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The structure of strong tournaments containing exactly one out-arc pancyclic vertex*

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Abstract

An arc in a tournament T with $n \geq 3$ vertices is called pancyclic if it belongs to a cycle of length l for all $3 \leq l \leq n$. We call a vertex u of T an out-arc pancyclic vertex of T if each out-arc of u is pancyclic in T. Yao, Guo and Zhang [*Discrete Appl. Math.* 99 (2000), 245–249] proved that every strong tournament contains at least one out-arc pancyclic vertex, and they gave an infinite class of strong tournaments, each of which contains exactly one out-arc pancyclic vertex. In this paper we give the structure of strong tournaments containing exactly one out-arc pancyclic vertex.

1 Introduction

Let D be a digraph with the vertex set $V(D)$ and the arc set $A(D)$. We denote the number of vertices in D by $|V(D)|$. A subdigraph induced by a subset $A \subseteq V(D)$ is denoted by $D[A]$. We also write $D - A$ for $D[V(D) - A]$.

A *tournament* is a digraph, where there is precisely one arc between every pair of distinct vertices. An *l-cycle* is a cycle of length l. An arc in a digraph D is said to be *pancyclic*, if it belongs to an l-cycle for all $3 \leq l \leq n$. An arc leaving from a vertex x in a digraph is called an *out-arc* of x. We call a vertex u of T an out-arc pancyclic vertex of T, if each out-arc of u is pancyclic in T.

A digraph D is said to be *strong*, if for every pair of vertices x and y, D contains a path from x to y and a path from y to x. A directed path from x to y in D is

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denoted by (x, y)-path. D is called *k-strong* if $|V(D)| \geq k + 1$ and $D - X$ is strong for any set $X \subseteq V(D)$ with $|X| < k$. If a digraph is strong, it is 1-strong. If a digraph D is k-strong, but not $(k+1)$-strong, then $\kappa(D) = k$ is defined as the *strong connectivity* of D.

In [4], Thomassen confirmed that every strong tournament contains a vertex x such that each out-arc of x is contained in a Hamilton cycle. In 2000, Yao et al.[5] extended the result of Thomassen and proved that every strong tournament T has an out-arc pancyclic vertex. In the same paper, they also found an infinite class of strong tournaments, each of which contains exactly one out-arc pancyclic vertex (see Figure 1). For a strong tournament T with minimum out-degree at least two, Guo et al. [3] proved that T contains at least three out-arc pancyclic vertices. This implies that a strong tournament containing exactly one out-arc pancyclic vertex can only be 1-strong.

Example. (Yao [5]). Let $n \geq 5$ be an integer and let T_n be a tournament with the vertex set $\{v_1, v_2, \ldots, v_n\}$ and the arc set

$$\{v_i v_j \mid 2 \leq i < j \leq n\} \cup \{v_{n-1}v_1, v_n v_1\} \cup \{v_1 v_j \mid 2 \leq j \leq n-2\}.$$

Then T_n contains exactly one vertex v_n whose out-arcs are pancyclic.

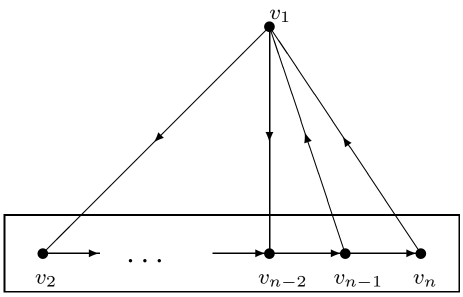

Figure 1: A strong tournament T containing exactly one out-arc pancyclic vertex, where $v_i \to v_j$ for all $2 \leq i < j \leq n$.

In this paper, we obtain a sufficient and necessary condition for a strong tournament to contain exactly one out-arc pancyclic vertex.

2 Terminology and Preliminaries

We assume that the reader is familiar with the standard terminology on digraphs and refer the reader to [1].

If uv is an arc of a digraph D, then we say that u *dominates* v or uv is an *out-arc* of u, and we write $u \to v$. For disjoint subsets X and Y of $V(D)$, if every vertex of X dominates every vertex of Y, we say X *dominates* Y and write $X \to Y$. If there is no arc from a vertex in Y to a vertex in X, we say $X \Rightarrow Y$. For a tournament, $X \to Y$ if and only if $X \Rightarrow Y$.

Let $x \in V(D)$ and H be a subdigraph of D. The set of all vertices in H dominating x (dominated by x, respectively) is denoted by $N_H^-(x)$ ($N_H^+(x)$, respectively). Furthermore, $d_H^-(x) = |N_H^-(x)|$ ($d_H^+(x) = |N_H^+(x)|$, respectively) is called the *in-degree* (*out-degree*, respectively) of x in H. We will omit the subscript if $H = D$. We use $\delta^+(D) = \min\{d_D^+(x) \mid x \in V(D)\}$ to stand for the *minimum out-degree* of D.

A path (cycle) containing every vertex of D is called a *Hamilton path (Hamilton cycle)* of D. For two vertices u, v on a path P, we let uPv denote the unique (u, v)-path on P.

A set $X \subseteq V(D)$ is called a *separating set* if $D - X$ is not strong. A vertex $u \in V(D)$ is also called a *cut vertex* of D if $D - \{u\}$ is not strong. A *strong component* of a digraph D is a maximal induced subdigraph of D which is strong. If D is not strong, then we can partition the vertices in D into strong components $T_1, T_2, \ldots, T_r$ ($r \geq 2$), such that $T_i \Rightarrow T_j$ if and only if $i < j$.

In the proofs of our main results, the following results are very useful.

Lemma 2.1 (Camion [2]). *A non-trivial tournament is strong if and only if it has a Hamilton cycle.*

Lemma 2.2 (Yao [5]). *Let T_n be a strong tournament on n vertices and assume that the vertices of T_n are labeled $u_1, u_2, \ldots, u_n$ such that $d^+(u_1) \leq d^+(u_2) \leq \ldots \leq d^+(u_n)$. If $d^+(u_1) = 1$, then all out-arcs of u_1 are pancyclic.*

Lemma 2.3 (Guo [3]). *Every strong tournament T with minimum out-degree at least two contains at least three out-arc pancyclic vertices.*

Lemma 2.4 (Yeo [6]). *Let D be a k-strong digraph, with $k \geq 1$, and let S be a separating set in D, such that $T = D - S$ is a tournament. Let $T_1, T_2, \ldots, T_r$ ($r \geq 2$) be the strong components of T, such that $T_1 \Rightarrow T_2 \Rightarrow \cdots \Rightarrow T_r$. Then, for every $1 \leq l \leq |V(T)| - 1$, $u \in T_1$ and $v \in T_r$, there exists a (u, v)-path of length l in T.*

3 Main results

Theorem 3.1. *Let T be a strong tournament with strong connectivity $\kappa(T) = 1$. Let v be a cut vertex of T and let $T_1, T_2, \ldots, T_r$ ($r \geq 2$) be the strong components of $T - v$, such that $T_i \Rightarrow T_j$ if and only if $i < j$. Let $x \in V(T_{r-1})$. Then the following hold:*

(1) If $V(T_{r-1}) = \{x\}$, then x is an out-arc pancyclic vertex of T if and only if $V(T_r) \to v \to x$.

(2) If $|V(T_{r-1})| > 1$, then x is an out-arc pancyclic vertex of T if and only if $V(T_r) \to v \to x$ and for each $y \in N^+_{T_{r-1}}(x)$, there exists a Hamilton path containing xy in T_{r-1} and there exists a 3-cycle containing xy in $D[V(T_{r-1}) \cup \{v\}]$.

Proof. Since T is strong, there is a vertex u in T_1 such that $v \to u$.

(1) (Sufficiency.) Suppose $V(T_{r-1}) = \{x\}$ and $V(T_r) \to v \to x$. Let $x_1 \in N^+(x)$ be arbitrary. Then $x_1 \in V(T_r)$ and $x_1 \to v$. It is clear that xx_1vx is a 3-cycle containing xx_1. Using Lemma 2.1 on each component, it is easy to prove that there exists a (u, x)-path P_l of length l in T for every $1 \leq l \leq |V(T)| - |V(T_r) \cup \{v\}| - 1$. Thus xx_1vuP_lx is a cycle of length $l + 3$ containing xx_1 for every $1 \leq l \leq |V(T)| - |V(T_r)| - 2$. Suppose that $x_1x_2 \dots x_{|V(T_r)|}x_1$ is a Hamilton cycle of T_r. Then $xx_1x_2 \dots x_ivuP_{|V(T)|-|V(T_r)|-2}x$ is a cycle of length $|V(T)| - |V(T_r)| + i$ containing xx_1 for every $2 \leq i \leq |V(T_r)|$. Therefore, xx_1 is contained in an m-cycle for every $m \in \{3, 4, \dots, |V(T)|\}$, we note that xx_1 is pancyclic and x is an out-arc pancyclic vertex of T.

(Necessity.) Now, let $V(T_{r-1}) = \{x\}$ and x is an out-arc pancyclic vertex of T. We will prove $V(T_r) \to v \to x$. Let $z \in V(T_r)$ be arbitrary. Then $z \in N^+(x)$ and xz is pancyclic in T. So xz is in a 3-cycle. Note that $T_i \Rightarrow T_j$ if and only if $i < j$. This implies that $z \to v \to x$. That is, we have that $V(T_r) \to v \to x$.

(2) (Sufficiency.) Suppose that $|V(T_{r-1})| > 1$, $V(T_r) \to v \to x$ and for each $y \in N^+_{T_{r-1}}(x)$, there exists a Hamilton path containing xy in T_{r-1} and there exists a 3-cycle containing xy in $D[V(T_{r-1}) \cup \{v\}]$. We will prove that x is an out-arc pancyclic vertex of T. Let $w \in N^+(x)$ be arbitrary. Then $w \in V(T_{r-1}) \cup V(T_r)$. We only need to prove that xw is pancyclic in T.

If $w \in V(T_{r-1})$, by assumption, there exists a Hamilton path containing xw in T_{r-1} and there exists a 3-cycle containing xw in $D[V(T_{r-1}) \cup \{v\}]$. Using Lemma 2.1 on each component except T_{r-1} and using the Hamilton path containing xw in T_{r-1}, it is easy to prove that, for each $z \in V(T_r)$, there exists a (u, z)-path P_l of length l in T containing xw for every $3 \leq l \leq |V(T)| - 2$. Thus uP_lzvu is a cycle of length $l + 2$ containing xw for every $3 \leq l \leq |V(T)| - 2$. Note that $xwzvx$ is a 4-cycle containing xw for any $z \in V(T_r)$ and there exists a 3-cycle containing xw in $D[V(T_{r-1}) \cup \{v\}]$. Therefore, xw is contained in an m-cycle for every $m \in \{3, 4, \dots, |V(T)|\}$, we note that xw is pancyclic.

If $w \in V(T_r)$, then $w \to v$. It is clear that $xwvx$ is a 3-cycle containing xw. Using Lemma 2.1 on each component except T_r, it is easy to prove that there exists a (u, x)-path P_l of length l in T for every $1 \leq l \leq |V(T)| - |V(T_r) \cup \{v\}| - 1$. Thus $xwvuP_lx$ is a cycle of length $l + 3$ containing xw for every $1 \leq l \leq |V(T)| - |V(T_r)| - 2$. Suppose that $x_1x_2 \dots x_{|V(T_r)|}$ is a Hamilton cycle of T_r with $x_1 = w$. Then $xx_1x_2 \dots x_ivuP_{|V(T)|-|V(T_r)|-2}x$ is a cycle of length $|V(T)| - |V(T_r)| + i$ containing xw for every $2 \leq i \leq |V(T_r)|$. Therefore xw is contained in an m-cycle for every $m \in \{3, 4, \dots, |V(T)|\}$, and we note that xw is pancyclic.

(Necessity.) Let $x \in V(T_{r-1})$ be an out-arc pancyclic vertex of T. We first prove $V(T_r) \to v \to x$. Let $z \in V(T_r)$ be arbitrary. Then $z \in N^+(x)$ and xz is pancyclic in T. So xz is in a 3-cycle which implies that $z \to v \to x$. That is, we have that $V(T_r) \to v \to x$.

For each $y \in N^+_{T_{r-1}}(x)$, we have that xy is pancyclic in T, and so xy is contained in an m-cycle for every $m \in \{3, 4, \dots, |V(T)|\}$. Let $C = vx_1x_2 \dots x_{|V(T)|-1}v$ be a

Hamilton cycle containing xy with some $x_k = x$ and $x_{k+1} = y$. Suppose that starting from v, x_i is the first vertex and x_j is the last one appearing in $V(T_{r-1})$ on C. Then $i \leq k < j$ and $x_i x_{i+1} \dots x_j$ is a Hamilton path containing xy of T_{r-1}. Since xy is in a 3-cycle, it is clear that such a 3-cycle can only be contained in the $D[V(T_{r-1}) \cup \{v\}]$. □

Theorem 3.2. *Let T be a strong tournament with n vertices and strong connectivity $\kappa(T)$. Then T contains exactly one out-arc pancyclic vertex if and only if all of the following hold.*

(1) $n \geq 5$.

(2) $\kappa(T) = 1$ *and there exists a cut vertex v of T and strong components $T_1, T_2, \dots, T_r$ $(r \geq 3)$ of $T - v$ such that $T_i \Rightarrow T_j$ if and only if $i < j$ and $|V(T_r)| = 1$.*

(3) There exists a vertex $z \in V(T_2) \cup \dots \cup V(T_{r-1})$ such that $v \to z$.

(4) If $V(T_{r-1}) = \{u_1\}$, then $u_1 \to v$. If $|V(T_{r-1})| > 1$, then, for any $x \in V(T_{r-1})$, we have $x \to v$ or there exists a vertex $y \in N^+_{T_{r-1}}(x)$ such that there is no Hamilton path containing xy in T_{r-1} or there exists a vertex $z \in N^+_{T_{r-1}}(x)$ such that there is no 3-cycle containing xz in $D[V(T_{r-1}) \cup \{v\}]$.

Proof. (Sufficiency.) Let T be a strong tournament with n vertices and strong connectivity $\kappa(T)$ and T satisfies (1)–(4). By (2), assume that v is a cut vertex of T and $T_1, T_2, \dots, T_r$ $(r \geq 3)$ are strong components of $T - v$ such that $T_i \Rightarrow T_j$ if and only if $i < j$ and $|V(T_r)| = 1$. Let $V(T_r) = \{u\}$. Then $d^+(u) = 1$ and Lemma 2.2 implies that u is an out-arc pancyclic vertex of T. We will prove that any vertex except u in T is not an out-arc pancyclic vertex of T. By (3), there exists a vertex $z \in V(T_2) \cup \dots \cup V(T_{r-1})$ such that $v \to z$. Then vz can not be contained in any Hamilton cycle of T and so v is not an out-arc pancyclic vertex of T. Let $x \in V(T_1) \cup \dots \cup V(T_{r-2})$ be arbitrary. It is easy to see that xu can not be contained in any Hamilton cycle of T and so x is also not an out-arc pancyclic vertex of T. We consider the vertices of T_{r-1}. By (4) and Theorem 3.1, we have that any vertex T_{r-1} is not an out-arc pancyclic vertex of T. Thus, u is an unique out-arc pancyclic vertex of T.

(Necessity.) Let T be a strong tournament with n vertices and strong connectivity $\kappa(T)$ which contains exactly one out-arc pancyclic vertex. By lemma 2.3, T is a strong tournament with minimum out-degree 1. Let M be the set of vertices with out-degree 1 in T. By Lemma 2.2, all vertices in M are out-arc pancyclic vertices of T. So we have $|M| = 1$. If $n \leq 4$, then it is clear that $|M| \geq 2$, a contradiction. Therefore, we have $n \geq 5$ and (1) is proved.

Let $M = \{u\}$ and $N^+(u) = \{v\}$. Then v is a cut vertex of T and $\kappa(T) = 1$. Let $T_1, T_2, \dots, T_r$ $(r \geq 2)$ be the strong components of $T - v$, such that $T_i \Rightarrow T_j$ if and only if $i < j$. Obviously, $V(T_r) = \{u\}$ and u is an out-arc pancyclic vertex of T. If $r = 2$, it is easy to see that v is also an out-arc pancyclic vertex of T, a contradiction. So we have $r \geq 3$. (2) is proved.

If $V(T_2) \cup \cdots \cup V(T_{r-1}) \Rightarrow \{v\}$, then $N^+(v) \subseteq V(T_1)$. Let $v_1 \in N^+(v)$ be arbitrary. By Lemma 2.4, there exists a (v_1, u)-path P_l of length l in T for every $1 \leq l \leq |V(T)| - 2$. So vv_1P_luv is an $(l + 2)$-cycle containing vv_1 for every $1 \leq l \leq |V(T)| - 2$, which implies that v is an out-arc pancyclic vertex of T, a contradiction. Therefore, there exists a vertex $z \in V(T_2) \cup \cdots \cup V(T_{r-1})$ such that $v \to z$ and (3) is proved.

Since u is the unique out-arc pancyclic vertex of T, we have that no vertex in T_{r-1} is an out-arc pancyclic vertex of T. Then (4) is obvious by using Theorem 3.1. □

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