

# Note: An inequality for the line-size sum in a finite linear space

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## Abstract

An inequality for finite linear spaces in relation with clique partitions of the complete graph  $K_n$  is given.

## 1 The inequality

For notions on finite linear spaces and clique partitions of graphs we refer the reader to [1] and [2] respectively.

Let  $(\mathcal{P}, \mathcal{L})$  be a (non-degenerate) finite linear space with  $v$  points and  $b$  lines. For every point  $p$  let  $r_p$  denote the number of lines containing  $p$  (the *degree* of  $p$ ) and for every line  $\ell$  let  $k_\ell$  denote the size of  $\ell$ . By an  $m$ -point we mean a point of degree  $m$ .

Recently in [2] a lower bound for the sum of line sizes of a finite linear space has been obtained.

**Theorem 1.1.** *Let  $(\mathcal{P}, \mathcal{L})$  be a non-degenerate finite linear space with  $v$  points, then we have*

$$\sum_{\ell \in \mathcal{L}} k_\ell \geq 3v - 3 \tag{1}$$

*and equality holds if and only if  $(\mathcal{P}, \mathcal{L})$  is a near-pencil.*

As a consequence of Theorem 1.1 we obtain the following result.

**Proposition 1.2.** *Let  $\mathcal{C}$  be a clique partition of the complete graph  $K_n$  whose cliques are of size at most  $n - 1$ . Then  $\sum_{C \in \mathcal{C}} |C| \geq 3n - 3$ .*

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In this note, a generalization of Theorem 1.1 is given.

**Theorem 1.3.** *Let  $(\mathcal{P}, \mathcal{L})$  be a non-trivial finite linear space on  $v$  points. Let  $m \geq 2$  denote the minimum point degree. Then*

$$\sum_{\ell \in \mathcal{L}} k_\ell \geq (v - m + 1)(m + 1).$$

*The equality holds if and only if  $m = 2$  and  $(\mathcal{P}, \mathcal{L})$  is a near-pencil.*

## 2 Proof of Theorem 1.3

In this section,  $(\mathcal{P}, \mathcal{L})$  is a finite linear space with minimal point degree  $m$ . As for any incidence structure, in a finite linear space the following equality holds:

$$\sum_{p \in \mathcal{P}} r_p = \sum_{\ell \in \mathcal{L}} k_\ell. \tag{2}$$

Assume  $\sum_{\ell \in \mathcal{L}} k_\ell \leq (v - m + 1)(m + 1)$ . Let  $x$  denote the number of points of degree  $m$ . Then

$$xm + (v - x)(m + 1) \leq \sum_{p \in \mathcal{P}} r_p = \sum_{\ell \in \mathcal{L}} k_\ell \leq (v - m + 1)(m + 1)$$

and so

$$x \geq m^2 - 1 \geq 3.$$

If there are three non-collinear  $m$ -points, then the size of each line is at most  $m$  and thus  $m^2 - 1 \leq x \leq v \leq m(m - 1) + 1$  and so  $m = 2$ , all the points have degree  $m$  and the linear space  $(\mathcal{P}, \mathcal{L})$  is the near-pencil on three points and so  $\sum_{\ell \in \mathcal{L}} k_\ell = (v - 1) \cdot (m + 1) = 6$ .

Hence we may assume that all the  $m$ -points are collinear and that there are points of degree different from  $m$ . Let  $L$  be the line containing all the points of degree  $m$ . Thus  $k_L \geq m^2 - 1$ . Counting the number of points of the linear space via the lines on an  $m$ -point, and since all lines other than  $L$  must have size at most  $m$ , we have

$$v \leq k_L + (m - 1)^2 \leq 2k_L - 2(m - 1)$$

and so

$$v - m + 1 \leq 2k_L - 3(m - 1).$$

Since the points outside  $L$  have degree at least  $k_L$ , it follows that

$$k_L m + (v - k_L)k_L \leq \sum_{p \in \mathcal{P}} r_p = \sum_{\ell \in \mathcal{L}} k_\ell \leq (2k_L - 3(m - 1))(m + 1)$$

and so

$$\begin{aligned} (v - k_L)k_L &\leq 2k_L + k_Lm - 3(m^2 - 1), \\ v - k_L &\leq m + 2 - \frac{3(m^2 - 1)}{k_L}, \\ v &\leq k_L + m + 1. \end{aligned}$$

Thus

$$v - m + 1 \leq k_L + 2,$$

from which it follows, since each line has size at least 2, that

$$k_L + 2(b - 1) \leq k_L + \sum_{\ell \in \mathcal{L}, \ell \neq L} k_\ell = \sum_{\ell \in \mathcal{L}} k_\ell \leq (k_L + 2)(m + 1),$$

and hence

$$2(b - 1) \leq k_Lm + 2(m + 1).$$

Counting lines meeting, but different from,  $L$  gives  $b - 1 \geq k_L(m - 1)$ ; thus

$$2k_Lm - 2k_L \leq k_Lm + 2(m + 1)$$

so

$$k_L(m - 2) \leq 2(m + 1).$$

But  $k_L \geq m^2 - 1$ , and therefore

$$(m - 1)(m - 2) \leq 2$$

and so either  $m = 2$  or  $m = 3$ .

If  $m = 3$  then  $k_L = 8 = m^2 - 1$ ,  $v \leq 12$  and all the points of  $L$  have degree  $m = 3$ . Moreover, from  $v \geq k_L + m - 1$  it follows that  $v \geq 10$ .

If  $v = 12$ , on each point of  $L$  there are  $L$  and two lines of length 3 contradicting the fact that the four points outside  $L$  may give rise to at most six lines of length 3.

If  $v = 11$ , on each point of  $L$  there are  $L$ , one line of length 3 and one of length 2, contradicting the fact that in such a case  $(\mathcal{P}, \mathcal{L})$  has at most three lines of length 3.

If  $v = 10$ ,  $(\mathcal{P}, \mathcal{L})$  is the union of  $L$  and a line of length 2 disjoint from  $L$  and the two points outside  $L$  have degree 9. Thus,  $42 = \sum_{p \in \mathcal{P}} r_p = \sum_{\ell \in \mathcal{L}} k_\ell \leq (10 - 3 + 1) \cdot 4 = 32$ , a contradiction.

Hence  $m = 2$  and  $(\mathcal{P}, \mathcal{L})$  is the near-pencil on  $v = k_L + 1$  points and  $\sum_{\ell \in \mathcal{L}} k_\ell = 3v - 3 = (v - m + 1)(m + 1)$ . This completes the proof of Theorem 1.3.  $\square$

Let  $(\mathcal{P}, \mathcal{L})$  be a finite linear space and let  $m$  be the minimum point degree. If  $m = 3$  then  $\sum_{\ell \in \mathcal{L}} k_\ell = \sum_{p \in \mathcal{P}} r_p \geq 3v > 3v - 3$  and if  $m = 2$  then by Theorem 1.3

$\sum_{\ell \in \mathcal{L}} k_\ell \geq (v - 1)3$ , so Theorem 1.1 and Proposition 1.2 follow from Theorem 1.3.

Let us end by observing that Theorem 2.5 of [2] can imply similar (and in some cases better) bounds than Theorem [2]. For instance, if we take the dual of the linear space (i.e. exchange the role of points and lines) and we assume that there is a point of degree  $b - c$  (where  $b$  is the number of lines of the linear space and  $c$  is a constant), then Inequality (8) in [2] gives a lower bound  $(2c + 1)b - c/2 - 5c^2/2$ . In contrast, Theorem 1.3 gives a lower bound less than  $2cb - (c - 1)v - c^2 + 1$ , which is worse for large  $v$ .

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## References

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