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# Recursive formulas for embedding distributions of cubic outerplanar graphs

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**Abstract**

Recently, the first author and his coauthor proved a $k^{\text{th}}$-order homogeneous linear recursion for the genus polynomials of any H-linear family of graphs (called path-like graph families by Mohar). Cubic outerplanar graphs are tree-like graph families. In this paper, we derive a recursive formula for the total embedding distribution of any cubic outerplanar graph. We also obtain explicit formulas for the number of embeddings of cubic outerplanar graphs into the plane, torus, projective plane and Klein bottle. In addition, we present a $O\left(n(h+\Delta)\right)$-time algorithm to compute the genus distribution and the crosscap number distribution of any cubic outerplanar graph, where $h$ and $\Delta$ are the *height* and *maximum degree* of the characteristic tree, respectively. We have written an efficient enumeration program in C++ for computing this recursive function and constructing tables of genus distributions of cubic outerplanar graphs. Our program is documented and available on request.

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# 1 Introduction

Counting graph embeddings on surfaces has frequently been investigated in the past quarter century. It has many connections with various other areas of mathematics, such as the characters of symmetric groups, geometry and topology. For the applications of genus distribution into physics, we may call attention to a paper of Visentin and Wieler [10]. It is well-known that the genus distribution is NP-complete. However, as noted by Gross [7], the genus distribution of a graph with bounded tree-width and bounded degree has a polynomial time algorithm. It is known that outerplanar graphs have tree-width at most 2. In [8], Gross presented a quadratic-time algorithm for computing the genus distribution of any cubic outerplanar graph. This is the first class of graphs whose genus distribution is known to be computable in polynomial time. The above results were obtained by the powerful techniques which were developed by Gross. In this paper, we focus our attention on the total embedding distributions of cubic outerplanar graphs.

A *graph* $G = (V(G), E(G))$ may have both loops and multiple edges. A *surface* is a compact 2-manifold without boundary. In topology theory, compact and connected surfaces are classified into the *orientable surfaces* $S_g$, with $g$ handles ($g \geq 0$), and the *nonorientable surfaces* $N_k$, with $k$ crosscaps ($k > 0$). A *graph embedding* into a surface means a *cellular embedding*. For any spanning tree of $G$, the number of cotree edges is called the *Betti number* of $G$, denoted by $\beta(G)$.

A *rotation at a vertex* $v$ of a graph $G$ is a cyclic order of all edge-ends incident with $v$. A *pure rotation system* $\rho$ of a graph $G$ is an assignment of a rotation to each vertex of $G$. As there are two rotations of each trivalent vertex, we color a vertex *black* if the rotation of the edge-ends incident on it is *clockwise*, and we color it *white* if the rotation is *counterclockwise*. We call any drawing of a graph that uses this convention to indicate a rotation system a *Gustin coloring* (called a Gustin representation in [4]). It follows that there is a bijection between the pure rotation systems of a cubic graph and the Gustin colorings of its vertices. Under this convention, we use Gustin colorings to indicate the pure rotation systems of a cubic graph.

A *general rotation system* for a graph $G$ is a pair $(\rho, \lambda)$, where $\rho$ is a pure rotation system and $\lambda$ is a mapping: $E(G) \longrightarrow \{0, 1\}$. The edge $e$ is said to be *twisted* (respectively, untwisted) if $\lambda(e) = 1$ (respectively, $\lambda(e) = 0$). It is well-known that every orientable embedding of a graph $G$ can be described uniquely by a pure rotation system. By allowing the parameter $\lambda$ to take non-zero values, we can describe the non-orientable embeddings of a graph $G$.

For any fixed spanning tree $T$, a *$T$-rotation system* $(\rho, \lambda)$ of $G$ is a general rotation system $(\rho, \lambda)$ such that $\lambda(e) = 0$ for all $e \in E(T)$. Two embeddings of $G$ are considered to be *equivalent* if their $T$-rotation systems are the same. Let $\Phi_G^T$ denote the set of all $T$-rotation systems of $G$. Suppose that among these $|\Phi_G^T|$ embeddings of $G$, there are $a_i$ embeddings, for $i = 0, 1, \ldots$, into the orientable surface $S_i$, and that there are $b_j$ embeddings, for $j = 1, 2, \ldots$, into the non-orientable surface $N_j$.

We call the bivariate polynomial

$$\mathbb{I}_G^T(x,y) = \sum_{i=0}^{\infty} a_i x^i + \sum_{j=1}^{\infty} b_j y^j$$

the *T-distribution polynomial* of $G$.

It should be noted that the $T$-distribution polynomial is independent of the choice of spanning tree $T$. Thus, we define the *total embedding polynomial* of $G$ to be the bivariate polynomial $\mathbb{I}_G(x,y) = \mathbb{I}_G^T(x,y)$, for any choice of a spanning tree $T$. The *genus distribution* and *crosscap number distribution* are defined to be the sequences $\{a_i(G)|i \geq 0\}$ and $\{b_j(G)|j \geq 1\}$, respectively. The sequence $\{a_i(G)|i \geq 0\} \cup \{b_j(G)|j \geq 1\}$ is called the *total embedding distribution* of the graph $G$. Also, we call the first and second parts of $\mathbb{I}_G(x,y)$ the *genus polynomial* of $G$ and the *crosscap number polynomial* of $G$, respectively, and we denote them by $g_G(x) = \sum_{i=0}^{\infty} a_i x^i$ and $f_G(y) = \sum_{i=1}^{\infty} b_i y^i$, respectively. Thus, we have $\mathbb{I}_G(x,y) = g_G(x) + f_G(y)$.

**Remark 1.1.** An example from Chen's thesis [1] shows that the crosscap number distribution is more difficult than the genus distribution. However the recent approach of Chen and Gross [3] shows this is just a good disguise, or alternatively, that there is not much difference in the difficulty.

A *bar-amalgamation* $G \oplus_e H$ of two disjoint graphs $H$ and $G$ is obtained by running an edge between a vertex of $G$ and a vertex of $H$. The following two theorems can be founded in [5] and [2].

**Theorem 1.2.** *(See [5])* $g_{G\oplus_e H}(x) = d_G(u)d_H(v)g_G(x)g_H(x)$, *where* $d_G(u)$ *is the vertex degree of* $u$ *in* $G$ *and* $d_G(v)$ *is the vertex degree of* $v$ *in* $H$.

**Theorem 1.3.** *(See [2])*

$$f_{G\oplus_e H}(y) = d_G(u)d_H(v)\Big[f_G(y)f_H(y) + f_G(y)g_H(y^2) + g_G(y^2)f_H(y)\Big],$$

*where* $d_G(u)$ *is the vertex degree of* $u$ *in* $G$ *and* $d_G(v)$ *is the vertex degree of* $v$ *in* $H$.

Note that a 3-regular graph $G$ is 2-edge connected if and only if $G$ is 2-connected. It follows from Theorem 1.2 and Theorem 1.3 that all cubic outerplanar graphs in the paper are 2-connected.

## 2 Total embedding polynomials for cubic outerplanar graphs

### 2.1 Characteristic tree

A graph is an *outerplanar graph* if it can be embedded in the plane without crossings in such a way that all of the vertices belong to the unbounded region $f_\infty$ of the embedding. An outerplane embedding is said to be *normalized* if all loops of the graph lie on the face-boundary walk of the unbounded region $f_\infty$. By an *outerplane graph* we mean an outerplanar graph with a fixed outerplane embedding. A tree is called a *rooted tree* if one vertex has been designated the root, in which case the edges have a natural orientation, towards or away from the root.

The *dual* $G^{**}$ of a planar graph $G$ is a graph defined as follows: each region in $G$ is a vertex in $G^{**}$, and two vertices in $G^{**}$ are adjacent if and only if the regions share an edge in $G$. The *weak dual* $G^*$ of $G$ is obtained from the dual $G^{**}$ by removing the vertex corresponding to the unbounded region $f_\infty$ in $G$. It is easy to see that the weak dual of an outerplanar graph is a plane tree, since a cycle in $G^*$ would represent a set of bounded regions in $G$ that separate a vertex $v$ from the unbounded region. We call this plane tree a *characteristic tree* of the outerplanar embedding of $G$, as shown in Figure 1. It follows that a characteristic tree $(T, \rho)$ is a tree $T$ with a rotation system $\rho$.

If $G = G_1 \cup G_2$ and $G_1 \cap G_2 = \{e\}$, then we say that $G$ is the *edge-amalgamation* of $G_1$ and $G_2$ on the edge $e$. The *preorder traversal* of a rooted tree $T$ with $n$ vertices is defined recursively as follows:

*Basis:* If $n = 1$, then the root is the only vertex, so we traverse the root.

*Recursive Step:* When $n > 1$, consider the subtrees $T_1, T_2, T_3, \ldots, T_k$ of $T$ whose roots are all the children of the root of $T$. Traverse each of these subtrees from left to right.

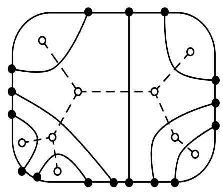

Figure 1: An outerplanar graph (solid lines) and its characteristic tree (dashed lines)

**Property 2.1.** *There is a mapping between all cubic outerplanar graphs and all trees $(T, \rho)$, where $\rho$ is a pure rotation system of $T$.*

*Proof.* Given a cubic outerplanar graph, by definition, its characteristic tree $(T, \rho)$ is uniquely determined.

Furthermore, once the plane tree $(T, \rho)$ is given, we replace each vertex $v$ of $(T, \rho)$ by the cycle graph $C_{2d(v)}$ with $2d(v)$ vertices. According to the preorder traversal, we can obtained an outerplanar graph by a series of edge amalgamations of the cycle graphs, as shown in Figure 2. The result follows. □

In the following discussion, a cubic outplanar graph will be denoted by $(T, \rho)$.

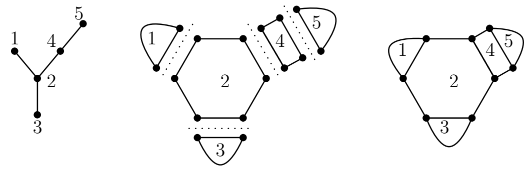


Figure 2: From a tree to a cubic outerplanar graph

## 2.2 Overlap matrices for cubic outerplanar graphs

Let $T$ be a spanning tree of a graph $G$ and let $(\rho, \lambda)$ be a $T$-rotation system. Let $e_1, e_2, \ldots, e_{\beta(G)}$ be the cotree edges of $T$, where $\beta(G)$ is the cycle rank of $G$. The *overlap matrix* of $(\rho, \lambda)$ is the $\beta(G) \times \beta(G)$ matrix $M = [m_{ij}]$ over $\mathbb{Z}_2 = \{0, 1\}$ such that

$$m_{ij} = \begin{cases} 1, & \text{if } i = j \text{ and } e_i \text{ is twisted;} \\ 1, & \text{if } i \neq j \text{ and the restriction of the underlying pure} \\ & \text{rotation system to the subgraph } T + e_i + e_j \text{ is nonplanar;} \\ 0, & \text{otherwise.} \end{cases}$$

When the restriction of the underlying pure rotation system to the subgraph $T+e_i+e_j$ is nonplanar, we say that edges $e_i$ and $e_j$ *overlap*. The following theorem is obtained by Mohar [9].

**Theorem 2.2.** *Let $(\rho, \lambda)$ be a general rotation system for a graph. Then the rank of any overlap matrix $M$ for the corresponding embedding equals twice the genus of the embedding surface, if that surface is orientable, and it equals the crosscap number otherwise. The rank is independent of the choice of a spanning tree.*

Note that the degree of a vertex in the dual graph $G^{**}$ of an outerplane graph $G$ is at most the size of the corresponding region in $G$, especially when $G$ is a cubic outerplanar graph, and the degree of a vertex in the characteristic tree is half the size of the corresponding region in $G$. Let $G$ be a cubic outerplanar graph and let $T$ be its characteristic tree. Suppose a vertex $v$ in $T$ has degree $d$; we divide the edges of the corresponding region $f_v$ into two parts, one part belonging to the intersection of $f_v$ and the unbounded region $f_\infty$, the other part belonging to the intersection of $f_v$ and the bounded regions. We call the part of edges that belongs to $f_\infty$ *adjoint edges*. Recall that a *chord* is an edge joining two non-adjacent vertices in a cycle. We have the following property.

**Property 2.3.** For a cubic outerplanar graph $G$, a vertex $v$ of degree $d(v)$ in the characteristic tree of $G$ corresponding to $d(v)$ adjoint edges in $G$.

*Proof.* Suppose the degree of vertex $v$ is $d(v)$. By the definition of a characteristic tree, the vertex $v$ crosses $d(v)$ chords of $G$. Note that each region of the outerplanar graph is a cycle, since all the chords are independent and the degree of $v$ in a characteristic tree is half the size of the corresponding region in $G$; so the property follows. □

In a Gustin coloring for a cubic graph, an edge is called *matched* if it has the same color at both endpoints; otherwise, it is called *unmatched*. In the following discussion, we label the adjoint edges of the vertex $v$ by $e_v^1, e_v^2, \ldots, e_v^{d(v)}$ in a counterclockwise order, as shown in Figure 3.

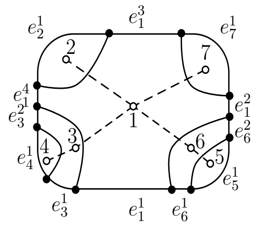


Figure 3: The adjoint edges of a cubic outerplanar graph

A *matching* in a graph is a set of edges having no common vertices. A *perfect matching* is a matching which matches all vertices of the graph. Let $G_{2n}$ be a cubic outerplanar graph with $2n$ vertices. By Euler's formula, the number of vertices of the weak dual $T$ of $G_{2n}$ is $n+1$. By the definition of weak dual of a cubic outerplanar graph, each chord intersects one edge of $T$; this means the number of chords is $n$. Furthermore, all the chords of $G_{2n}$ form a perfect match. Let $v_1, v_2, \ldots, v_{n+1}$ be vertices of $T$. By Property 2.3, each vertex $v_i$ corresponds to $d(v_i)$ adjoint edges in $G$. For each vertex $v_i$ $(1 \le i \le n+1)$, we delete an adjoint edge $e_{v_i}^1$ of $v_i$, and the resulting graph is a spanning tree of $G_{2n}$. Thus we assume $e_{v_1}^1, e_{v_2}^1, \ldots, e_{v_{n+1}}^1$ are the cotree edges of $G_{2n}$. Let $M_n(e_{v_1}^1, e_{v_2}^1, \ldots, e_{v_{n+1}}^1)$ be the overlap matrices of $G_{2n}$ over $\mathbb{Z}_2$. By face-tracing, we have the following result.

**Lemma 2.4.** *Two cotree edges $e_{v_i}^1$ and $e_{v_j}^1$ overlap if and only if $v_i$ and $v_j$ are adjacent in $T$ and the edge crossing $v_iv_j$ is unmatched $(i \neq j)$.*

The following property follows directly from the lemma above. We give a detailed proof here.

**Lemma 2.5.** *For a fixed overlap matrix of the form $M_n(e_{v_1}^1, e_{v_2}^1, \ldots, e_{v_{n+1}}^1)$, corresponding to a spanning tree $T$ in a cubic outerplanar graph $G_{2n}$ there are exactly $2^n$ different $T$-rotation systems corresponding to the matrix.*

*Proof.* Note that there are four different assignments of colors to a chord of $G_{2n}$; two of them are matched while the other two are unmatched. Furthermore, all the $n$ chords form a perfect match of $G_{2n}$ (i.e. they are independent). From Lemma 2.4, each matrix $M_n(e_{v_1}^1, e_{v_2}^1, \ldots, e_{v_{n+1}}^1)$ corresponds to $2^n$ different Gustin colorings. The property follows. □

## 2.3 A characterization

In graph theory, an *isomorphism* of simple graphs $G$ and $H$ is a bijection between the vertex sets of $G$ and $H$, $f : V(G) \longrightarrow V(H)$, such that any two vertices $u$ and $v$ of $G$

are adjacent in $G$ if and only if $f(u)$ and $f(v)$ are adjacent in $H$. If an isomorphism exists between two graphs, then the graphs are called *isomorphic* and are denoted by $G \cong H$.

**Theorem 2.6.** *Let $(T_i, \rho_i)$ be the characteristic tree of the cubic outerplanar graph $G_i$, $i = 1, 2$. If $T_1$ and $T_2$ are isomorphic, then the two graphs $G_1$ and $G_2$ have the same total embedding distributions.*

*Proof.* Suppose $V(T_1) = \{v_1, v_2, \ldots, v_{n+1}\}$ and $V(T_2) = \{u_1, u_2, \ldots, u_{n+1}\}$. Let $e^1_{v_i}$ be the adjoint edge of $v_i$ and $e^1_{u_i}$ be the adjoint edge of $u_i$, for $i = 1, 2, \ldots, n+1$. Let $M_{n+1}(e^1_{v_1}, e^1_{v_2}, \ldots, e^1_{v_{n+1}})$ be the overlap matrices of $G_1$ over $\mathbb{Z}_2$, and let $M_{n+1}(e^1_{u_1}, e^1_{u_2}, \ldots, e^1_{u_{n+1}})$ be the overlap matrices of $G_2$ over $\mathbb{Z}_2$. Since $T_1$ and $T_2$ are isomorphic, there exists an isomorphism mapping such that $f : V(G_1) \longrightarrow V(G_2)$. For simplicity, we may assume that $f(v_i) = u_i$. By Lemma 2.4, the following conditions are equivalent:

Two cotree edges $e^1_{v_i}$ and $e^1_{v_j}$ overlap $\Longleftrightarrow$ $v_i$ and $v_j$ are adjacent in $T_1$ $\Longleftrightarrow$ $u_i$ and $u_j$ are adjacent in $T_2$ $\Longleftrightarrow$ two cotree edges $e^1_{u_i}$ and $e^1_{u_j}$ overlap.

In other words, $M_{n+1}(e^1_{v_1}, e^1_{v_2}, \ldots, e^1_{v_{n+1}}) = M_{n+1}(e^1_{u_1}, e^1_{u_2}, \ldots, e^1_{u_{n+1}})$. By Lemma 2.5, the result follows. □

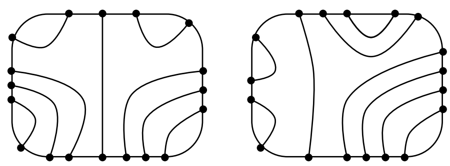


Figure 4: Two non-isomorphic Cubic outerplanar graphs have non-isomorphic characteristic trees

However, the reverse of Theorem 2.6 is not true. We illustrate with an example. Let $G$ and $H$ be two graphs of Figure 4. Although we have

$$\begin{aligned} I_G(x,y) =& I_H(x,y) \\ =& 2^9(1 + 21x + 122x^2 + 240x^3 + 128x^4) \\ &+ 2^9(19y + 183y^2 + 1432y^3 + 6990y^4 + 25536y^5 \\ &+ 66192y^6 + 122368y^7 + 151296y^8 + 112384y^9 + 37376y^{10}), \end{aligned}$$

the two graphs $G$ and $H$ still have non-isomorphic characteristic trees.

Theorem 2.6 shows that the embedding distribution of any cubic outerplanar graph is related to its characteristic tree. In the following discussion, we shall use $g_T(x)$ and $f_T(y)$ to denote the genus polynomial and crosscap number polynomial of the cubic outerplanar graph $G_{2n}$, respectively.

### 2.4 Recursive formula for the rank-distribution polynomial

Let $G_{2n}$ be a cubic outerplanar graph of order $2n$ and let $T$ be a characteristic tree of $G_{2n}$. Suppose $V(T) = \{v_1, v_2, \ldots, v_{n+1}\}$ and $E(T) = \{e_1, e_2, \ldots, e_n\}$. Let $e^1_{v_1}, e^1_{v_2}, \ldots, e^1_{v_{n+1}}$ be cotree edges of $G_{2n}$ described in the previous section. Let $M_{n+1}(e^1_{v_1}, e^1_{v_2}, \ldots, e^1_{v_{n+1}}) = M^{(X,Y)}_{n+1}$ be the overlap matrix corresponding to a given general rotation system of $G_{2n}$, where $X = (x_1, x_2, \ldots, x_{n+1})$, $Y = (y_{e_1}, y_{e_2}, \ldots, y_{e_n})$. Recall that $x_i = 1$ if and only if the edge $e^1_{v_i}$ is twisted, for $i = 1, 2, \ldots, n+1$ and that $y_{e_i} = 1$ if and only if the chord $f$ of $G_{2n}$ which crosses $e_i$ is unmatched, for $i = 1, 2, \ldots, n$.

We now consider the set $\mathcal{C}_{n+1}$ of all $(n+1)\times(n+1)$ matrices $M^{(X,Y)}_{n+1}$ of $G_{2n}$ over $\mathbb{Z}_2$. We define the *rank-distribution polynomial* of the set $\mathcal{C}_{n+1}$ as

$$\mathcal{N}_T(z) = \sum_{j=0}^{n+1} N_{n+1}(j) z^j,$$

where $N_n(j)$ is the number of different assignments of the variables $x_i$ and $y_{e_k}$, with $1 \le i \le n+1$ and $1 \le k \le n$, for which the matrix $M^{X,Y}_{n+1}$ in $\mathcal{C}_{n+1}$ has rank $j$. Similarly, we consider the set

$$\mathcal{O}_{n+1} \;=\; \{M^{0,Y}_{n+1} \mid Y \in \mathbb{Z}^n_2\},$$

and define the *rank-distribution polynomial* of $\mathcal{O}_T$ to be the polynomial

$$\mathcal{O}_T(z) \;=\; \sum_{j=0}^{n+1} O_{n+1}(j) z^j, \tag{1}$$

where $O_{n+1}(j)$ is the number of different assignments of the variables $y_{e_1}, y_{e_2}, \ldots, y_{e_n}$ for which the matrix $M^Y_{n+1} = M^{0,Y}_{n+1}$ in $\mathcal{O}_{n+1}$ has rank $j$.

A *leaf* of $T$ is a vertex of degree 1. Let the vertex $u$ be a leaf of $T$ and let $v$ be its neighbor. Let $u, v_1, v_2, \ldots, v_{d(v)-1}$ be the neighbors of $v$. Let $T_1, T_2, \ldots, T_{d(v)-1}$, ($d(v) \ge 2$), and $u$ be the connected components of $T - v$, as shown in Figure 5. Similarly, we define the rank distribution polynomials of $\mathcal{O}_{T_i}(z)$ and $\mathcal{N}_{T_i}(z)$. We have the following recursive formula for the rank distribution polynomial $\mathcal{O}_T(z)$ of $G_{2n}$.

**Theorem 2.7.** *Let $u$ be a leaf of $T$ and let $v$ be its neighbor. Let $T_1, T_2, \ldots, T_{d(v)-1}$ and $u$ be the connected components of $T - v$. Then the rank-distribution polynomial $\mathcal{O}_T$ for $(n+1)\times(n+1)$ matrices $M^{(O,Y)}_{n+1}$ satisfies the recurrence relation*

$$\mathcal{O}_T(z) \;=\; \mathcal{O}_{T-u}(z) \;+\; 2^{d(v)-1} z^2 \prod_{i=1}^{d(v)-1} \mathcal{O}_{T_i}(z). \tag{2}$$

*with the initial conditions*

$$\mathcal{O}_{P_0}(z) = \mathcal{O}_{P_1}(z) = 1 \text{ and } \mathcal{O}_{P_2}(z) = z^2 + 1. \tag{3}$$

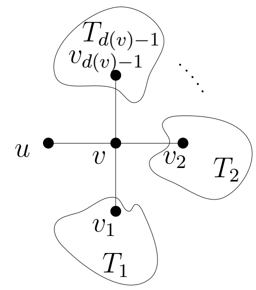


Figure 5: A tree $T$ and its connected components $T_1, T_2, \ldots, T_{d(v)-1}$ and $u$

*Proof.* Suppose the overlap matrix $M_{n+1}^{(O,Y)}$ has the following form

$$\begin{pmatrix} 0 & x_{uv} & 0 & \cdots & 0 \\ x_{uv} & 0 & * & \cdots & * \\ 0 & * & & & \\ \vdots & \vdots & & A & \\ 0 & * & & & \end{pmatrix}$$

where $x_{uv} = 1$ if and only if $e_u^1$ and $e_v^1$ overlap. The following two cases are considered.

- Case 1: $x_{uv} = 0$. In this case, we delete the first row and the first column; the resulting matrix is the overlap matrix of the outerplanar graph whose characteristic tree is $T - u$. This case contributes a term $\mathcal{O}_{T-u}(z)$ to the polynomial $\mathcal{O}_T(z)$.
- Case 2: $x_{uv} = 1$. In this case, we can transfer the above matrix to the following form.

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & A & \\ 0 & 0 & & & \end{pmatrix}$$

By Lemma 2.4, the cotree edge $e_v^1$ can overlap the edges $e_{v_1}^1, e_{v_2}^1, \ldots, e_{v_{d(v)-1}}^1$. It will be convenient to use variable $x_{vv_i} = 1$ or $x_{vv_i} = 0$ to denote that $e_v^1$ overlaps $e_{v_i}^1$, or does not, respectively, for $i = 1, 2, \ldots, d(v) - 1$. There are a total of $2^{d(v)-1}$ different combinations of values for the variables $x_{vv_1}, x_{vv_2}, \ldots, x_{vv_{d(v)-1}}$. Note that $A$ is the overlap matrix of the outerplanar graph whose overlap matrix is $T_1 \cup T_2 \cup \cdots \cup T_{d(v)-1}$. By using Lemma 2.4 again, we have the following form of $A$:

$$A = \begin{pmatrix} A_1 & & & \mathbf{0} \\ & A_2 & & \\ & & \ddots & \\ \mathbf{0} & & & A_{d(v)-1} \end{pmatrix}$$

where $A_i$ is the overlap matrix of the cubic outer plane graph whose characteristic tree is $T_i$, for $i = 1, 2, \ldots, d(v)-1$. Note that $\text{rank}(M_{n+1}^{(O,Y)}) = \text{rank}(A)+2 = \sum_{i=1}^{d(v)-1} rank(A_i) + 2$; this case contributes in all a term $2^{d(v)-1}z^2\prod_{i=1}^{d(v)-1} \mathcal{O}_{T_i}(z)$ to the polynomial $\mathcal{O}_T(z)$.

Summarizing the two cases above, the proof is now complete. □

For the nonorientable case, we have the following theorem.

**Theorem 2.8.** *Let $u$ be a leaf of $T$ and let $v$ be its neighbor. Let $T_1, T_2, \ldots, T_{d(v)-1}$ and $u$ be the connected components of $T-v$. The rank-distribution polynomial $N_T(z)$ for $(n+1)\times(n+1)$ matrices $M_{n+1}^{(X,Y)}$ satisfies the recurrence relation*

$$N_T(z) \;=\; (1+2z)N_{T-u}(z) \;+\; 2^{d(v)}z^2 \prod_{i=1}^{d(v)-1} \mathcal{N}_{T_i}(z). \tag{4}$$

*with the initial conditions*

$$\mathcal{N}_{P_0}(z) = 1, \mathcal{N}_{P_1}(z) = 1+z \text{ and } \mathcal{N}_{P_2}(z) = 4z^2+3z+1. \tag{5}$$

*Proof.* Suppose that overlap matrix $M_{n+1}^{(X,Y)}$ has the following form

$$\begin{pmatrix} y_1 & x_{uv} & 0 & \cdots & 0 \\ x_{uv} & y_2 & * & \cdots & * \\ 0 & * & & & \\ \vdots & \vdots & & A & \\ 0 & * & & & \end{pmatrix}$$

where $x_{uv} = 1$ if and only if $e_u^1$ and $e_v^1$ overlap, $y_1 = 1$ if and only if $e_u^1$ is twisted, and $y_2 = 1$ if and only if $e_v^1$ is twisted. The following four cases can be proved in the same way as Cases 1 and 2 of Theorem 2.7 above.

| Cases | Contributions to $N_T(z)$ |
|---|---|
| $x_{uv} = 0, y_1 = 0$ | $N_{T-u}(z)$ |
| $x_{uv} = 0, y_1 = 1$ | $zN_{T-u}(z)$ |
| $x_{uv} = 1, y_1 = 1$ | $zN_{T-u}(z)$ |
| $x_{uv} = 1, y_1 = 0$ | $2^{d(v)}z^2\prod_{i=1}^{d(v)-1} \mathcal{N}_{T_i}(z)$ |

Combining the cases above, we have the desired result. □

### 2.5 Total embedding polynomial

By Lemma 2.5, we have the following two theorems.

**Theorem 2.9.** *Let $G_{2n}$ be a cubic outerplanar graph and $T$ be its characteristic tree. The genus polynomial $g_T(x)$ of $G_{2n}$ equals*

$$g_T(x) = 2^n \mathcal{O}_T(\sqrt{x}). \tag{6}$$

**Theorem 2.10.** *Let $G_{2n}$ be a cubic outerplanar graph and $T$ be its characteristic tree. The crosscap polynomial $f_T(y)$ of $G_{2n}$ equals*

$$f_T(y) = 2^n(\mathcal{N}_T(y) - \mathcal{O}_T(y).) \tag{7}$$

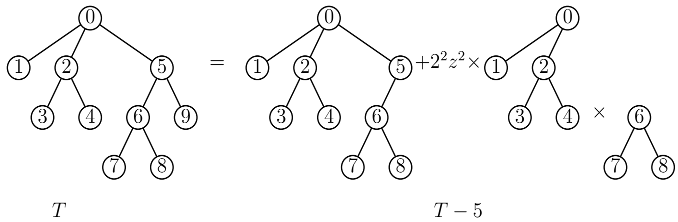


Figure 6: A tree and its decomposition

**Example 2.11.** Suppose $T$ is the tree in Figure 6. Let us find the exact values for the polynomial $\mathcal{O}_T(z)$. By applying Theorem 2.7 recursively, we obtain

$$\mathcal{O}_T(z) = (1 + 25z^2 + 162z^4 + 252z^6 + 72z^8)$$

From Theorem 2.9, $g_T(x) \;=\; 2^9\mathcal{O}_T(\sqrt{x}) = 2^9(1 + 25x + 162x^2 + 252x^3 + 72x^4)$.

### 2.6 Embeddings of $G_{2n}$ into surfaces of small genus

An *internal vertex* of a tree is a vertex of degree at least 2. We now go on to find explicit formulas for embeddings of $G_{2n}$ into a plane, torus, projective plane and Klein bottle.

**Theorem 2.12.** *Let $T$ be the characteristic tree of $G_{2n}$. Let $d_i$ be the number of internal vertices of degree $i$ in $T$, for $2 \le i \le n$. Then the number of embeddings of cubic outerplanar graphs $G_{2n}$ into a plane and torus are $2^n$ and $2^n\left(\sum_{i=2}^{n} d_i(2^i-2)+1\right)$, respectively.*

*Proof.* Let the genus polynomial of the cubic outerplanar graph $G_{2n}$ be $g_T(z) = \sum_{i\ge 0} g_i(T)z^i$. Let the rank distribution of $G_{2n}$ be $\mathcal{O}_T(z) \;=\; \sum_{j=0}^{n+1} O_{n+1}(j)z^j$.

By the recursive formula (2) in Theorem 2.7, we have

$$O_{n+1}(0) = O_n(0) = \cdots = O_1(0) = 1 \tag{8}$$

$$\begin{aligned} O_{n+1}(2) &= 1 + d_2 \times 2 + d_3 \left(2^2 + 2\right) + \cdots + d_n \left(2^{n-1} + 2^{n-2} + \cdots + 2\right) \\ &= \sum_{i=2}^{n} d_i(2^i - 2) + 1. \end{aligned} \tag{9}$$

By formula (6) in Theorem 2.9, we have

$$g_0(T) = 2^n O_{n+1}(0) = 2^n$$

$$g_1(T) = 2^n O_{n+1}(2) = 2^n \left( \sum_{i=2}^{n} d_i(2^i - 2) + 1 \right).$$

The result follows.

□

Similarly, we have the following theorem.

**Theorem 2.13.** *Let $T$ be the characteristic tree of $G_{2n}$. Let $d_i$ be the number of internal vertices of degree $i$ in $T$, for $2 \leq i \leq n$. Then the number of embeddings of cubic outerplanar graphs $G_{2n}$ into a projective plane and Klein bottle are $2^n(2n+1)$ and $2^n \left( 2n^2 + 1 + \sum_{i=2}^{n} d_i(2^i - 2) \right)$, respectively.*

# 3 The algorithm

## 3.1 Tree structures of Theorem 2.7 and Theorem 2.8

We now restate Theorems 2.7 and 2.8 with tree structures. We need to use some definitions of rooted trees. Each element of a tree is called a *node* of the tree. Every node in a tree defines a *subtree*, namely the tree defined by this node and all its children. The vertex $v$ is an *ancestor* of $w$ if there is a path from $v$ to $w$; we then also call $w$ a descendent of $v$. Two nodes are called *brothers* if they are sons of the same father.

Given a tree $T$, we choose a vertex as its root, and then label the vertex of $T$ by preorder traversal; the labeling set is $\{0, 1, \ldots, n\}$. Suppose $T$ is a labeled tree. Let $A(i)$ be *the set of all ancestors* of $i$ and let $B(i)$ be *the set of left brothers* of $i$. Let $F(i)$ be the *father* of node $i$ and let $T(i)$ be the *subtree* of node $i$. Let $L(i)$ be the *maximum labeling* of $T(i)$. Figure 7 gives an example to illustrate this concept, where $A(5) = \{0\}$, $B(5) = \{1, 2\}$, $F(5) = 0$ and $L(5) = 9$.

For any integers $i, k$, $0 \leq i, k \leq n$, if node $i$ is the ancestor of $k$ then we denote by $T[i, k]$ the subtree which contains the nodes $i, i+1, \ldots, k$. In particular, $T[i, i]$ and $T[k, k]$ are isolated vertex sets. Let $G[i, k]$ be the rank distribution polynomial of $T[i, k]$. We have $G[i, i] = G[k, k] = 1$. We also have the following property.

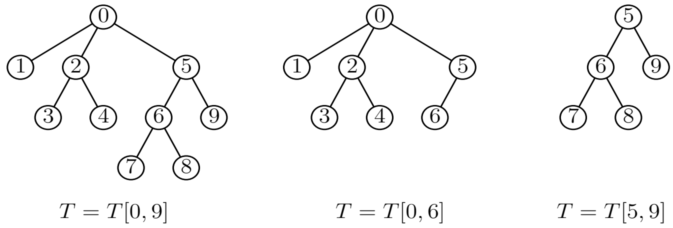


Figure 7: A preorder traverse labeling of a tree

**Property 3.1.** *For any* $i \in A(k+1)$ *we have* $i \in A(k) \cup \{k\}$, $(0 \le k \le n-1)$.

**Property 3.2.** *For any* $i \in A(k+1)$, *the tree* $T[i,k]$ *can be obtained from* $T[i,k+1]$ *by deleting edges which are incident to the node* $k+1$, $(0 \le k \le n-1)$.

By Properties 3.1 and 3.2, and Theorem 2.7, we restate Theorem 2.7 as the following recursive formula.

**Theorem 3.3.** *We have that*

$$G[i,k] = G[i,k-1] + 2^{|B(k)|+1} z^2 G[i,F(k)-1] \prod_{j \in B(k)} G[j,L(j)].$$

*In particular, when* $F(k) = i$,

$$G[i,k] = G[i,k-1] + 2^{|B(k)|} z^2 \prod_{j \in B(k)} G[j,L(j)].$$

Also, we have another expression of Theorem 2.8.

**Theorem 3.4.** *We have*

$$G[i,k] = (1+2z)G[i,k-1] + 2^{|B(k)|+2} z^2 G[i,F(k)-1] \prod_{j \in B(k)} G[j,L(j)].$$

*In particular, when* $F(k) = i$, *then*

$$G[i,k] = (1+2z)G[i,k-1] + 2^{|B(k)|+1} z^2 \prod_{j \in B(k)} G[j,L(j)].$$

### 3.2 Algorithm aspect

With the help of the analysis and notation above, we propose the following algorithm that computes the rank distribution polynomial $G[0,n]$ efficiently.

**Begin Algorithm 1**
*Input:* A tree $T$ with pre-order labels $0, 1, \ldots, n$.
*Output:* The rank distribution polynomial $G[0,n]$ of $T$.
// $A(i)$ is the set of all ancestors of $i$

// $B(i)$ is the set of left brothers of $i$.
// $F(i)$ is the father of node $i$
// $T(i)$ is the subtree of node $i$.
// $L(i)$ is the maximum labeling of $T(i)$.
// $T[i,k]$ is the subtree which contains the nodes $i, i+1, \ldots, k$.
// $G[i,k]$ is the rank distribution polynomial of $T[i,k]$.
// Let $G$ be an $(n+1)*(n+1)$ matrix

1 Pre-processing: for every node $v$ in $T$, compute the sets $A(v), B(v)$ and $L(v)$;
2 $G[0,0] = 1$;
3 **for** $k = 1$ to $n$ **do**
4 $\quad G[k,k] = 1$;
5 $\quad a = z * z$;
6 $\quad$ **for** all $j \in B(k)$ **do**
7 $\qquad a = aG[j, L(j)]$;
8 $\quad$ **end for**
9 $\quad$ **for** all $i \in A(k)$ **do**
10 $\qquad$ **if** $i = F(k)$ **then**
11 $\qquad G[i,k] = G[i,k-1] + a * 2^{|B(k)|}$;
12 $\qquad$ **else**
13 $\qquad\quad G[i,k] = G[i,k-1] + a * 2^{|B(k)|+1} * G[i, F(k)-1]$;
14 $\qquad$ **end if**
15 $\quad$ **end for**
16 **end for**
17 **return** $G[0,n]$;

**End Algorithm 1**

The pre-processing in Algorithm 1 (line 1) can be finished in $O(n)$ time. This algorithm contains a 2-loop: the outer loop runs $n$ times, and computes a column in matrix $G$ each time. There are two inner loops: the first loop (lines 6 to 8) is to compute the item $\prod_{j\in B(k)} G[j, L(j)]$ described in Theorem 3.3, and it runs $\Delta$ times in the worst case (where $\Delta$ is the maximum degree of $T$); the second loop (lines 9 to 15) is to compute elements in the $k$-th column of matrix $G$, and it runs $h$ times in the worst case (where $h$ is the height of $T$). So the complexity of this algorithm is $O(n(h+\Delta))$ (on average, $h$ and $\Delta$ are far less than $n$). Similarly, we can encode Theorem 3.4 as an $O(n(h+w))$-time algorithm and the details are omitted. According to the algorithm above, we build the *embedding-distribution computer program*.

**Example 3.5.** Let $T$ be the tree of Figure 8. Suppose $T$ is the characteristic tree of the cubic outerplanar graph $G_{124}$. The computer program shows that the genus polynomial $g_T(x)$ of $G_{124}$ equals.

$$\begin{aligned}\frac{g_T(x)}{2^{62}} =& 1 + 183x + 15504x^2 + 808392x^3 + 29090400x^4 + 767987472x^5\\ &+ 15435671040x^6 + 241930272384x^7 + 3004569543168x^8\\ &+ 29877836739840x^9 + 239410877976576x^{10}\\ &+ 1550357283483648x^{11} + 8111720263237632x^{12}\\ &+ 34174289079189504x^{13} + 115119139150430208x^{14}\\ &+ 306551896452071424x^{15} + 634256725446623232x^{16}\\ &+ 993604844419743744x^{17} + 1132860542331912192x^{18}\\ &+ 881351152818978816x^{19} + 415275675304329216x^{20}\\ &+ 88557124429283328x^{21}\end{aligned}$$

Note that the running time of the program is less than 1 second. We also input two trees: one tree is 226 vertices, the other is 280 vertices. The running time is about 1 second and 2 seconds, respectively.

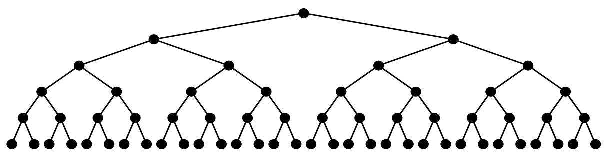

Figure 8: A complete binary tree with 63 vertices

In [6], Gross, Robbins and Tucker conjectured that the genus distribution of a graph is log-concave. With the help of an embedding-distribution computer program, we also verified the log-concavity for the genus distribution of any cubic outerplanar graph with fewer than 33 vertices. This provides further evidence that the genus distribution of any graph is log-concave.

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