

Fans are cycle-antimagic

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Abstract

A simple graph $G = (V, E)$ admits an H -covering if every edge in E belongs at least to one subgraph of G isomorphic to a given graph H . Then the graph G admitting an H -covering is (a, d) - H -antimagic if there exists a bijection $f : V \cup E \rightarrow \{1, 2, \dots, |V| + |E|\}$ such that, for all subgraphs H' of G isomorphic to H , the H' -weights, $wt_f(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$, form an arithmetic progression with the initial term a and the common difference d . Such a labeling is called *super* if the smallest possible labels appear on the vertices.

This paper is devoted to studying the existence of super (a, d) - H -antimagic labelings for fans when subgraphs H are cycles.

1 Introduction

We consider finite and simple graphs. Let the vertex and edge sets of a graph G be denoted by $V = V(G)$ and $E = E(G)$, respectively. An *edge-covering* of G is a family of subgraphs H_1, H_2, \dots, H_t such that each edge of E belongs to at least one of the subgraphs H_i , $i = 1, 2, \dots, t$. Then it is said that G admits an (H_1, H_2, \dots, H_t) -*(edge) covering*. If every subgraph H_i is isomorphic to a given graph H , then the

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graph G admits an H -covering. Note that in this case all subgraphs of G isomorphic to H must be in the H -covering. A bijective function $f : V \cup E \rightarrow \{1, 2, \dots, |V| + |E|\}$ is an (a, d) - H -antimagic labeling of a graph G admitting an H -covering whenever, for all subgraphs H' isomorphic to H , the H' -weights

$$wt_f(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$$

form an arithmetic progression $a, a + d, a + 2d, \dots, a + (t - 1)d$, where $a > 0$ and $d \geq 0$ are two integers, and t is the number of all subgraphs of G isomorphic to H . Such a labeling is called *super* if the smallest possible labels appear on the vertices. A graph that admits a (super) (a, d) - H -antimagic labeling is called (super) (a, d) - H -antimagic. For $d = 0$ it is called H -magic and H -supermagic, respectively.

The notion of H -supermagic graphs was introduced by Gutiérrez and Lladó [8] as an extension of the edge-magic and super edge-magic labelings introduced by Kotzig and Rosa [11] and Enomoto, Lladó, Nakamigawa and Ringel [7], respectively. They proved that some classes of connected graphs are H -supermagic, such as the stars $K_{1,n}$ and the complete bipartite graphs $K_{n,m}$ are $K_{1,h}$ -supermagic for some h . They also proved that the path P_n and the cycle C_n are P_h -supermagic for some h . More precisely they proved that the cycle C_n is P_h -supermagic for any $2 \leq h \leq n - 1$ such that $\gcd(n, h(h - 1)) = 1$. Lladó and Moragas [12] studied the cycle-(super)magic behavior of several classes of connected graphs. They proved that wheels, windmills, books and prisms are C_h -magic for some h . Maryati, Salman, Baskoro, Ryan and Miller [16] and also Salman, Ngurah and Izzati [18] proved that certain families of trees are path-supermagic. Ngurah, Salman and Susilowati [17] proved that chains, wheels, triangles, ladders and grids are cycle-supermagic. Maryati, Salman and Baskoro [15] investigated the G -supermagicness of a disjoint union of c copies of a graph G and showed that the disjoint union of any paths is cP_h -supermagic for some c and h .

The (a, d) - H -antimagic labeling was introduced by Inayah, Salman and Simanjuntak [9]. In [10] the authors investigate the super (a, d) - H -antimagic labelings for some families of connected graphs H . In [19] was proved that wheels W_n , $n \geq 3$, are super (a, d) - C_k -antimagic for every $k = 3, 4, \dots, n - 1, n + 1$ and $d = 0, 1, 2$.

The (super) (a, d) - H -antimagic labeling is related to a super d -antimagic labeling of type $(1, 1, 0)$ of a plane graph that is the generalization of a face-magic labeling introduced by Lih [13]. Further information on super d -antimagic labelings can be found in [2, 5].

For $H \cong K_2$, (super) (a, d) - H -antimagic labelings are also called (super) (a, d) -edge-antimagic total labelings and have been introduced in [20]. More results on (a, d) -edge-antimagic total labelings, can be found in [4, 14]. The vertex version of these labelings for generalized pyramid graphs is given in [1].

The existence of super (a, d) - H -antimagic labelings for disconnected graphs is studied in [6] and there is proved that if a graph G admits a (super) (a, d) - H -antimagic labeling, where $d = |E(H)| - |V(H)|$, then the disjoint union of m copies

of the graph G , denoted by mG , admits a (super) (b, d) - H -antimagic labeling as well. In [3] is shown that the disjoint union of multiple copies of a (super) $(a, 1)$ -tree-antimagic graph is also a (super) $(b, 1)$ -tree-antimagic. A natural question is whether the similar result holds also for another differences and another H -antimagic graphs.

A fan F_n , $n \geq 2$, is a graph obtained by joining all vertices of the path P_n to a further vertex, called the *centre*. The vertices on the path we will call the *path vertices*. The edges adjacent to the central vertex we will call the *spokes* and the remaining edges we will call the *path edges*. Thus F_n contains $n + 1$ vertices, say, v_1, v_2, \dots, v_{n+1} , and $2n - 1$ edges, say, $v_{n+1}v_i$, $1 \leq i \leq n$, and $v_i v_{i+1}$, $1 \leq i \leq n - 1$.

In this paper we investigate the existence of super (a, d) - H -antimagic labelings for fans when subgraphs H are cycles.

2 Super (a, d) -cycle-antimagic labeling of fan

Let C_k be a cycle on k vertices. Every cycle C_k in F_n is of the form $C_k^j = v_j v_{j+1} v_{j+2} \dots v_{j+k-2} v_{n+1} v_j$, where $j = 1, 2, \dots, n - k + 2$. It is easy to see that each edge of F_n belongs to at least one cycle C_k^j if $k = 3, 4, \dots, \lfloor \frac{n}{2} \rfloor + 2$.

For the C_k -weight of the cycle C_k^j , $j = 1, 2, \dots, n - k + 2$, under a total labeling f we get

$$\begin{aligned} wt_f(C_k^j) &= \sum_{v \in V(C_k^j)} f(v) + \sum_{e \in E(C_k^j)} f(e) \\ &= \sum_{s=0}^{k-3} \left(f(v_{j+s}) + f(v_{j+s} v_{j+s+1}) \right) + \left(f(v_{j+k-2}) + f(v_{j+k-2} v_{n+1}) \right) \\ &\quad + f(v_{n+1}) + f(v_j v_{n+1}). \end{aligned} \tag{1}$$

2.1 Differences $d = 1, 3$

The next theorem shows that F_n admits super cycle-antimagic labelings for differences $d = 1$ and $d = 3$.

Theorem 1. *Let $n \geq 3$ be a positive integer and $3 \leq k \leq \lfloor \frac{n}{2} \rfloor + 2$. Then the fan F_n admits a super (a, d) - C_k -antimagic labeling for $d = 1, 3$.*

Proof. Let us consider the total labelings f_1 and f_2 of F_n defined in the following way

$$\begin{aligned} f_1(v_i) = f_2(v_i) &= i, && \text{for } i = 1, 2, \dots, n + 1 \\ f_1(v_i v_{n+1}) &= 2n + 2 - i, && \text{for } i = 1, 2, \dots, n \\ f_2(v_i v_{n+1}) &= n + 1 + i, && \text{for } i = 1, 2, \dots, n \\ f_1(v_i v_{i+1}) = f_2(v_i v_{i+1}) &= 3n + 1 - i, && \text{for } i = 1, 2, \dots, n - 1. \end{aligned}$$

It is easy to see that f_1 and f_2 are super labelings as the vertices of F_n are labeled by the labels $1, 2, \dots, n + 1$.

Under both labelings the spokes attain the labels $n + 2, n + 3, \dots, 2n + 1$ and the path edges are labeled by the numbers $2n + 2, 2n + 3, \dots, 3n$.

The sum of the path vertex label and the corresponding incident path edge label is a constant. More precisely, for every $i = 1, 2, \dots, n - 1$ and for $m = 1, 2$ we have

$$f_m(v_i) + f_m(v_i v_{i+1}) = i + (3n + 1 - i) = 3n + 1. \tag{2}$$

Under the labeling f_1 the sum of the path vertex label and the incident spoke label is a constant, that is, for every $i = 1, 2, \dots, n$

$$f_1(v_i) + f_1(v_i v_{n+1}) = i + (2n + 2 - i) = 2n + 2. \tag{3}$$

On the other side under the labeling f_2 the sums of the path vertex label and corresponding spoke label form an arithmetic sequence with difference 2, that is, for every $i = 1, 2, \dots, n$

$$f_2(v_i) + f_2(v_i v_{n+1}) = i + (n + 1 + i) = n + 1 + 2i. \tag{4}$$

According to (1), (2) and (3) we obtain

$$\begin{aligned} wt_{f_1}(C_k^j) &= (k - 2)(3n + 1) + (2n + 2) + (n + 1) + (2n + 2 - j) \\ &= (k - 2)(3n + 1) + 5n + 5 - j \end{aligned}$$

and with respect to (1), (2) and (4) we obtain

$$\begin{aligned} wt_{f_2}(C_k^j) &= (k - 2)(3n + 1) + (n + 1 + 2(j + k - 2)) + (n + 1) + (n + 1 + j) \\ &= (k - 1)(3n + 3) + 3j. \end{aligned}$$

Thus under the labeling f_1 the set of all the C_k -weights consists of consecutive integers and under the labeling f_2 the C_k -weights form the arithmetic sequence with the difference 3. This concludes the proof. \square

2.2 Differences depending on the length of cycle

The following theorem proves the existence of super cycle-antimagic labelings for differences $2k - 5, 2k - 1$ and $3k - 1$.

Theorem 2. *Let $n \geq 3$ be a positive integer and $3 \leq k \leq \lfloor \frac{n}{2} \rfloor + 2$. Then the fan F_n admits a super (a, d) - C_k -antimagic labeling for $d = 2k - 5, 2k - 1, 3k - 1$.*

Proof. Let us consider the total labelings f_3, f_4 and f_5 of F_n defined in the following

way

$$\begin{aligned}
 f_m(v_i) &= i, && \text{for } i = 1, 2, \dots, n + 1 \text{ and } m = 3, 4, 5 \\
 f_m(v_i v_{n+1}) &= \begin{cases} 3n + 1 - i, & \text{for } i = 1, 2, \dots, n \text{ and } m = 3 \\ 2n + i, & \text{for } i = 1, 2, \dots, n \text{ and } m = 4 \\ n + 2i, & \text{for } i = 1, 2, \dots, n \text{ and } m = 5 \end{cases} \\
 f_m(v_i v_{i+1}) &= \begin{cases} n + 1 + i, & \text{for } i = 1, 2, \dots, n - 1 \text{ and } m = 3, 4 \\ n + 1 + 2i, & \text{for } i = 1, 2, \dots, n - 1 \text{ and } m = 5. \end{cases}
 \end{aligned}$$

It is easy to see that f_m is a super labeling for every $m = 3, 4, 5$. Under the labelings f_3 and f_4 the path edges are labeled with the numbers $n + 2, n + 3, \dots, 2n$ and under the labeling f_5 they attain the numbers $n + 3, n + 5, \dots, 3n - 1$. The labelings f_3 and f_4 assign to spokes the labels $2n + 1, 2n + 2, \dots, 3n$ and the labeling f_5 assigns labels $n + 2, n + 4, \dots, 3n$.

For every $i = 1, 2, \dots, n - 1$ we have

$$f_m(v_i) + f_m(v_i v_{i+1}) = i + (n + 1 + i) = n + 1 + 2i, \quad \text{if } m = 3, 4 \tag{5}$$

$$f_5(v_i) + f_5(v_i v_{i+1}) = i + (n + 1 + 2i) = n + 1 + 3i. \tag{6}$$

For every $i = 1, 2, \dots, n$ we get

$$f_3(v_i) + f_3(v_i v_{n+1}) = i + (3n + 1 - i) = 3n + 1, \tag{7}$$

$$f_4(v_i) + f_4(v_i v_{n+1}) = i + (2n + i) = 2n + 2i, \tag{8}$$

$$f_5(v_i) + f_5(v_i v_{n+1}) = i + (n + 2i) = n + 3i. \tag{9}$$

For C_k -weights from (1), (5) and (7) it follows

$$\begin{aligned}
 wt_{f_3}(C_k^j) &= \sum_{s=0}^{k-3} (n + 1 + 2(j + s)) + (3n + 1) + (n + 1) + (3n + 1 - j) \\
 &= (k - 2)(n + k - 2) + 7n + 3 + j(2k - 5),
 \end{aligned}$$

by (1), (5) and (8) we obtain

$$\begin{aligned}
 wt_{f_4}(C_k^j) &= \sum_{s=0}^{k-3} (n + 1 + 2(j + s)) + (2n + 2(j + k - 2)) + (n + 1) + (2n + j) \\
 &= (k - 2)(n + k) + 5n + 1 + j(2k - 1),
 \end{aligned}$$

and by (1), (6) and (9) we get

$$\begin{aligned}
 wt_{f_5}(C_k^j) &= \sum_{s=0}^{k-3} (n + 1 + 3(j + s)) + (n + 3(j + k - 2)) + (n + 1) + (n + 2j) \\
 &= (k + 1)(n + 4) + \frac{3(k - 3)(k - 2)}{2} - 11 + j(3k - 1).
 \end{aligned}$$

Thus under the labelings f_m , $m = 3, 4, 5$, the C_k -weights form the arithmetic sequence with the differences $2k - 5$, $2k - 1$ and $3k - 1$, respectively. \square

The existence of super cycle-antimagic labelings of a fan for differences $3k - 9$, $k - 7$ and $k + 1$ follows from the next theorem.

Note that for some of these values of difference d is negative, which only means that the cycle-weights form decreasing sequence, or alternatively the difference in the corresponding increasing arithmetic sequence is $|d|$. Note that if $d = 0$ then the cycle-weights are the same.

Theorem 3. *Let $n \geq 3$ be a positive integer and $3 \leq k \leq \lfloor \frac{n}{2} \rfloor + 2$. Then the fan F_n is super (a, d) - C_k -antimagic for $d = 3k - 9, k - 7, k + 1$.*

Proof. Define the total labeling f_m , $m = 6, 7, 8$, of F_n as follows

$$f_m(v_i) = \begin{cases} i, & \text{for } i = 1, 2, \dots, n + 1 \text{ and } m = 6 \\ n + 1 - i, & \text{for } i = 1, 2, \dots, n \text{ and } m = 7, 8 \\ n + 1, & \text{for } i = n + 1 \text{ and } m = 7, 8 \end{cases}$$

$$f_m(v_i v_{n+1}) = \begin{cases} 3n + 2 - 2i, & \text{for } i = 1, 2, \dots, n \text{ and } m = 6, 7 \\ n + 2i, & \text{for } i = 1, 2, \dots, n \text{ and } m = 8 \end{cases}$$

$$f_m(v_i v_{i+1}) = n + 1 + 2i, \quad \text{for } i = 1, 2, \dots, n - 1 \text{ and } m = 6, 7, 8.$$

Since the labelings f_m , $m = 6, 7, 8$, assign the smallest possible labels to the vertices of F_n , they are super. For each labeling f_m , $m = 6, 7, 8$, the path edges attain the labels $n+3, n+5, \dots, 3n-1$ and the spokes are labeled by the labels $n+2, n+4, \dots, 3n$.

For every $i = 1, 2, \dots, n - 1$ we get

$$f_6(v_i) + f_6(v_i v_{i+1}) = i + (n + 1 + 2i) = n + 1 + 3i, \tag{10}$$

$$f_m(v_i) + f_m(v_i v_{i+1}) = (n + 1 - i) + (n + 1 + 2i) = 2n + 2 + i, \text{ if } m = 7, 8 \tag{11}$$

For every $i = 1, 2, \dots, n$ we have

$$f_6(v_i) + f_6(v_i v_{n+1}) = i + (3n + 2 - 2i) = 3n + 2 - i, \tag{12}$$

$$f_7(v_i) + f_7(v_i v_{n+1}) = (n + 1 - i) + (3n + 2 - 2i) = 4n + 3 - 3i, \tag{13}$$

$$f_8(v_i) + f_8(v_i v_{n+1}) = (n + 1 - i) + (n + 2i) = 2n + 1 + i. \tag{14}$$

According to (1), (10) and (12)

$$wt_{f_6}(C_k^j) = \sum_{s=0}^{k-3} \left(n + 1 + 3(j + s) \right) + (3n + 2 - (j + k - 2)) + (n + 1) + (3n + 2 - 2j) = n(k + 5) + \frac{3(k - 3)(k - 2)}{2} + 5 + j(3k - 9),$$

with respect to (1), (11) and (13)

$$\begin{aligned} wt_{f_7}(C_k^j) &= \sum_{s=0}^{k-3} (2n + 2 + (j + s)) + (4n + 3 - 3(j + k - 2)) + (n + 1) \\ &\quad + (3n + 2 - 2j) = (2n - 1)(k + 2) + \frac{(k - 3)(k - 2)}{2} + 10 \\ &\quad + j(k - 7). \end{aligned}$$

and from (1), (11) and (14) it follows

$$\begin{aligned} wt_{f_8}(C_k^j) &= \sum_{s=0}^{k-3} (2n + 2 + (j + s)) + (2n + 1 + (j + k - 2)) + (n + 1) \\ &\quad + (n + 2j) = (2n + 3)k + \frac{(k - 3)(k - 2)}{2} - 4 + j(k + 1). \end{aligned}$$

One can see that under the labelings f_m , $m = 6, 7, 8$, the C_k -weights constitute the arithmetic sequences with the differences $3k - 9$, $k - 7$ and $k + 1$, respectively. \square

3 C_3 -antimagicness of fans

In [13] Lih proved that F_n is C_3 -supermagic for every n except when $n \equiv 2 \pmod{4}$. Ngurah, Salman and Susilowati [17] completed this result and they proved that for any integer $n \geq 2$ the fan F_n is C_3 -supermagic.

Immediately from Theorems 1 through 3 we obtain that if F_n satisfies the necessary condition for covering by C_3 , then there exist super (a, d) - C_k -antimagic labelings of F_n for every $d \in \{0, 1, 3, 4, 5, 8\}$. Moreover, in the next theorem we are able to prove also that differences $d = 2$ and $d = 6$ are feasible.

Theorem 4. *The fan F_n , $n \geq 4$, is super (a, d) - C_3 -antimagic for $d = 0, 1, 2, 3, 4, 5, 6, 8$.*

Proof. The existence of such labelings for $d = 0, 1, 3, 4, 5, 8$ immediately follows from Theorems 1 through 3. For $d = 2, 6$ let us consider the following.

Construct the total labelings g_m , $m = 1, 2$, of F_n such that

$$\begin{aligned} g_m(v_i) &= \begin{cases} \frac{i+1}{2}, & \text{for } 1 \leq i \leq n, i \equiv 1 \pmod{2} \text{ and } m = 1 \\ \lceil \frac{n}{2} \rceil + \frac{i}{2}, & \text{for } 2 \leq i \leq n, i \equiv 0 \pmod{2} \text{ and } m = 1 \\ n + 1, & \text{for } i = n + 1 \text{ and } m = 1 \\ i, & \text{for } i = 1, 2, \dots, n + 1 \text{ and } m = 2 \end{cases} \\ g_m(v_i v_{n+1}) &= \begin{cases} 2n + i, & \text{for } i = 1, 2, \dots, n \text{ and } m = 1 \\ n + 1 + i, & \text{for } 1 \leq i \leq n, i \equiv 1 \pmod{2} \text{ and } m = 2 \\ n + 2 \lceil \frac{n}{2} \rceil + i, & \text{for } 2 \leq i \leq n, i \equiv 0 \pmod{2} \text{ and } m = 2 \end{cases} \\ g_m(v_i v_{i+1}) &= \begin{cases} 2n + 1 - i, & \text{for } i = 1, 2, \dots, n - 1 \text{ and } m = 1 \\ n + 1 + 2i, & \text{for } i = 1, 2, \dots, n - 1 \text{ and } m = 2. \end{cases} \end{aligned}$$

The labelings g_1 and g_2 are super as the vertices of F_n are labeled with the smallest possible labels. Under the labeling g_1 or g_2 the path edges attain the labels $n + 2, n + 3, \dots, 2n$ or $n + 3, n + 5, \dots, 3n - 1$, respectively, and the spokes admit the labels $2n + 1, 2n + 2, \dots, 3n$ or $n + 2, n + 4, \dots, 3n$, respectively.

For the C_3 -weights of the cycle $C_3^j = v_j v_{j+1} v_{n+1} v_j$, $j = 1, 2, \dots, n - 1$, we get

$$wt_{g_1}(C_3^j) = g_1(v_j) + g_1(v_j v_{j+1}) + g_1(v_{j+1}) + g_1(v_{j+1} v_{n+1}) + g_1(v_{n+1}) + g_1(v_j v_{n+1})$$

$$= \begin{cases} \frac{j+1}{2} + (2n + 1 - j) + \left(\lceil \frac{n}{2} \rceil + \frac{j+1}{2}\right) + (2n + (j + 1)) + (n + 1) \\ \quad + (2n + j) = 7n + \lceil \frac{n}{2} \rceil + 4 + 2j \\ \quad \text{for } 1 \leq j \leq n - 1, j \equiv 1 \pmod{2} \\ \left(\lceil \frac{n}{2} \rceil + \frac{j}{2}\right) + (2n + 1 - j) + \frac{j+2}{2} + (2n + (j + 1)) + (n + 1) \\ \quad + (2n + j) = 7n + \lceil \frac{n}{2} \rceil + 4 + 2j \\ \quad \text{for } 2 \leq j \leq n - 1, j \equiv 0 \pmod{2} \end{cases}$$

and

$$wt_{g_2}(C_3^j) = g_2(v_j) + g_2(v_j v_{j+1}) + g_2(v_{j+1}) + g_2(v_{j+1} v_{n+1}) + g_2(v_{n+1}) + g_2(v_j v_{n+1})$$

$$= \begin{cases} j + (n + 1 + 2j) + (j + 1) + (n + 2 \lceil \frac{n}{2} \rceil + (j + 1)) + (n + 1) \\ \quad + (n + 1 + j) = 4n + 2 \lceil \frac{n}{2} \rceil + 5 + 6j \\ \quad \text{for } 1 \leq j \leq n - 1, j \equiv 1 \pmod{2} \\ j + (n + 1 + 2j) + (j + 1) + (n + 1 + (j + 1)) + (n + 1) \\ \quad + (n + 2 \lceil \frac{n}{2} \rceil + j) = 4n + 2 \lceil \frac{n}{2} \rceil + 5 + 6j \\ \quad \text{for } 2 \leq j \leq n - 1, j \equiv 0 \pmod{2}. \end{cases}$$

For $j = 1, 2, \dots, n - 1$ that is

$$wt_{g_1}(C_3^j) = 7n + \lceil \frac{n}{2} \rceil + 4 + 2j$$

and

$$wt_{g_2}(C_3^j) = 4n + 2 \lceil \frac{n}{2} \rceil + 5 + 6j.$$

This means that under the labelings g_1 and g_2 the C_3 -weights form the arithmetic sequences with the differences 2 and 6, respectively. \square

4 C_4 -antimagicness of fans

Every cycle C_4 in F_n is of the form $C_4^j = v_j v_{j+1} v_{j+2} v_{n+1} v_j$, $j = 1, 2, \dots, n - 2$, and for $n \geq 4$, each edge of F_n belongs to at least one cycle of C_4^j . For the C_4 -weight of

the cycle C_4^j , $j = 1, 2, \dots, n - 2$, under a total labeling f we have

$$wt_f(C_4^j) = f(v_j) + f(v_j v_{j+1}) + f(v_{j+1}) + f(v_{j+1} v_{j+2}) + f(v_{j+2}) + f(v_{j+2} v_{n+1}) + f(v_{n+1}) + f(v_j v_{n+1}). \tag{15}$$

From Theorems 1 through 3 it follows that F_n , provided necessary condition for the covering by C_4 cycles is met, then there exist super (a, d) - C_4 -antimagic labelings for every $d \in \{1, 3, 5, 7, 11\}$. The following theorem shows also that differences $d = 0, 2, 4$ and $d = 6$ are feasible.

Theorem 5. *The fan F_n , $n \geq 4$, is super (a, d) - C_4 -antimagic for $d = 0, 1, 2, 3, 4, 5, 6, 7, 11$.*

Proof. For $d = 1, 3, 5, 7, 11$ the results follow from Theorems 1 through 3. If $d = 0, 2, 4, 6$ let us consider the following.

For F_n , $n \geq 4$, define the total labelings h_m , $1 \leq t \leq 4$, in the following way

$$h_m(v_i) = \begin{cases} n + 1 - i, & \text{for } i = 1, 2, \dots, n \text{ and } m = 1, 3 \\ n + 1, & \text{for } i = n + 1 \text{ and } m = 1, 3 \\ i, & \text{for } i = 1, 2, \dots, n + 1 \text{ and } m = 2, 4 \end{cases}$$

$$h_m(v_i v_{n+1}) = \begin{cases} 2n + i, & \text{for } i = 1, 2, \dots, n \text{ and } m = 1, 4 \\ 3n + 1 - i, & \text{for } i = 1, 2, \dots, n \text{ and } m = 2, 3 \end{cases}$$

$$h_m(v_i v_{i+1}) = \begin{cases} n + 1 + \frac{i+1}{2}, & \text{for } 1 \leq i \leq n - 1, i \equiv 1 \pmod{2}, \\ & \text{and } m = 1, 2, 3, 4 \\ n + \lfloor \frac{n}{2} \rfloor + 1 + \frac{i}{2}, & \text{for } 2 \leq i \leq n - 1, i \equiv 0 \pmod{2}, \\ & \text{and } m = 1, 2, 3, 4. \end{cases}$$

It is easy to see that h_m is a super labeling for every $m = 1, 2, 3, 4$. Under all labelings the path edges attain the labels $n + 2, n + 3, \dots, 2n$ and the spokes are labeled by the labels $2n + 1, 2n + 2, \dots, 3n$.

According to (15)

$$wt_{h_1}(C_4^j) = \begin{cases} (n + 1 - j) + (n + 1 + \frac{j+1}{2}) + (n + 1 - (j + 1)) \\ \quad + (n + \lfloor \frac{n}{2} \rfloor + 1 + \frac{j+1}{2}) + (n + 1 - (j + 2)) \\ \quad + (2n + (j + 2)) + (n + 1) + (2n + j) \\ \quad = 10n + \lfloor \frac{n}{2} \rfloor + 6 \\ \hspace{10em} \text{for } 1 \leq j \leq n - 2, j \equiv 1 \pmod{2} \\ (n + 1 - j) + (n + \lfloor \frac{n}{2} \rfloor + 1 + \frac{j}{2}) + (n + 1 - (j + 1)) \\ \quad + (n + 1 + \frac{(j+1)+1}{2}) + (n + 1 - (j + 2)) \\ \quad + (2n + (j + 2)) + (n + 1) + (2n + j) \\ \quad = 10n + \lfloor \frac{n}{2} \rfloor + 6 \\ \hspace{10em} \text{for } 2 \leq j \leq n - 2, j \equiv 0 \pmod{2}, \end{cases}$$

$$\begin{aligned}
 wt_{h_2}(C_4^j) &= \begin{cases} j + \left(n + 1 + \frac{j+1}{2}\right) + (j + 1) \\ \quad + \left(n + \lfloor \frac{n}{2} \rfloor + 1 + \frac{j+1}{2}\right) + (j + 2) \\ \quad + (3n + 1 - (j + 2)) + (n + 1) + (3n + 1 - j) \\ \quad = 9n + \lfloor \frac{n}{2} \rfloor + 7 + 2j \\ \quad \text{for } 1 \leq j \leq n - 2, j \equiv 1 \pmod{2} \\ j + \left(n + \lfloor \frac{n}{2} \rfloor + 1 + \frac{j}{2}\right) + (j + 1) \\ \quad + \left(n + 1 + \frac{(j+1)+1}{2}\right) + (j + 2) \\ \quad + (3n + 1 - (j + 2)) + (n + 1) + (3n + 1 - j) \\ \quad = 9n + \lfloor \frac{n}{2} \rfloor + 7 + 2j \\ \quad \text{for } 2 \leq j \leq n - 2, j \equiv 0 \pmod{2}, \end{cases} \\
 wt_{h_3}(C_4^j) &= \begin{cases} (n + 1 - j) + \left(n + 1 + \frac{j+1}{2}\right) + (n + 1 - (j + 1)) \\ \quad + \left(n + \lfloor \frac{n}{2} \rfloor + 1 + \frac{j+1}{2}\right) + (n + 1 - (j + 2)) \\ \quad + (3n + 1 - (j + 2)) + (n + 1) + (3n + 1 - j) \\ \quad = 12n + \lfloor \frac{n}{2} \rfloor + 4 - 4j \\ \quad \text{for } 1 \leq j \leq n - 2, j \equiv 1 \pmod{2} \\ (n + 1 - j) + \left(n + \lfloor \frac{n}{2} \rfloor + 1 + \frac{j}{2}\right) + (n + 1 - (j + 1)) \\ \quad + \left(n + 1 + \frac{(j+1)+1}{2}\right) + (n + 1 - (j + 2)) \\ \quad + (3n + 1 - (j + 2)) + (n + 1) + (3n + 1 - j) \\ \quad = 12n + \lfloor \frac{n}{2} \rfloor + 4 - 4j \\ \quad \text{for } 2 \leq j \leq n - 2, j \equiv 0 \pmod{2}, \end{cases}
 \end{aligned}$$

and

$$\begin{aligned}
 wt_{h_4}(C_4^j) &= \begin{cases} j + \left(n + 1 + \frac{j+1}{2}\right) + (j + 1) \\ \quad + \left(n + \lfloor \frac{n}{2} \rfloor + 1 + \frac{j+1}{2}\right) + (j + 2) \\ \quad + (2n + (j + 2)) + (n + 1) + (2n + j) \\ \quad = 7n + \lfloor \frac{n}{2} \rfloor + 9 + 6j \\ \quad \text{for } 1 \leq j \leq n - 2, j \equiv 1 \pmod{2} \\ j + \left(n + \lfloor \frac{n}{2} \rfloor + 1 + \frac{j}{2}\right) + (j + 1) \\ \quad + \left(n + 1 + \frac{(j+1)+1}{2}\right) + (j + 2) \\ \quad + (2n + (j + 2)) + (n + 1) + (2n + j) \\ \quad = 7n + \lfloor \frac{n}{2} \rfloor + 9 + 6j \\ \quad \text{for } 2 \leq j \leq n - 2, j \equiv 0 \pmod{2}. \end{cases}
 \end{aligned}$$

This means that under the labelings h_m , $m = 1, 2, 3, 4$, the C_4 -weights form the arithmetic sequences with the differences $d = 0, 2, 4$ and 6 , respectively. \square

5 Conclusion

In this paper we examined the existence of super (a, d) - C_k -antimagic labelings for fans. We proved that the fan F_n , $n \geq 3$, admits a super (a, d) - C_k -antimagic labeling for $k = 3, 4, \dots, \lfloor \frac{n}{2} \rfloor + 2$ and $d \in \{1, 3, k - 7, k + 1, 2k - 5, 2k - 1, 3k - 9, 3k - 1\}$. We showed that there exists a super (a, d) - C_3 -antimagic labeling for $d = 0, 1, 2, 3, 4, 5, 6, 8$ and a super (a, d) - C_4 -antimagic labeling for $d = 0, 1, 2, 3, 4, 5, 6, 7, 11$ of F_n , $n \geq 4$.

For further investigation we propose the following open problem.

Open Problem 1. Find a super (a, d) - C_k -antimagic labeling of the fan F_n for $d \neq 1, 3, k - 7, k + 1, 2k - 5, 2k - 1, 3k - 9, 3k - 1$.

Acknowledgements

The research for this article was supported by APVV-15-0116.

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(Received 16 Mar 2016; revised 7 Mar 2017)