2vg, 2g, 2, 1)$-PACDF on Z_{2vg}:

$$\begin{aligned} \mathcal{F}_1^* \ &: \ \{x_{1i}, y_{1i}\}, \{x_{2i}, y_{2i} + \delta_i vg\} \\ \mathcal{F}_2^* \ &: \ \{x_{2i}, y_{2i} + \delta_i vg\}, \{x_{1i} + vg, y_{1i} + vg\} \end{aligned}$$

for $1 \le i \le g(v-1)/2$, where $\delta_i = 1$ or 0 according as $|y_{2i} - x_{2i}| < vg/2$ or otherwise. In fact, let $\{x^*_{hj}, y^*_{hj}\}$ be the jth subset of $\mathcal{F}^*_h$ for $1 \le j \le g(v-1)$ and $h = 1, 2$. Then the definition of δ_i implies that $\cup_{j=1}^{g(v-1)}\{\pm(x^*_{hj} - y^*_{hj})\} = \{\pm(x_{1i} - y_{1i}), \pm(x_{2i} - y_{2i} - \delta_i vg) \mid 1 \le i \le g(v-1)/2\}$ contains every element of $Z_{2vg} - vZ_{2vg}$ exactly once for each $h = 1, 2$. Furthermore, it can be checked that $\cup_{j=1}^{g(v-1)}\{\pm(x^*_{1j} - x^*_{2j}), \pm(y^*_{1j} - y^*_{2j}), \pm(x^*_{1j} - y^*_{2j}), \pm(y^*_{1j} - x^*_{2j})\} = \{\pm(x_{1i} - x_{2i} + nvg), \pm(y_{1i} - y_{2i} + nvg), \pm(x_{1i} - y_{2i} + nvg), \pm(y_{1i} - x_{2i} + nvg) \mid 1 \le i \le g(v-1)/2, 0 \le n \le 1\}$ contains every element of $Z_{2vg} - vZ_{2vg}$ exactly four times. Thus it is seen that both $\mathcal{F}^*_1$ and $\mathcal{F}^*_2$ are $(2vg, 2g, 2, 1)$-CDFs, and the family of set-unions $\{x^*_{1j}, y^*_{1j}\} \cup \{x^*_{2j}, y^*_{2j}\}$, $1 \le j \le g(v-1)$, yields a $(2vg, 2g, 4, 6)$-CDF. The proof is complete. □

The results obtained here will be used in the next section.

4 Existence of $(vg, g, 2, 1)$-PACDFs

In this section, the discussion on existence of $(vg, g, 2, 1)$-PACDFs is made by use of direct and recursive methods.

Throughout Sections 4 and 5, let P be any odd integer such that $\gcd(P, 6) = 1$ and $P \ge 5$. Then any prime factor of P is not less than 5.

At first, two classes of $(vg, g, 2, 1)$-PACDFs are produced by use of direct constructions as the following shows.

Lemma 4.1 Let $P \geq 5$ be an odd integer with $\gcd(P, 6) = 1$. Then there exists a $(2P, 2, 2, 1)$-PACDF.

Proof. Since $\gcd(2, P) = 1$, the following two families on $Z_2 \times Z_P$ can yield the required $(2P, 2, 2, 1)$-PACDF on Z_{2P}, by corresponding the element j for $0 \leq j \leq 2P - 1$ to (z, w), where $j \equiv z \pmod 2$ and $j \equiv w \pmod P$:

$$\begin{aligned} \mathcal{F}_1 &: \{(0,0),(1,a)\}, \{(0,0),(0,a)\} \\ \mathcal{F}_2 &: \{(1,2a),(1,4a)\}, \{(0,2a),(1,4a)\} \end{aligned}$$

for any integer a with $1 \leq a \leq (P-1)/2$. □

Lemma 4.2 Let $P \geq 5$ be an odd integer with $\gcd(P, 6) = 1$. Then there exists a $(3P, 3, 2, 1)$-PACDF.

Proof. Since $\gcd(3, P) = 1$, the following two families on $Z_3 \times Z_P$ can yield the required $(3P, 3, 2, 1)$-PACDF on Z_{3P}, by corresponding the element j for $0 \leq j \leq 3P - 1$ to (z, w), where $j \equiv z \pmod 3$ and $j \equiv w \pmod P$:

$$\begin{aligned} \mathcal{F}_1 &: \{(0,0),(1,a)\}, \{(0,a'),(0,-a')\} \\ \mathcal{F}_2 &: \{(0,2a),(1,3a)\}, \{(1,2a'),(1,-2a')\} \end{aligned}$$

for any integers a and a' with $1 \leq a \leq P-1$ and $1 \leq a' \leq (P-1)/2$. □

Next, some results on the existence of $(vg, g, 2, 1)$-PACDFs obtained from $(vg, g, 4, 1)$-CDFs are shown as follows.

Lemma 4.3 There exists a $(2^{4m+n}, 2^n, 2, 1)$-PACDF for any $n \in \{2, 3, 4, 5\}$ and any positive integer m.

Proof. Lemma 3.4 with the $(2^{4m+n}, 2^{4m+n-4}, 4, 1)$-CDF obtained by Lemma 3.2 can provide a $(2^{4m+n}, 2^{4m+n-4}, 2, 1)$-PACDF for any $m \geq 1$ and any $n \in \{2, 3, 4, 5\}$. Hence, for $m = 1$ the result can be shown. Furthermore, by Lemma 3.5 with a $(2^{4(m+1)+n}, 2^{4m+n}, 2, 1)$-PACDF, the existence of a $(2^{4m+n}, 2^n, 2, 1)$-PACDF implies the existence of a $(2^{4(m+1)+n}, 2^n, 2, 1)$-PACDF for $m \geq 1$. Thus, the proof is complete by mathematical induction on m. □

Lemma 4.4 There exist a $(3^n, 9, 2, 1)$-PACDF and a $(3^{n'}, 3, 2, 1)$-PACDF for any even integer $n \geq 4$ and any odd integer $n' \geq 3$.

Proof. By applying Lemma 3.4 with the $(81, 9, 4, 1)$-CDF and the $(243, 27, 4, 1)$-CDF given in Lemma 3.3, it is shown that there are a $(81, 9, 2, 1)$-PACDF and a $(243, 27, 2, 1)$-PACDF. Furthermore, a $(27, 3, 2, 1)$-PACDF is given in Example A.9. Hence, for any $n \geq 3$, a $(3^n, 3^{n-2}, 2, 1)$-PACDF can be obtained by applying Lemma 3.6 with the CDM$(4, 27)$ given by Lemma 3.1. Thus, by applying Lemma 3.5 with a $(3^n, 3^{n-2}, 2, 1)$-PACDF and a $(3^m, 3^{m-2}, 2, 1)$-PACDF for $3 \leq m \leq n-2$ repeatedly, a $(3^n, 9, 2, 1)$-PACDF and a $(3^{n'}, 3, 2, 1)$-PACDF can be obtained for any even integer $n \geq 4$ and any odd integer $n' \geq 3$, respectively. □

Finally, some results on the existence of $(vg, g, 2, 1)$-PACDFs are shown by use of recursive constructions as follows.

Lemma 4.5 There exist a $(2 \cdot 3^n, 18, 2, 1)$-PACDF and a $(2 \cdot 3^{n'}, 6, 2, 1)$-PACDF for any even integer $n \geq 4$ and any odd integer $n' \geq 3$.

Proof. By applying Lemma 3.7 with the $(3^n, 9, 2, 1)$-PACDF and $(3^{n'}, 3, 2, 1)$-PACDF obtained by Lemma 4.4, the proof is complete. □

Lemma 4.6 There exists a $(2^m 3, 2^{m-1}, 2, 1)$-PACDF for any integer $m \geq 2$.

Proof. By applying Lemma 3.7 with the $(12, 2, 2, 1)$-PACDF given in Example A.8 repeatedly, the proof is complete. □

Lemma 4.7 There exists a $(2^m 3^n, 2^m, 2, 1)$-PACDF for any integers $m \geq 1$ and $n \geq 2$.

Proof. It follows that a family of initial blocks of the 2-PACB$(3^n, 2, 1)$ for $n \geq 2$ obtained by Theorem 1.1 yields a $(3^n, 1, 2, 1)$-PACDF. Hence, by applying Lemma 3.7 repeatedly, the proof is complete. □

Lemma 4.8 Let $P \geq 5$ be an odd integer with $\gcd(P, 6) = 1$. Then there exists a $(2Pq, 2q, 2, 1)$-PACDF for any odd prime $q \geq 5$.

Proof. By applying Lemma 3.6 with the CDM$(4, q)$ for a prime q and the $(2P, 2, 2, 1)$-PACDF obtained by Lemmas 3.1 and 4.1, respectively, the proof is complete. □

Lemma 4.9 Let $P \geq 5$ be an odd integer with $\gcd(P, 6) = 1$. Then there exists a $(2^n P, 2^n, 2, 1)$-PACDF for any positive integer n.

Proof. It follows that a family of initial blocks of the 2-PACB$(P, 2, 1)$ obtained by Theorem 1.1 yields a $(P, 1, 2, 1)$-PACDF. Hence, by applying Lemma 3.7 repeatedly, the proof is complete. □

Lemma 4.10 Let $P \geq 5$ be an odd integer with $\gcd(P, 6) = 1$. Then there exists a $(3^n P, P, 2, 1)$-PACDF for any integer $n \geq 2$.

Proof. For any $n \geq 2$, it follows that a family of initial blocks of the 2-PACB$(3^n, 2, 1)$ obtained by Theorem 1.1 yields a $(3^n, 1, 2, 1)$-PACDF. Hence, by applying Lemma 3.6 with the CDM$(4, P)$ obtained by Lemma 3.1, the proof is complete. □

Lemma 4.11 Let $P \geq 5$ be an odd integer with $\gcd(P, 6) = 1$. Then there exists a $(2^m 3P, 2^m 3, 2, 1)$-PACDF for any positive integer m.

Proof. By applying Lemma 3.7 with the $(3P, 3, 2, 1)$-PACDF obtained by Lemma 4.2 repeatedly, the proof is complete. □

Lemma 4.12 Let $P \geq 5$ be an odd integer with $\gcd(P, 6) = 1$. Then there exists a $(2^m 3^n P, 2^m P, 2, 1)$-PACDF for any integers $m \geq 1$ and $n \geq 2$.

Proof. By applying Lemma 3.7 with the $(3^n P, P, 2, 1)$-PACDF obtained by Lemma 4.10 repeatedly, the proof is complete. □

Each of the above-mentioned results will play an important role to show the existence of a 2-PARB$(v, 2, 1)$ in the next section.

5 Proof of Theorem 1.3

In this section, Theorem 1.3 as the main result of this paper is established. At first a class of 2-PARB$(v, 2, 1)$'s is formed.

Lemma 5.1 There exists a 2-PARB$(v, 2, 1)$ for any $v \geq 6$ with $\gcd(v-1, 6) = 1$.

Proof. First note that the condition $\gcd(v-1, 6) = 1$ implies $\{\pm ta \mid 2 \leq a \leq (v-2)/2\} = Z_{v-1} \setminus \{0, \pm t\}$ on Z_{v-1} for any $t \in \{1, 2, 3\}$. Then, it can be shown that the following block families on $Z_{v-1} \cup \{\infty\}$ yield the required 2-PARB$(v, 2, 1)$ having

$$\begin{aligned}
\mathcal{B}_1 &: \{0,1\}, \{0,\infty\}, \{0,a\} \mod v-1 \\
\mathcal{B}_2 &: \{2,\infty\}, \{2,3\}, \{2a,3a\} \mod v-1
\end{aligned}$$

for any integer a with $2 \leq a \leq (v-2)/2$. □

Note that Lemma 5.1 reveals a generalization of Theorem 2.5 in [11], since any odd prime $v - 1$ satisfies $\gcd(v-1, 6) = 1$.

Next, a class of 2-PARB$(v, 2, 1)$'s can be produced as the following shows.

Lemma 5.2 There exists a 2-PARB$(2p + 1, 2, 1)$ for any odd prime p.

Proof. When $p = 3, 5, 7$, Examples A.2, A.4 and A.6 yield the required designs. Next let $p \geq 11$. Then it can be shown that the following block families yield a 2-PAB$(v, 2, 1)$ on $Z_2 \times Z_p \cup \{\infty\}$:

$$\begin{aligned}
\mathcal{B}_1 : \;& \{(0,2),(1,1)\}, \{(0,4),(1,2)\}, \{(0,0),(1,3)\}, \{(0,0),(1,4)\}, \\
& \{(0,0),\infty\}, \{(0,0),(1,a)\}, \{(0,0),(0,a')\}, \\
& \{(0,0),(1,0)\}\mathrm{PC}(p) \mod (2,p) \\
\mathcal{B}_2 : \;& \{(1,2),(1,4)\}, \{(1,4),(1,8)\}, \{(1,6),(1,12)\}, \{(0,12),\infty\}, \\
& \{(1,8),(1,16)\}, \{(1,2a),(1,4a)\}, \{(0,2a'),(1,4a')\}, \\
& \{(0,4),(1,4)\}\mathrm{PC}(p) \mod (2,p)
\end{aligned}$$

for any integers a and a' with $5 \leq a \leq (p-1)/2$ and $1 \leq a' \leq (p-1)/2$. Since $\gcd(2,p) = 1$ implies $Z_2 \times Z_p \cong Z_{2p}$, the required 2-PARB$(2p+1,2,1)$ on $Z_{2p} \cup \{\infty\}$ can be constructed, by corresponding the element j for $0 \leq j \leq 2p-1$ to (z,w), where $j \equiv z \pmod 2$ and $j \equiv w \pmod p$. □

Next, some results on the existence of a 2-PARB$(v,2,1)$ are shown by use of the observation on $(vg,g,2,1)$-PACDFs given in Section 4 and the following recursive construction.

Lemma 5.3 The existence of a $(vg,g,2,1)$-PACDF and a 2-PARB$(g+1,2,1)$ implies the existence of a 2-PARB$(vg+1,2,1)$.

Proof. Let $\mathcal{F}_1, \mathcal{F}_2$ be two families of a $(vg,g,2,1)$-PACDF. Further let two families of initial blocks of a 2-PARB$(g+1,2,1)$ be

$$\mathcal{F}'_h \;:\; \{x_i^{(h)}, y_i^{(h)}\}$$

for $1 \leq i \leq \lfloor (g+2)/2 \rfloor$ and $h = 1,2$. Then $\mathcal{F}^*_h = \mathcal{F}_h \cup v\mathcal{F}'_h$, $h = 1,2$, can yield a 2-PARB$(vg+1,2,1)$ with

$$v\mathcal{F}'_h \;:\; \{vx_i^{(h)}, vy_i^{(h)}\}$$

on $Z_{vg} \cup \{\infty\}$ for $1 \leq i \leq \lfloor (g+2)/2 \rfloor$ and $h = 1,2$. □

The following example illustrates Lemma 5.3 with $v = 9$ and $g = 3$.

Example 5.4 Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two families of the $(27,3,2,1)$-PACDF given in Example A.9. Furthermore, two families of initial blocks on $Z_{27} \cup \{\infty\}$ obtained from the 2-PARB$(4,2,1)$ given in Example A.1 can be

$$\begin{aligned} 9\mathcal{F}'_1 &\;:\; \{0,\infty\}, \{9,18\} \\ 9\mathcal{F}'_2 &\;:\; \{9,18\}, \{0,\infty\}. \end{aligned}$$

Then combined families $\mathcal{F}^*_h = \mathcal{F}_h \cup 9\mathcal{F}'_h$, $h = 1,2$, yield a 2-PARB$(28,2,1)$.

For a 2-PACB$(v,2,1)$ with families $\mathcal{B}_1, \mathcal{B}_2$ of blocks, two initial blocks $\{a, a+t\} \in \mathcal{B}_1$ and $\{b, b+t\} \in \mathcal{B}_2$, $a,b,t \in Z_v, t \neq v/2$, such that a set-union of the two initial blocks is an initial block of $\mathcal{B}^*_{12}$, are now called *friend initial blocks*.

Lemma 5.5 There exists a 2-PARB$(2^n+1,2,1)$ for any integer $n \geq 2$.

Proof. When $n = 2,3,4$, the respective existence of a 2-PACB$(2^n,2,1)$ with friend initial blocks can be seen in [12], i.e., Example 3.4 with $\{0,1\}, \{2,3\}$, Example 3.5 with $\{0,1\}, \{4,5\}$ and Example 3.9 with $\{0,7\}, \{5,12\}$. When $n = 5$, Lemma 3.2 in [15] gives a 2-PACB$(2^5,2,1)$ with friend initial blocks $\{0,11\}, \{19,30\}$.

By replacing the friend initial blocks $\{a, a+t\}$ with $\{a, a+t\}$ and $\{a, \infty\}$, and also $\{b, b+t\}$ with $\{b, \infty\}$ and $\{b, b+t\}$, it is shown that there exists a 2-PARB$(2^n+1,2,1)$ for $n = 2,3,4,5$.

On the other hand, for any $n' \in \{2,3,4,5\}$ and any integer $s \geq 1$, a $(2^{4s+n'}, 2^{n'}, 2, 1)$-PACDF can be obtained by Lemma 4.3. Hence, by applying Lemma 5.3 with a 2-PARB$(2^{n'}+1, 2, 1)$, the proof is complete. □

Lemma 5.6 There exists a 2-PARB$(3^n + 1, 2, 1)$ for any positive integer n.

Proof. When $n = 1, 2$, the existence of the required design is given in Examples A.1 and A.3. On the other hand, Lemma 4.4 shows the existence of a $(3^n, 9, 2, 1)$-PACDF and a $(3^{n'}, 3, 2, 1)$-PACDF for any even integer $n \geq 4$ and any odd integer $n' \geq 3$, respectively. Hence, based on these PACDFs, by applying Lemma 5.3 with a 2-PARB$(10, 2, 1)$ and a 2-PARB$(4, 2, 1)$, the proof is complete. □

Lemma 5.7 There exists a 2-PARB$(2^m 3^n + 1, 2, 1)$ for any positive integers m and n.

Proof. When $(m, n) = (1, 1), (2, 1), (1, 2)$, Examples A.2, A.5 and A.7 show the result, respectively.

Let $m = 1$. Then Lemma 4.5 shows the existence of a $(2 \cdot 3^n, 18, 2, 1)$-PACDF and a $(2 \cdot 3^{n'}, 6, 2, 1)$-PACDF for any even integer $n \geq 4$ and any odd integer $n' \geq 3$, respectively. Hence, based on these PACDFs, Lemma 5.3 with a 2-PARB$(19, 2, 1)$ and a 2-PARB$(7, 2, 1)$ shows the existence of a 2-PARB$(2 \cdot 3^n + 1, 2, 1)$ for any integer $n \geq 3$.

Let $m \geq 3$ and $n = 1$. Then the $(2^m \cdot 3, 2^{m-1}, 2, 1)$-PACDF obtained by Lemma 4.6 and the 2-PARB$(2^{m-1} + 1, 2, 1)$ as in Lemma 5.5 show the existence of a 2-PARB$(2^m \cdot 3 + 1, 2, 1)$, by applying Lemma 5.3.

Finally, let $m \geq 2, n \geq 2$. Then a 2-PARB$(2^m \cdot 3^n + 1, 2, 1)$ can be obtained by applying Lemma 5.3 with the $(2^m \cdot 3^n, 2^m, 2, 1)$-PACDF and the 2-PARB$(2^m + 1, 2, 1)$ obtained by Lemmas 4.7 and 5.5, respectively. □

Lemma 5.8 Let $P \geq 5$ be an odd integer with $\gcd(P, 6) = 1$. Then there exists a 2-PARB$(2^n P + 1, 2, 1)$ for any positive integer n.

Proof. Let $p \geq 5$ be a prime factor of P and $P/p = Q$. Then $Q \geq 1$.

When $n = 1$, Lemma 5.2 itself shows the result for $Q = 1$. Next, for $Q \geq 5$, a $(2P, 2p, 2, 1)$-PACDF can be obtained by applying Lemma 4.8. Hence, Lemmas 5.2 and 5.3 show the existence of a 2-PARB$(2P + 1, 2, 1)$.

When $n \geq 2$, a $(2^n P, 2^n, 2, 1)$-PACDF can be obtained by Lemma 4.9. Hence, the existence of a 2-PARB$(2^n + 1, 2, 1)$, on account of Lemma 5.5, implies the existence of a 2-PARB$(2^n P + 1, 2, 1)$. □

Lemma 5.9 Let $P \geq 5$ be an odd integer with $\gcd(P, 6) = 1$. Then there exists a 2-PARB$(3^n P + 1, 2, 1)$ for any positive integer n.

Proof. Let $n = 1$. Then the existing $(3P, 3, 2, 1)$-PACDF obtained by Lemma 4.2 and the 2-PARB$(4, 2, 1)$ given in Example A.1 show the existence of the required design by Lemma 5.3.

When $n \geq 2$, Lemma 4.10 can provide a $(3^nP, P, 2, 1)$-PACDF. On the other hand, a 2-PARB$(P+1, 2, 1)$ can be given by Theorem 5.1. Hence, Lemma 5.3 can be used to show the existence of a 2-PARB$(3^nP+1, 2, 1)$. □

Lemma 5.10 Let $P \geq 5$ be an odd integer with $\gcd(P, 6) = 1$. Then there exists a 2-PARB$(2^m3^nP+1, 2, 1)$ for any positive integers m and n.

Proof. Let $m \geq 1$ and $n = 1$. Then a $(2^m \cdot 3P, 2^m \cdot 3, 2, 1)$-PACDF can be given by Lemma 4.11. Furthermore Lemmas 5.3 and 5.7 show the existence of a 2-PARB$(2^m \cdot 3P+1, 2, 1)$.

When $m \geq 1$ and $n \geq 2$, a $(2^m \cdot 3^nP, 2^mP, 2, 1)$-PACDF can be obtained by Lemma 4.12. Hence, Lemmas 5.3 and 5.8 can be used to show the existence of a 2-PARB$(2^m \cdot 3^nP+1, 2, 1)$. □

Finally, the main result is now established as in Theorem 1.3 by taking Theorem 2.2 and Lemmas 5.1 and 5.5 to 5.10.

Proof of Theorem 1.3. When $\gcd(v-1, 6) = 1$, Lemma 5.1 shows the existence of a 2-PARB$(v, 2, 1)$. If $\gcd(v-1, 6) \neq 1$, then $v - 1 = 2^m3^n$ or 2^m3^nP, where $m \geq 0, n \geq 0, (m, n) \neq (0, 0)$ and $P \geq 5$ is any odd integer such that $\gcd(P, 6) = 1$. Then by using Lemmas 5.5 to 5.10 the existence of a 2-PARB$(v, 2, 1)$ is shown for any $v \geq 4$. This fact with Theorem 2.2 completes the proof. □

Remark. Some results on the existence of a 2-PACB$(v, 2, 1)$ are obtained in [12, 15]. Furthermore, some methods of constructing a 2-PARB$(v, 2, 1)$ given in this paper can be used to construct 2-PACB$(v, 2, 1)$'s. As a result, Theorem 1.2 on the existence of 2-PACB designs would be improved. Even so, we cannot show the existence of a 2-PACB$(v, 2, 1)$ for *any* v. The existence problem of this cyclic type will be discussed in a forthcoming paper.

Appendix

Some individual examples which can be found by use of a computer are presented. Note that each of these examples cannot be given by the construction methods provided in this paper.

Example A.1 An ARB$(4, 2, 1)$ on $Z_3 \cup \{\infty\}$:

$$\begin{aligned} \mathcal{B}_1 &: \{0, \infty\}, \{1, 2\} \quad \text{mod } 3 \\ \mathcal{B}_2 &: \{1, 2\}, \{0, \infty\} \quad \text{mod } 3. \end{aligned}$$

Example A.2 A 3-PARB$(7, 2, 1)$ on $Z_6 \cup \{\infty\}$:

$$\begin{aligned} \mathcal{B}_1 &: \{0, \infty\}, \{0, 1\}, \{0, 2\}, \{0, 3\}\text{PC}(3) \quad \text{mod } 6 \\ \mathcal{B}_2 &: \{1, 3\}, \{2, \infty\}, \{4, 5\}, \{1, 4\}\text{PC}(3) \quad \text{mod } 6 \\ \mathcal{B}_3 &: \{4, 5\}, \{3, 5\}, \{3, \infty\}, \{2, 5\}\text{PC}(3) \quad \text{mod } 6. \end{aligned}$$

Example A.3 A 2-PARB$(10, 2, 1)$ on $Z_9 \cup \{\infty\}$:

$$\begin{aligned} \mathcal{B}_1 &: \{0,\infty\}, \{0,1\}, \{0,2\}, \{0,3\}, \{0,4\} \quad \text{mod } 9 \\ \mathcal{B}_2 &: \{4,7\}, \{2,4\}, \{3,4\}, \{4,\infty\}, \{3,7\} \quad \text{mod } 9. \end{aligned}$$

Example A.4 A 2-PARB$(11, 2, 1)$ on $Z_{10} \cup \{\infty\}$:

$$\begin{aligned} \mathcal{B}_1 &: \{0,\infty\}, \{0,1\}, \{0,2\}, \{0,3\}, \{0,4\}, \{0,5\}\text{PC}(5) \quad \text{mod } 10 \\ \mathcal{B}_2 &: \{4,7\}, \{7,9\}, \{8,9\}, \{4,8\}, \{5,\infty\}, \{2,7\}\text{PC}(5) \quad \text{mod } 10. \end{aligned}$$

Example A.5 A 2-PARB$(13, 2, 1)$ on $Z_{12} \cup \{\infty\}$:

$$\begin{aligned} \mathcal{B}_1 &: \{0,\infty\}, \{0,1\}, \{0,2\}, \{0,3\}, \{0,4\}, \{0,5\}, \\ &\quad \{0,6\}\text{PC}(6) \quad \text{mod } 12 \\ \mathcal{B}_2 &: \{6,8\}, \{2,11\}, \{6,\infty\}, \{10,5\}, \{7,3\}, \{8,9\}, \\ &\quad \{1,7\}\text{PC}(6) \quad \text{mod } 12. \end{aligned}$$

Example A.6 A 2-PARB$(15, 2, 1)$ on $Z_{14} \cup \{\infty\}$:

$$\begin{aligned} \mathcal{B}_1 &: \{0,\infty\}, \{0,1\}, \{0,2\}, \{0,3\}, \{0,4\}, \{0,5\}, \{0,6\}, \\ &\quad \{0,7\}\text{PC}(7) \quad \text{mod } 14 \\ \mathcal{B}_2 &: \{7,10\}, \{3,12\}, \{1,9\}, \{9,11\}, \{1,\infty\}, \{9,10\}, \{4,8\}, \\ &\quad \{6,13\}\text{PC}(7) \quad \text{mod } 14. \end{aligned}$$

Example A.7 A 2-PARB$(19, 2, 1)$ on $Z_{18} \cup \{\infty\}$:

$$\begin{aligned} \mathcal{B}_1 &: \{0,\infty\}, \{0,1\}, \{0,2\}, \{0,3\}, \{0,4\}, \{0,5\}, \{0,6\}, \\ &\quad \{0,7\}, \{0,8\}, \{0,9\}\text{PC}(9) \quad \text{mod } 18 \\ \mathcal{B}_2 &: \{7,10\}, \{4,12\}, \{7,\infty\}, \{9,16\}, \{1,5\}, \{9,10\}, \{2,8\}, \\ &\quad \{1,3\}, \{2,15\}, \{1,10\}\text{PC}(9) \quad \text{mod } 18. \end{aligned}$$

The following examples of PACDFs are used for recursive constructions in Section 4.

Example A.8 A $(12, 2, 2, 1)$-PACDF:

$$\begin{aligned} \mathcal{F}_1 &: \{0,1\}, \{0,2\}, \{0,3\}, \{0,4\}, \{0,5\} \\ \mathcal{F}_2 &: \{5,10\}, \{7,11\}, \{2,4\}, \{8,11\}, \{2,3\}. \end{aligned}$$

Example A.9 A $(27, 3, 2, 1)$-PACDF:

$$\begin{aligned} \mathcal{F}_1 &: \{0,1\}, \{0,2\}, \{0,3\}, \{0,4\}, \{0,5\}, \{0,6\}, \{0,7\}, \{0,8\}, \\ &\quad \{0,10\}, \{0,11\}, \{0,12\}, \{0,13\} \\ \mathcal{F}_2 &: \{6,7\}, \{7,26\}, \{8,13\}, \{2,17\}, \{17,24\}, \{1,7\}, \{19,23\}, \{14,24\}, \\ &\quad \{3,14\}, \{15,17\}, \{1,4\}, \{11,25\}. \end{aligned}$$

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