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Automorphisms of S_6 and the color cubes puzzle

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Abstract

Given six colors, a *color cube* is one where each face is single-colored and each color appears on some face. The *Color Cubes puzzle* is a variation of a classic problem due to P. MacMahon: one starts with an arbitrary collection of color cubes of unit length and tries to find a subset that can be arranged into an $n \times n \times n$ cube where each face is a single color. In this paper we determine the minimum size of a set of cubes that, regardless of its composition, guarantees the construction of an $n \times n \times n$ cube's *frame*, its corners and edges. We do this for all n, and find that for $n \geq 4$ one has the best possible result, that as long as there are enough cubes to build a frame it can always be done. Part of our analysis involves the S_6 action on the set of color cubes. In addition to the problem simplification it provides, this action also gives another way to visualize the outer automorphism of S_6.

1 Introduction

Given a palette of six colors, a pleasant combinatorial argument shows that there are 30 distinct ways to color a cube so that each cube face is one color and all six colors appear on some face. The resulting set of 30 cubes can be used to construct a number of puzzles:

1. Select one color cube. Find eight other color cubes with distinct colorings that can be assembled into a $2 \times 2 \times 2$ larger version of the selected cube.

2. Proceed as above, but select the cubes so that all touching internal faces have matching colors as well.

3. Find 27 distinct color cubes that can be used to construct a $3 \times 3 \times 3$ cube where each face is one color.

Percy MacMahon asked all three questions in [16, pp. 42–46]. Solutions to the first two problems appear in MacMahon's text, and a solution to the third appears on the website [14]. MacMahon (1854 – 1929) may be best known for authoring one of the first books on enumerative combinatorics [15] and for serving as the President of the London Mathematical Society. However, he also had a proclivity towards mathematical recreations, and his cube puzzles in particular have become classic– they appear in Martin Gardner's writings a number of times, including in "Thirty Color Cubes" (Chapter 6 of [9]) and "The 24 Color Squares and the 30 Color Cubes" (Chapter 16 of [10]).

The work in this paper is motivated by the third problem on the list. It is also an extension of results in [4], where one takes n^3 *arbitrary* color cubes and determines when it is possible to construct an $n \times n \times n$ cube with single-color faces. (This is the formal Color Cubes puzzle.) The following is the main theorem from [4].

Theorem 1.1. *Let $n > 2$. Given n^3 arbitrary color cubes, it is always possible to solve the Color Cubes puzzle.*

As noted in [4], there is another way to look at this puzzle that is a better indicator of its difficulty. Under the assumption that every color appears on every cube, any cube can be used to make up the $(n-2) \times (n-2) \times (n-2)$ interior of the $n \times n \times n$ cube as well as the $(n-2) \times (n-2)$ interior of its six faces. The only cubes that cannot be chosen arbitrarily are those that lie on its corners and edges, the *frame* of the $n \times n \times n$ cube. Therefore, once a cube's frame has been constructed the Color Cubes puzzle is solved. The number of cubes in the frame grows linearly with n, so it is an interesting question to determine the minimum number of cubes necessary to complete a frame. The focus of this paper is Conjecture 5.4 in [4], which posits that for n sufficiently large, when there are enough cubes to fill in a frame then one can actually construct it. Our main results, which are the best possible, are:

Theorem A. When $n = 2$ or 3, any set of 24 color cubes is sufficient to build a frame.

Theorem B. When $n \geq 4$, any set of $12n - 16$ color cubes is sufficient to build a frame.

The proof of the minimum number of cubes needed to guarantee construction of the frame depends on the frame's size, so we consider separately the cases $n = 2$, $n = 3$, and $n \geq 4$. The proof for the case $n = 2$ already exists in the literature as a paper by Haraguchi [11], and is essentially computational. We include an alternative proof that also relies on some computer searches, but which we feel is more explicit. Many of the arguments in this paper and [11] both involve a particular arrangement of the 30 distinct color cubes—what we call the *tableau*—due to John H. Conway. During our proof of the $n = 2$ case we investigate the S_6 action induced by color permutation on the tableau and determine a number of its properties. This action allows us to simplify many of our arguments. As a bonus, the S_6 action on the tableau provides another concrete demonstration of the action of an outer automorphism of S_6.

The Color Cubes puzzle is just one of a number of cube stacking puzzles. A particular instance of the Color Cubes puzzle with $n = 2$ is known as Eric Cross's "Eight Blocks to Madness," where one tries to arrange a collection of eight cubes into a $2 \times 2 \times 2$ cube. Another well-known cube stacking puzzle is Instant Insanity®, a 4-color puzzle whose elegant graph theoretic solution is presented in many introductory texts on combinatorics (see [6], for example).

This paper is organized as follows. In the next section we introduce notation and a formal statement of the Color Cubes puzzle. In Section 3 we describe the tableau and some of its properties. We also analyze the S_6 action on the tableau and show that it is related to the outer automorphism of S_6. In Section 4 we cover the case $n = 2$, in Section 5 the case $n = 3$, and in Section 6 the cases $n \geq 4$. In Section 7 we address the complexity of a generalization of the Color Cubes puzzle with n colors, and show that it can be solved in polynomial time. We conclude with some open questions in Section 8.

2 Definitions and Problem Statement

In what follows, we will assume that all cubes are colored with six colors unless we explicitly state otherwise. To distinguish between different cubes, we set up an equivalence and say that cubes that have the same coloring up to direct isometry of $\mathbb{R}^3$ are of the same *variety*. As mentioned in the Introduction, there are 30 distinct varieties of color cubes. A *solution* is an arrangement of n^3 cubes that forms an $n \times n \times n$ cube with single-color faces. A *corner solution* is a set of eight cubes, appropriately oriented, that fills the corner positions of a solution. We say that an $n \times n \times n$ solution is *modeled* on a cube if the solution and its model are of the same variety. We note that a solution and its corner solution are always of the same variety.

We also track relative positions of face colors on the cube. The unordered pair of colors that are opposite each other on a cube form an *opposite pair*. Similarly, the unordered pair of colors on faces that share an edge of a cube are an *adjacent pair*.

We call the three colors on the faces that meet at the corner of cube a *corner triple*. Since order matters for corner triples, we read the colors clockwise around the corner. Furthermore, two corner triples are equivalent if one is a cyclic permutation of the other. That is, $(2, 3, 6) \sim (6, 2, 3) \not\sim (6, 3, 2)$. Corner triples are *mirror image* corner triples if they contain the same colors but are not equivalent. On a given cube, every possible unordered pair of distinct colors appears either as an adjacent pair or an opposite pair. Therefore, every cube contains 12 of the 15 possible adjacent pairs, and 8 of the 40 possible corner triples.

In general, we will denote a generic cube by c. One variety that very closely resembles c is its *mirror image* c^*; the varieties c and c^* together form a pair of *mirror cubes*. Mirror cubes have the same opposite and adjacent pairs, but have mirror image corner triples. We have the following useful characterization.

Lemma 2.1. *The varieties c and c^* are related through the exchange of an opposite pair. Therefore, c and c^* are the only two varieties with the same opposite pairs.*

Proof. The first claim follows by visualizing a mirror placed parallel to some face of a variety c. For the second claim, if two varieties have the same opposite pairs, then they can be oriented so at least two of the opposite pairs are in the same orientation. □

As mentioned in the Introduction, the Color Cubes puzzle is solved once the frame is complete. In an $n \times n \times n$ cube, the *n-frame* consists of $12(n-2)+8 = 12n-16$ cubes. We incorporate this definition into a more refined version of our puzzle.

The Color Cubes Puzzle (Frame Version): Given a set of cubes, determine whether it is possible to construct some n-frame.

The solution to the Color Cubes puzzle in [4] involved the construction of a frame given an arbitrary collection of n^3 cubes. Since the number of cubes in the frame grows linearly with n, starting with n^3 cubes seems generous. In fact, since there are only 30 distinct cube varieties, when $12n - 16 < \lceil \frac{n^3}{30} \rceil$, the pigeonhole principle guarantees a solution for the frame using just one variety of cube. This happens for $n \geq 19$. We are interested in the other direction, determining small sets of cubes that can still be used to build a frame.

Definition 2.2. *Let the frame number, denoted by* $\mathrm{fr}(n)$*, be the smallest value so that given a set of* $\mathrm{fr}(n)$ *cubes, one is guaranteed to be able to build an n-frame, regardless of the set's component cube varieties.*

Knowing the value of $\mathrm{fr}(n)$ gets to the heart of the Color Cubes puzzle, so that's where we focus our attention.

3 The cube tableau and automorphisms of S_6

We describe the tableau, an arrangement attributed to J. H. Conway [14], of the 30 cube varieties into a 6×6 matrix with blank diagonal entries. A copy of this tableau can be found in the Appendix. Each of the 30 slots contains the net of a cube variety, and under each net are three pairs of numbers in braces which represent the variety's opposite faces. J. H. Conway used the tableau to provide a complete answer to puzzle 2 in the Introduction [14]. In Conway's notation, tableau rows are labeled A through F and tableau columns are labeled a through f, giving tableau entries the following "coordinates."

	Ab	Ac	Ad	Ae	Af
Ba		Bc	Bd	Be	Bf
Ca	Cb		Cd	Ce	Cf
Da	Db	Dc		De	Df
Ea	Eb	Ec	Ed		Ef
Fa	Fb	Fc	Fc	Fe	

The collection of the 30 distinct cube varieties contains a great deal of structure, much of which is visible in the tableau. P. Cameron describes the construction of the tableau in a WordPress blog dealing with S_6, where he also lists some of its properties [5].

1. Each of the 15 possible color pairs appears exactly once in each row and each column of the tableau. Geometrically, this means that two colors are an opposite pair on exactly one variety per row and per column.

2. The varieties exhibit mirror symmetry across the diagonal line of the tableau. (In Conway's nomenclature, varieties Xy and Yx are mirror cubes.) This can also be seen in the tableau in the Appendix by Lemma 2.1, since mirror cubes have the same opposite pairs.

3. There are 15 combinations of six colors into three sets of two, and all 15 are represented in the nets above the diagonal and in the nets below the diagonal.

There is a natural S_6 action on the tableau that arises from permuting the six face colors. Furthermore, the action provides a way to visualize the outer automorphism of S_6. Following Cameron and using terminology that goes back to Sylvester [18], we call an unordered pair of distinct colors a *duad* and a collection of three duads a *syntheme*. A collection of five synthemes such that each of the 15 duads appears exactly once is a *pentad*. There are six distinct pentads. Duads, synthemes, and pentads correspond to opposite pairs, the three opposite pairs in a cube, and the collection of such pairs in each row and column of the tableau, respectively.

Lemma 3.1. *A permutation of the color palette induces a permutation that sends rows to rows and columns to columns. Once the permutation is determined on the top row of the tableau, the action on the rest of the tableau is uniquely determined.*

Proof. A color permutation of the palette takes pentads to pentads, so it is sufficient to show that a row isn't taken to a column. Since all permutations in S_6 are generated by transpositions, we will show that applying a transposition sends rows to rows. Let (a_1, a_2) be the transposition. There is precisely one variety in each row with the duad $\{a_1, a_2\}$ as part of its syntheme; fix a row and call this variety c. By Lemma 2.1, exchanging colors a_1 and a_2 on c has the geometric effect of sending it to its mirror image c^*. That means that c's row pentad is sent either to the row pentad of c^* or the column pentad of c^*. Take another variety d in c's row with syntheme $\{a_1a_3\}\{a_2a_4\}\{a_5a_6\}$. Applying (a_1, a_2) to d yields the syntheme $\{a_2a_3\}\{a_1a_4\}\{a_5a_6\}$, which by Lemma 2.1 is not the same as d^*. Since d^* is part of c^*'s column pentad and already contains the duad $\{a_5a_6\}$, (a_1, a_2) must send d to c^*'s row pentad. Finally, since the tableau has mirror symmetry across the diagonal, once the action on the rows is determined, so is the action on the columns. □

Lemma 3.2. *Consider the set of permutations of the color palette that fix a distinguished variety in the top row of the tableau. This set is isomorphic to S_4, and it acts faithfully on the non-distinguished cubes in the top row.*

Proof. The stabilizer of the distinguished variety is its group of direct isometries in $\mathbb{R}^3$ which is isomorphic to S_4, and by Lemma 3.1, any palette permutation that fixes the distinguished variety sends its row to itself. A simple check shows that at least one element of the stabilizer acts non-trivially on the tableau. Further, if a permutation fixes every variety in the first row, by Lemma 3.1 it fixes every variety in the tableau and must be in the tableau stabilizer, which is a normal subgroup of S_6. The only possible non-trivial normal subgroup, A_6, is clearly too large to be this stabilizer. We conclude that the action is faithful. □

Corollary 3.3. *Let σ be the map that takes $\alpha \in S_6$ to its action on the six pentads via permutation of the color palette. Then σ is an outer automorphism of S_6.*

Proof. Given a transposition (a_1, a_2), since every row and column in the tableau contains exactly one cube with the duad $\{a_1, a_2\}$, each of these cubes gets sent to a different row, the row containing its mirror image. Therefore, the effect of (a_1, a_2) on the tableau is to swap three pairs of rows. This map cannot be an inner automorphism since it does not preserve cycle structure. □

To our knowledge Lemmas 3.1 and 3.2, and Corollary 3.3 have not appeared in print. Together, they show that the tableau provides a particularly nice demonstration of the action of the outer automorphism of S_6. The S_6 action on the tableau joins other recent results of this type [13], and we believe it deserves to be better known.

There are additional relationships between varieties in the rows and columns which we will use in subsequent sections. Before we state these results we recall two lemmas from [4].

Lemma 3.4. *[4, Lemma 2.6] Two cubes share exactly nine, ten, or twelve adjacent pairs according to whether they share exactly zero, one, or three opposite pairs, respectively.*

Lemma 3.5. *[4, Lemma 2.7] Given two varieties:*

1. *If they have no opposite pairs in common, then they share zero or two corner triples.*

2. *If they have one opposite pair in common, then they share exactly two corner triples.*

3. *If they have three opposite pairs in common, then they share zero or eight corner triples.*

The proof of Lemma 3.5, which we summarize here, follows from the description of how a variety c is related to the other 29 cube varieties. There are eight varieties that arise from choosing a corner triple of c and cyclically permuting its colors clockwise and counterclockwise. These varieties share two corner triples but no opposite pairs with c. Their mirror images form eight new varieties which share no corner triples and no opposite pairs with c. There are twelve varieties that come from exchanging an adjacent pair on c. These varieties all share two adjacent corners and one opposite pair with c. The last variety is c^*, which shares three opposite pairs and no corner triples with c. This characterization also provides additional structure to the tableau, as well as a means for its construction.

Corollary 3.6. *Fix a distinguished variety c in position Ab in the tableau. Then*

1. *The variety Ba is c^*.*

2. *The eight varieties related to variety c by cyclically permuting a corner followed by a mirror image are in column b and row A.*

3. *The eight varieties related to variety c by cyclically permuting a corner are in column a and row B.*

4. *The twelve varieties related to variety c by edge flipping are in columns c, d, and e, and rows C and D, and E.*

Proof. Since the tableau has mirror symmetry, c^* is in position Ba. Column b and row A are pentads containing variety c, so these eight varieties share no corners with c. From the characterization in Lemma 3.5 and the discussion afterwards, these varieties are formed from c by cyclically permuting a corner followed by a mirror image. By checking cases, one finds that the varieties that are formed using a clockwise cyclic permutation constitute one pentad, and the varieties that are formed using a counterclockwise cyclic permutation form the other. Using mirror symmetry, column a and row B are formed from c by cyclically permuting a corner. That leaves the last twelve varieties in the claimed positions. □

Corollary 3.7. *Let Xy be an arbitrary variety in the tableau. Then the varieties that share no corners with variety Xy are precisely the variety Yx, the varieties in row X, and the varieties in column y.*

Proof. The variety Yx is the mirror image of Xy, so shares no corner triples with it. The statement about the varieties in row X and column y follows from Corollary 3.6, once we note by Lemma 3.1 that there is a permutation of the color palette that moves variety Xy to the distinguished position. □

4 Building a $2 \times 2 \times 2$ solution: $\mathrm{fr}(2) = 24$

In this section we calculate fr(2). This calculation is also a main result, Theorem 3, in Haraguchi's paper [11]. Haraguchi uses mirror symmetry of the tableau and the existence of pentads along with constraint programming and integer programming to provide what he describes as a "computer-assisted proof" of the value of fr(2). In addition, his Theorem 2 is the same as our example below of a collection of 23 cubes without a corner solution. Our proofs also incorporate some computational results (namely Lemmas 4.1 and 4.2), but most differ from those in [11] in that they utilize properties of the tableau to a greater extent, and provide a more detailed look into why sets of 24 cubes always have solutions.

In [4] it was (incorrectly) conjectured that $\mathrm{fr}(2) = 23$. Using properties of the tableau from Section 3, one can quickly construct a counterexample. We first describe a way to keep track of the cubes in an arbitrary set using the tableau. A set may contain many cubes of one variety; if variety Xy occurs k times in the set, put k into Xy's position in the tableau. We call k the *multiplicity* of the variety; if $k \geq 8$ then there is a corner solution modeled on Xy. If variety Xy is not part of the set, we put a dot in its position in the tableau.

Consider the following collection of 23 cubes, which is also given in Theorem 2 in [11]. Within the five cubes of one row pentad, take seven copies of three varieties and one copy of the other two. By Lemma 3.1 there is a color automorphism that moves one variety with multiplicity seven to position Ab in the tableau. By Lemma 3.2 there is another automorphism that puts the row A into the following form:

$$7 \quad 7 \quad 7 \quad 1 \quad 1$$

This collection cannot be used to build a corner solution modeled on any variety in the top row of the tableau since no two varieties in this row share corners. One also can't build a corner solution modeled on any variety Xa in column a. Variety Xa shares no corners with variety Ax, and Lemma 3.5 implies that the other varieties in row A can contribute at most two corners to the frame. However, at least one variety only appears once in the collection. In a similar way, a variety Xy in columns b through f shares no corners with variety Ay, and the remaining four cube varieties are insufficient to construct a corner solution.

We claim that $\mathrm{fr}(2) = 24$. Our argument uses partitions of the set of 24 cubes into k subsets, where the size of a subset determines a variety's multiplicity. Depending on context we write our partitions in one of two forms, as 54^3321^2 or $(5, 4, 4, 4, 3, 2, 1, 1)$. Both tell us that the set of 24 cubes is divided into eight distinct varieties, one variety with multiplicity 5, three with multiplicity 4, etc. Once we fix a partition, we still need to assign varieties to the multiplicities, and with a set of 30 possibilities the number of ways to do this is quite large. For example, the number of cases we have to check to ensure that the partition 54^3321^2 of 24 cubes always has a solution is about 1.97×10^{10}. Instead of checking them all, we determine conditions on the structure of the partition that will always yield a solution. This is the focus of the next two lemmas.

Lemma 4.1. *Any collection that contains ten distinct varieties always has a corner solution.*

Proof. The proof is by computer search, written in *Mathematica*. One builds the *cube-corner* bipartite graph, where one set of vertices is the set of 30 possible cubes and the other set is the 40 possible corners. Edges in the cube-corner graph connect varieties in the first vertex set with their corners in the second. The algorithm proceeds by choosing a subset of ten varieties from the 30 possible. Rather than work with the entire bipartite graph each time, for each subset the algorithm loops through all 30 possible corner solutions, building the subgraph of the cube-corner graph where one set of vertices is the subset of ten varieties, and other set is the set of eight corners from the variety of the potential corner solution. *Mathematica* then determines a maximum matching, where a matching of size eight means the distinguished corner solution can be assembled from the subset. For every fixed subset, the algorithm always finds a variety that yields a maximum matching. □

At first glance, it appears that the proof of Lemma 4.1 requires checking $\binom{30}{10} \approx 3.0 \times 10^7$ cases, but we can use the tableau to reduce this number somewhat. Given ten distinct varieties in the tableau, at least two are in the same row. This row can be moved to the top of the tableau using the S_6 action, and furthermore we may assume that the two varieties are Ab and Ac. This leaves $\binom{28}{8} \approx 3.1 \times 10^6$ cases, about one tenth of the previous number. This shorter computation still requires almost four hours of CPU time using a 2.70 GHz Intel Core i7-3740QM CPU with a 64-bit operating system and 8 GB of memory.

The result in Lemma 4.1 implies that it is sufficient to consider subsets of 24 cubes comprised of between four and nine different varieties, a total of 354 partitions. We use the next lemma, which is also proven by computer search and is a small extension of a result in [4], to show that the vast majority of these contain a subset that forms a corner solution, regardless of the varieties that appear.

Lemma 4.2. *(See [4], Lemma 4.1.) Given the following sets of cubes, one can always construct a corner solution.*

- *18 cubes consisting of two varieties with multiplicity* 7 *and any four other varieties.*
- *16 cubes consisting of two varieties with multiplicity* 6 *and two varieties with multiplicity* 2*.*
- *19 cubes consisting of two varieties with multiplicity* 5 *and three varieties with multiplicity* 3*.*
- *16 cubes consisting of three varieties with multiplicity* 4 *and two varieties with multiplicity* 2*.*
- *18 cubes consisting of six varieties with multiplicity* 3*.*
- *14 cubes consisting of seven varieties with multiplicity* 2*.*

□

We note that Lemma 4.1 in [4] erroneously claimed that one can always construct a corner solution from two varieties with multiplicity 4 and four with multiplicity 2. The correct statement is above, that three varieties with multiplicity 4 and two with multiplicity 2 suffices. Interested readers may contact the first author for copies of the code that was used to prove Lemmas 4.1 and 4.2.

Lemma 4.2 immediately implies the existence of a corner solution for all but 25 partitions. (One partition for which the lemma does not apply, for example, is $753^2 21^4$.) The next three lemmas provide tools to handle many of these remaining cases.

Lemma 4.3. *Given a set of eight cubes consisting of four varieties from any pentad, each with multiplicity* 2*, one can always construct a corner solution.*

Proof. Taking mirror images if necessary, we may assume the pentad is a row pentad. Then by Lemmas 3.1 and 3.2 we may assume that the varieties are in the top row (row A) and in columns b through e of the tableau, like so:

$$2 \quad 2 \quad 2 \quad 2 \quad \cdot$$

By Corollary 3.7, each of these varieties shares exactly two corners with varieties Bf, Cf, Df, or Ef. Furthermore, the corners are distinct, since the top row is a pentad. So there are at least four possible corner solutions. □

The next result is taken from [4] and has a slightly different flavor, focusing on how many corners of a solution can be filled.

Lemma 4.4. *(See [4], Lemma 4.4.) Fix a variety c. Given $k \leq 4$ distinct varieties that share corners with c, one can always use these varieties for k distinct corners in a solution modeled on c.*

Lemma 4.5. *Given a decreasing partition $(a_1, a_2, \ldots, a_n)$ with $a_2 \geq 4$, if $a_1 = 7$ and $a_2 + n \geq 13$, then there is a subset of eight cubes that forms a corner solution. The same is true if $a_1 = 6$ and $a_2 + n \geq 14$.*

Proof. For simplicity, we will refer to the variety with multiplicity a_i as variety i. The main idea of this proof is to determine conditions when there are enough varieties that share corners with variety 2 to complete a corner solution modeled on it.

By using the S_6 action, we may assume that variety 1, which appears 7 times, is in position Ab in the tableau. Then by Corollary 3.7, there is automatically a solution unless the remaining cube varieties are among the nine in row A, column b, and the mirror variety Ba. Therefore, variety 2 is either in a pentad with variety 1 (we may assume it is the row pentad) or is variety 1^*. By using a color automorphism, we may further assume that variety 2 is in position Ac or Ba as in the diagrams below. Boxes represent the cube varieties that do not automatically result in a corner solution as noted in Corollary 3.7.

	a_1	a_2	□	□	□
□		·	·	·	·
·	□		·	·	·
·	□	·		·	·
·	□	·	·		·
·	□	·	·	·	

Case Ac

	a_1	□	□	□	□
a_2		·	·	·	·
·	□		·	·	·
·	□	·		·	·
·	□	·	·		·
·	□	·	·	·	

Case Ba

Case Ac: Variety 2 shares two corners with variety 1* and all the varieties in the column's boxed positions. By Lemma 4.4, when $k \leq 4$ of these varieties are present, then they can be used to construct k distinct corners of a corner solution modeled on variety 2. Since $a_2 \geq 4$ and there are only five slots in the top row of the tableau, we can always build a corner solution when $a_2 + (n-5) \geq 8$.

Case Ba: Here, variety 2 shares two corners with every boxed variety, so $a_2 + (n-2) \geq 8$ (equivalently, $a_2 + n \geq 10$) varieties guarantees a corner solution. This condition automatically holds when $a_2 + n \geq 13$.

Finally, if $a_1 = 6$, then one variety with multiplicity 1 can be in the dotted positions in the tableau without yielding a corner solution. Now there are 7 distinct corners available to build a corner solution modeled on variety 1, and we're in the prior case. □

We apply Lemma 4.5 to the 25 partitions and end up with nine remaining cases to consider:

$$75^31^2,\ 75^241^3,\ 65^31^3,\ 743^321^2,\ 65^241^4,\ 5^41^4,\ 73^421^3,\ 643^321^3,\ 5^341^5$$

These cases can all be handled by ad hoc methods, although broadly speaking the techniques are similar to those in the proof of Lemma 4.5. The main difference is that we consider when multiple copies of a variety contribute more than one corner to a solution. In general, one places variety 1 in position Ab. Variety 2 is then in either position Ac or Ba. The case of position Ba is usually easy to analyze. When $a_1 = 7$ (respectively, 6), we apply the condition from the Ba case of Lemma 4.5 and confirm that $a_2 + n \geq 10$ (respectively, 11) holds. Also, Lemma 4.3 implies that if there are four varieties in the same pentad with multiplicity greater than 1 then there is a solution, so we avoid that situation too. Keeping these observations in mind, we sketch arguments for the two most involved of the remaining cases. The other seven cases are similar, but easier.

1. The partition 73^421^3: Let $a_1 = 7$ and $a_2 = 3$ as in the Case Ac. We note that all varieties in positions Ac – Af in row A share two distinct corners with all varieties in positions Cb – Fb in column b. By Lemma 4.3, four distinct varieties with multiplicity at least 2 in variety 1's row or column pentad are sufficient to build a corner solution. The only other possibilities, up to order in the row and column, are below. (Variety 9 can be in either of the boxed positions.)

	7	3	3	1	□
2		·	·	·	·
·	3		·	·	·
·	3	·		·	·
·	1	·	·		·
·	□	·	·	·	

Case 1

	7	3	3	1	□
3		·	·	·	·
·	3		·	·	·
·	2	·		·	·
·	1	·	·		·
·	□	·	·	·	

Case 2

In both cases, the three varieties in positions Cb, Db, and Eb can contribute five distinct corners to a corner solution modeled on the variety in position Ac, which has multiplicity 3. These are enough copies to complete the corner solution.

2. The partition $5^4 1^4$. In this case, there can be up to two varieties in dotted positions in the tableau. However, if one of those varieties has multiplicity 5 then there is a corner solution modeled on that variety; variety 1 shares two corners with it, as do all except for two varieties in variety 1's row and column pentads. Similarly, if variety 2 is in position Ba then there is a solution; arguing as in Lemma 4.5, $a_2 + n = 13 \geq 12$. Therefore, the three remaining varieties with multiplicity 5 are in variety 1's row or column. If they are all in the same pentad, then Lemma 4.3 implies a solution. Otherwise, the remaining varieties split two and one (say into variety 1's row and column respectively). Then the two varieties in variety 1's row provide three corners for a corner solution modeled on the variety in the column.

Since there are corner solutions for all possible partitions we have the main result of this section and the first part of Theorem A. (See also Theorem 3 from [11].)

Theorem 4.6 (Theorem A, part 1)**.** *Given any set of* 24 *cubes, there is always a subset from which one can construct a corner solution. Consequently,* $\mathrm{fr}(2) = 24$.

5 Building a $3 \times 3 \times 3$ solution: $\mathrm{fr}(3) = 24$

In contrast to the $2 \times 2 \times 2$ case, determining $\mathrm{fr}(n)$ for $n > 2$ requires knowledge of more than the corner solution. This is reflected in the proofs in this section, which have a different flavor than the case $n = 2$. In order to determine how to fill the edges of the frame, we need to know a bit more about how cube varieties are related to each edge. This is the subject of the next few results, which describe how to construct partial frames given cubes of a particular type, how to place cubes into edges of the frame, and how cubes that share an opposite pair are related. In addition, these results provide a number of conditions which, when satisfied, guarantee the existence of a corner solution. We use these results both in this section and in Section 6.

Lemma 5.1. *Given a corner solution and* $k(n-2)$ *cubes that share* k *edge pairs with the corner solution, then it is possible to place all* $k(n-2)$ *cubes into the* n*-frame.*

Proof. By Lemma 3.4, $k = 9, 10$, or 12. Consider a bipartite graph where one set of vertices, $\{u\}$, represents the $k(n-2)$ cubes, and the other set of vertices, $\{v\}$, represents the $12(n-2)$ edge positions on the frame. One connects vertices u and v whenever u can be used to fill position v in the frame. By hypothesis, all vertices u have degree $k(n-2)$. Therefore, by Hall's theorem there is a matching of size $k(n-2)$, the size of the vertex set $\{u\}$. This is the required assignment. □

Lemma 5.2. *Fix c, one of the six cube varieties that share one opposite pair. The variety c shares no corners with c^*, and shares exactly two unique corners with each of the other four varieties.*

Proof. Assume that the opposite pair are the colors 5 and 6 (with 5 facing up), and that the colors around the girth of c are given by the cyclic permutation (1234). The cyclic permutations for the other five varieties are (1243), (1324), (1342), (1423), and (1432). In the girth, the first four share exactly one ordered adjacent pair with c, namely $\{1, 2\}$, $\{4, 1\}$, $\{3, 4\}$, and $\{2, 3\}$ respectively. Each ordered adjacent pair gives rise to a pair of corners that match c's. The last cube is c^*. □

Corollary 5.3. *Take four copies of c and two copies each of two other varieties that share the same opposite pair. If the two other varieties are not c^*, then the eight cubes can be assembled into a corner solution modeled on c. The same is true for six copies of c and two cubes that are not of the same variety as c^*.*

Lemma 5.4. *Given seven cubes each of varieties c and c^* along with one cube of any other variety, one can find a subset of eight cubes that forms a corner solution modeled on either variety c or variety c^*.*

Proof. In the tableau, assume that c is the distinguished variety. By Corollary 3.7, any variety that isn't c or c^* shares two corners with either c or c^*. □

Lemma 5.5. *[4, Lemma 3.2] Given 15 cubes that share one opposite pair, it is always possible to find a subset of eight that forms a corner solution.*

Proof. A sketch is as follows: Find the variety, c, that occurs with largest multiplicity in the set of 15. If variety c occurs seven times, then a set of seven copies each of c and c^* does not have a subset that forms a corner solution, but the set formed by adding any other variety does by Lemma 5.4. The cases where variety c occurs few than seven times are similar, and have lower thresholds. □

From Section 4 we know that $\text{fr}(2) = 24$, so $\text{fr}(3) \geq 24$. We now show that this inequality is sharp, which is the second part of Theorem A. We base our proof on the technique used in Theorem 3.4 of [4].

Theorem 5.6 (Theorem A, part 2)**.** *The frame of a $3 \times 3 \times 3$ puzzle can always be completed given 24 arbitary cubes, so $\text{fr}(3) = 24$.*

Proof. Given 24 cubes, there is always a $2 \times 2 \times 2$ solution by Theorem 4.6. If we can place twelve of the remaining 16 cubes into the twelve edge positions (which we also identify as adjacent pairs) then we have a solution. We assume this is not possible and show that we can find a different corner solution whose frame can be completed. Denote by S_1 the set of 16 remaining cubes. Label the twelve edges of the corner solution as e_1 through e_{12}, and let n_i denote the number of cubes in S_1 that have e_i as an adjacent pair. We may assume that the edges $\{e_i\}$ are labeled so that $n_1, n_2, \ldots n_{12}$ are in ascending order. We note that $n_i \geq i$ is a sufficient condition for a solution to the frame–pick any cube that has adjacent pair e_1 as its representative in the frame and continue the process in ascending order. If this cannot be done, then there is a largest index j with $n_j < j$.

We bound the total number of adjacent pairs in two ways. On the low end, each of the 16 cubes in S_1 shares at least nine adjacent pairs with the solution by Lemma 3.4. On the high end, the j edges $e_1, \ldots, e_j$ may occur no more than $j - 1$ times each, whereas the $12 - j$ edges $e_{j+1}, \ldots, e_{12}$ can occur 16 times each. That is,

$$16 \times 9 \leq \sum_{i=1}^{12} n_i \leq (j)(j-1) + (12-j)(16).$$

When we solve this quadratic inequality over the integers, we find that $j \leq 3$ or $j \geq 15$. The latter case is impossible as $j \leq 12$. We conclude that if we cannot complete the frame, it is because at most three edges cannot be matched. Although this is consistent with the results of Lemma 5.1, we are now able to identify three situations where there fails to be a solution: $n_1 = 0$; $n_1 = n_2 = 1$; or $n_1 = n_2 = n_3 = 2$.

Case 1: $n_1 = n_2 = n_3 = 2$. If these adjacent pairs are from at least three cubes then by Hall's theorem we can pick representatives for the three edges e_1, e_2, e_3, and complete the $3 \times 3 \times 3$ frame. Otherwise, the adjacent pairs are from the same two cubes, and the three pairs of adjacent colors are opposite on the other 14 cubes in S_1, i.e., they form a syntheme. Therefore, Lemma 2.1 implies that the 14 cubes in S_1 are of a variety and its mirror image. Let c_M be a variety which occurs the most, with multiplicity at least seven.

We claim that we can always construct a frame modeled on variety c_M. That there is a corner solution modeled on c_M follows immediately if c_M occurs eight or more times. Otherwise, c_M and c_M^* both have multiplicity seven, and Lemma 5.4 implies the existence of a corner solution. By Lemma 5.1, we can use the eight cubes from the original corner solution to fill in eight edges of the 3-frame. From Lemma 3.4, we know that c_M and c_M^* have the same adjacent pairs, so any of the remaining copies of c_M and/or c_M^* from S_1 complete the four unfilled frame edges.

Case 2: $n_1 = n_2 = 1$. As in Case 1, we may assume that the adjacent pairs e_1 and e_2 are from the same cube. In addition, since two adjacent pairs are opposite on the remaining 15 cubes in S_1, e_1 and e_2 have no colors in common. Finally, we note that two opposite pairs uniquely determine the colors of the final opposite pair. This implies that the 15 cubes in S_1 again consist of a variety and its mirror image. Let c_M be the variety which occurs the most, with multiplicity at least eight. We now proceed as in Case 1.

Case 3: $n_1 = 0$. This is the most involved case. There is one adjacent pair, say $\{1, 2\}$, that is opposite on all 16 cubes in S_1. We can build a corner solution from eight cubes in S_1 by Lemma 5.5, and this corner solution will have $\{1, 2\}$ as an opposite pair. Let S_2 consist of the following 16 cubes: the eight from S_1 not used in the new corner solution and the eight from the original corner solution. In particular, $|S_1 \cap S_2| = 8$. We repeat the edge enumerating process with the new corner solution and S_2. Each cube in $S_1 \cap S_2$ shares at least ten adjacent pairs with the corner solution by Lemma 3.4, so

$$8 \times 9 + 8 \times 10 \leq \sum_{i=1}^{12} n_i \leq (j)(j-1) + (12-j)(16).$$

This implies that $j \leq 2$. If there is no solution, then there are two new possibilities. If $j = 2$, then $n_1 = n_2 = 1$, which was addressed in Case 2 above. The last possibility is that again $n_1 = 0$. In this case, all 16 cubes in S_2 have one opposite pair in common.

We claim that the adjacent pair e_1 can contain neither color 1 nor color 2. Assume, for example, that e_1 is the pair $\{1, x\}$. Then 1 and x are opposite in all cubes in S_2, particularly those from S_1. This implies x is color 2, which is impossible since $\{1, 2\}$ is an opposite pair in the new corner solution and cannot be e_1. Therefore, we may assume that e_1 is the adjacent pair $\{3, 4\}$. Then all 16 cubes in S_2 have $\{3, 4\}$ as an opposite pair. In addition, at least eight cubes, including the ones from S_1, have $\{1, 2\}$ and $\{3, 4\}$ as opposite pairs, and hence $\{5, 6\}$ as well. This subset of S_2, which we denote S_3, consists of cubes of some variety and its mirror. Note that by Lemma 3.4, the cubes in S_3 all have the same adjacent pairs. Let c_M be the variety that occurs most often in S_3.

We consider subcases based on the multiplicity of variety c_M in S_3. We note that by Lemma 5.1, any collection of nine cubes can be placed into edge positions in a frame. Furthermore, by Lemma 3.4, any additional cube which shares an opposite pair with the corner solution can be fit into a tenth edge position. Our approach for the rest of the proof is to demonstrate the existence of a set of cubes from which we can both construct a corner solution and fill in any of the two or three edge positions that are not covered by Lemma 5.1.

Subcase: Six or more cubes of variety c_M in S_3: By Lemma 5.2, all cubes in $S_2 \setminus S_3$ share exactly two corners with c_M. If $|S_2 \setminus S_3| \geq 2$, then by Corollary 5.3 we can take two of these cubes and six copies of c_M to complete a corner solution modeled on c_M. By Lemma 5.1, any nine cubes can be fit into the frame, and since all cubes in S_2 share at least one opposite pair with c_M, we can fill a tenth edge position too. There are at least two unused cubes remaining from S_3, which are guaranteed to fill any of the remaining two edge positions in the frame.

Otherwise, $|S_2 \setminus S_3| < 2$, so at least 15 cubes in S_2 are of varieties c_M and c_M^*, and c_M has multiplicity eight or more. Then it is easy to build a corner solution modeled on variety c_M and complete the frame.

Subcase: Five cubes of variety c_M in S_3: Say we can find three cubes which, with five copies of c_M, complete a corner solution modeled on c_M. Then nine edges of the frame can be filled by any collection of cubes, and the three cubes of variety c_M^* from S_3 fill in the remaining edges. By Lemma 5.2, a sufficient condition for this corner solution to exist is for $S_2 \setminus S_3$ to have three cubes of two or more different varieties.

If this does not happen, since $|S_3| \leq 10$, Lemma 5.2 implies that $S_2 \setminus S_3$ contains at least six cubes, all of some third variety c_1. We build a frame modeled on this variety. By Corollary 5.3, four cubes of variety c_1 and two each of c_M and c_M^* can be used to construct a corner solution modeled on c_1. We fit any eight cubes into arbitrary edge positions of the frame. One cube each of c_M and c_M^* fill in edges nine and ten, and two remaining copies of c_1 complete the frame.

Subcase: Four cubes of variety c_M in S_3: In this case c_M and c_M^* both have multiplicity 4 in S_3. We assume first that there are cubes in $S_2 \setminus S_3$ which can be used with the four copies of c_M to build a corner solution modeled on c_M. Any eight cubes will fill eight edges of the frame, and the four copies of c_M^* complete the final four unfilled edges.

If we cannot find the four cubes, then by Lemma 5.2, the eight remaining cubes in S_2 are either all of a third variety c_2, or are seven of variety c_2 and one of variety c_3. Either way, by Corollary 5.3 we can construct a corner solution modeled on c_2 using four copies of c_2 and two each of c_M and c_M^*. Any nine cubes will fit into the edge positions, and the three remaining copies of c_2 finish the frame. ☐

6 Building an $n \times n \times n$ solution: $\mathrm{fr}(n) = 12n - 16$ for $n \geq 4$

In this section we prove Theorem B by showing that for $n \geq 4$ we can always build a frame with the smallest possible number of cubes. In [4], this was possible by tweaking the argument from Section 5. Unfortunately, that does not work in our setting, since in [4] it was assumed that there were roughly n^3 cubes available to build the frame, whereas here the number only grows linearly with n.

Theorem 6.1 (Theorem B)**.** *If $n \geq 4$, then* $\mathrm{fr}(n) = 12n - 16$.

Proof. We proceed using induction. For the base case, we note that the 4-frame contains 32 cubes. By Theorem 5.6, any collection of 24 cubes contains a $3 \times 3 \times 3$ solution. Therefore, any collection with enough cubes for a 4-frame contains a subset of 20 cubes that forms a 3-frame.

Now assume that we can build the frame for the $(n-1) \times (n-1) \times (n-1)$ cube. The difference between an $(n-1)$-frame and an n-frame is twelve cubes, one for each edge. If the 12 cubes cannot be inserted to extend the frame, we will show that it is possible to construct another frame using a different corner solution. Referencing Lemma 5.1, we see that we will be able to fit at least nine of the twelve cubes into the frame. We proceed by cases.

Case 1: exactly eleven edges of the n-frame are complete. Assume that the unfilled position in the frame is the adjacent pair $\{1, 2\}$. In a worst case, the unplaced cube c shares only nine edges with the corner solution. Note that $\{1, 2\}$ is an opposite pair on variety c, so colors 1 and 2 are adjacent to the other four colors. This implies that of the three edges which c doesn't share with the corner solution, the adjacent pair $\{1, 2\}$ is the only one with either color 1 or 2. (In fact, the three non-shared adjacent pairs form a syntheme.)

There are some quick potential ways to complete the frame. If a cube on one of the nine edges that the frame shares with c contains the adjacent pair $\{1, 2\}$, move it to the unfilled position and put c in its place to complete the puzzle. If such a swap is not possible, then $\{1, 2\}$ must be an opposite pair in all $9(n - 2)$ cubes as well as on the unused cube.

Next, assume there is a cube, c_1, among the $2(n-2)$ cubes on the other two non-shared frame edges that has $\{1, 2\}$ as a (hidden) adjacent pair. Denote c_1's adjacent pair contribution to the frame by $\{x, y\}$ (with $x, y \neq 1, 2$). If there is another cube, c_2, on one of the nine shared edges that also has the adjacent pair $\{x, y\}$, put c_1 into the $\{1, 2\}$ edge, replace c_1 with c_2, and replace c_2 with c to complete the frame.

If this cannot be done, then $\{x, y\}$ is an opposite pair on each of the $9(n - 2)$ cubes on the shared edges. Since $\{1, 2\}$ is already an opposite pair on these cubes, by Lemma 3.5, it follows that all $9(n-2)$ cubes are of two mirror varieties, and have the same adjacent pairs. Since $n \geq 4$, we can pick eight of one variety for a corner solution. Start building the frame using the cubes from the first corner solution, the two unshared edges, and the incomplete edge. This is possible by Lemma 5.1. Complete the frame with the remainder of the $9(n - 2)$ cubes, any of which can be used in any edge position.

The other possibility is that there is indeed no cube among the $2(n - 2)$ on the other two edges that has $\{1, 2\}$ as an adjacent pair, implying that $\{1, 2\}$ is an opposite pair on at least $11(n - 2)$ cubes and the unplaced cube. Denote this set of $11n - 21$ cubes by S, and the remaining $n - 3$ cubes of the frame by T. Now $11n - 21 \geq 23$ for $n \geq 4$. Partition S into three subsets, each consisting of a variety and its mirror image (which have the same adjacent pairs). One of these subsets will have size at least eight, so at least one cube variety, say c_3, has multiplicity no less than 4.

Assume first that there are at least two cubes in each of the other two subsets. Then by Corollary 5.3, these can be assembled into a corner solution modeled on c_3. We note that

$$\left\lceil \frac{11n - 21}{3} \right\rceil - 4 \geq 2(n - 2) \text{ for } n \geq 4,$$

so after building the corner solution, there are enough cubes with the same adjacent pairs as c_3 to complete any two edges. Use Lemmas 3.4 and 5.1 to fill in as much as possible of ten edges of the n-frame using cubes from T and the other subsets. Then the remaining $\lceil \frac{11n-21}{3} \rceil - 4$ cubes can be used in any remaining open position in the n-frame.

Next, assume that there is only one cube in one of the subsets of S. Move that cube to T; now S consists of four varieties in two subsets. Since $11n - 22 \geq 22$ for $n \geq 4$, there is at least one subset with 11 cubes and six of some variety, say c_4. As above,

$$\left\lceil \frac{11n - 21}{2} \right\rceil - 4 \geq 2(n - 2) \text{ for } n \geq 4.$$

As long as there are two cubes in the smaller subset, we can apply Corollary 5.3 to build a corner solution modeled on c_4. The rest of the construction is as before. Finally, if there is no more than one cube in the smaller subset, then $11n-23$ cubes are one of two varieties. Use the one that appears most frequently for the corner solution, and the construction of the n-frame is straightforward.

Case 2: exactly ten edges of the n-frame are complete. There are two cubes of the twelve that remain to be placed. Since neither cube can be placed into the edges, the colors that are adjacent in the unfilled positions of the frame are opposite pairs on the two cubes. This implies that the two unfilled adjacent pairs do not share a common color. In a worst case the two cubes share the same nine edges with the corner solution. As in Case 1, if some cube in one of the nine corresponding edges of the frame also has one of the missing adjacent pairs, then move it to the unfilled position and put one of the two unused cubes in its place. This leaves one position left to fill, which was covered in Case 1.

If such a swap cannot be done, then none of the $9(n-2)$ cubes from the completed edges nor the two unused cubes have the two unfilled adjacent pairs in the corner solution. These $9n-16$ cubes must therefore share the same two, and hence three, opposite pairs. Call the set of these cubes S, and the remaining $3n-8$ cubes in the frame the set T. We note that each cube in S is one of two varieties of mirror cubes, and all the cubes in S share the same adjacent pairs. Since $9n-16 \geq 20$ for $n \geq 4$, at least ten of the $9n-16$ cubes are of one variety. Pick eight of these to make the corner solution, then start filling in the edges of the frame using cubes from the set T. Since $|T| < 9(n-2)$, by Lemma 5.1 all the cubes in T can be placed into the new frame. The cubes from S share the same adjacent pairs as the corner solution, so they can be used to complete the n-frame.

Case 3: exactly nine edges of the n-frame are complete. There are three cubes of the twelve that remain to be placed. Since none of the three fit into the existing frame, by Lemma 3.4 all three share the same nine adjacent pairs with the n-frame. If some cube in the corresponding edges of the frame can be used to complete an unfilled edge, swap it out and put one of the three unused cubes in its place. We are now in Case 2, so there is a solution.

If no such swap is possible, then the $9(n-2)$ cubes from the completed edges in the frame and three unused cubes share the same three opposite pairs. We call the set of these $9n-15$ cubes S, the remaining $3n-9$ cubes in the frame the set T, and proceed as in Case 2. □

7 Complexity

An analysis of the complexity of games is popular in the literature, and this popularity extends to geometric puzzles as well. (See [1] and some of its references, or [12].) Robertson and Munro showed in [17] that to determine a solution of the generalized Instant Insanity puzzle with n cubes and n colors is an NP-complete problem. More recent work by Demaine et al. in [8] studies variations of Instant Insanity with several types of prisms; some of these puzzles are NP-complete to solve, while others can be solved in polynomial time. For the n-Color Cubes puzzle, one of the most restrictive situations is to determine, given $12n-16$ n-color cubes, whether one can build some n-frame. We show that this problem can be answered in polynomial time. We thank Adam Hesterberg for sketching out the following argument, based on matching.

Given a palette of n colors, there are $O(n^6)$ distinct cube varieties. For a fixed variety, we can construct a bipartite matching problem. One set of vertices is the set of $12n-16$ cubes, and the other set of vertices consists of the $12n-16$ frame positions, both edges and corners, required to build a frame modeled on the fixed variety. We draw an edge between sets if the cube in the first set contains the edge or corner in the second set. The bipartite graph has $O(n)$ vertices, and since it may happen that a cube can be used in every position in the frame (if it is a cube of that variety), there are $O(n^2)$ edges. There is a solution to the n-Color Cubes puzzle if the graph has a perfect matching, that is, a maximum matching of size $12n-16$. The complexity of matching algorithms is well-studied, and it is known that this type of bipartite matching is solvable in polynomial time [7, Section 27.3]. The result follows.

8 Open Questions and Final Remarks

Although MacMahon's original questions on the set of color cubes are now nearly a century old, they are still generating fruitful investigations. In this section we describe a number of open questions for the interested reader to pursue. We start with the tableau, whose associated S_6 action made it very useful in reducing the number and type of cases we needed to consider in this paper. We believe that there is additional structure in the tableau still to be realized that would further reduce the amount of computation required to complete the arguments. This motivates our first problem.

Problem 8.1. *Refine the analysis of the S_6 action on the tableau, and determine which of the results in Lemma 4.2 follow from this finer understanding.*

The Color Cubes puzzle whose solution is in this paper is just one member of a larger family of related puzzles. A pretty generalization in the spirit of MacMahon's Problem 2 in the Introduction is to determine if it is possible to solve the $3\times3\times3$ puzzle so that all of the internal faces also have matching colors. Another way to generalize the problem is by changing the number of colors. There are two variations of this problem, depending on whether $k\le 6$ or $k>6$. When $k\le 6$, one might start by assuming a regularity condition, and say that a *k-color cube* is one where each cube face has a single color and all k colors appear on at least one face of the cube.

Define $g(n,k)$ to be the minimum number of k-color cubes required to fill all necessary positions in an $n\times n\times n$ color cubes puzzle. For $k\le 6$ colors, the necessary positions are the frame. When $k>6$, there are too many colors for all of them to appear on each cube, although we can still apply a regularity condition that no color appear more than once on a face of any cube. We note, however, that the successful construction of a frame no longer implies that the rest of the $n\times n\times n$

cube can be completed; the necessary positions have to be expanded to include the six $(n-2) \times (n-2)$ face centers as well.

Problem 8.2. *For $n > 1$, determine $g(n, k)$, the minimum number of cubes colored with k colors required to solve the $n \times n \times n$ k-Color Cubes puzzle.*

These calculations have been completed for all n in the cases $k = 2$ and $k = 3$ in [2], and $k = 4$ in [3]. In particular, the results in [2] show that $g(n, 3) = 16n - 17$ for $n \geq 4$. The case $k = 5$ is open, and may be the most challenging of the cases with $k \leq 6$, in large part because of the large number of distinct cubes. The total number of distinct cubes up to rigid rotation can be determined with the aid of a Polya counting argument, and are given in Table 1 for $k \leq 6$.

Number of Colors	2	3	4	5	6
Distinct Cube Varieties	8	30	68	75	30

Table 1: Distinct Cube Varieties on k Colors ($k \leq 6$)

We define $\text{fr}(n, k)$ to be the frame analog of $\text{fr}(n)$ using k colors. Although $g(n, k) = \text{fr}(n, k)$ for $k \leq 6$, we expect that $g(n, k) > \text{fr}(n, k)$ for $k > 6$ and sufficiently large n (probably $n = 3$!). In addition, we also believe that for fixed n, $g(n, k) - \text{fr}(n, k)$ should increase with k, the number of colors. This leads into the third problem.

Problem 8.3. *Determine asymptotic bounds, both upper and lower, on the sizes of $\text{fr}(n, k)$ and/or $g(n, k)$.*

Finally, for all values of k, there are analogous problems that arise when the regularity condition is relaxed. This means that some colors might not appear or certain (or any) cubes, and for $k \geq 6$, a color may also appear more than once on a cube. We expect that removing the regularity condition makes the puzzles considerably more difficult.

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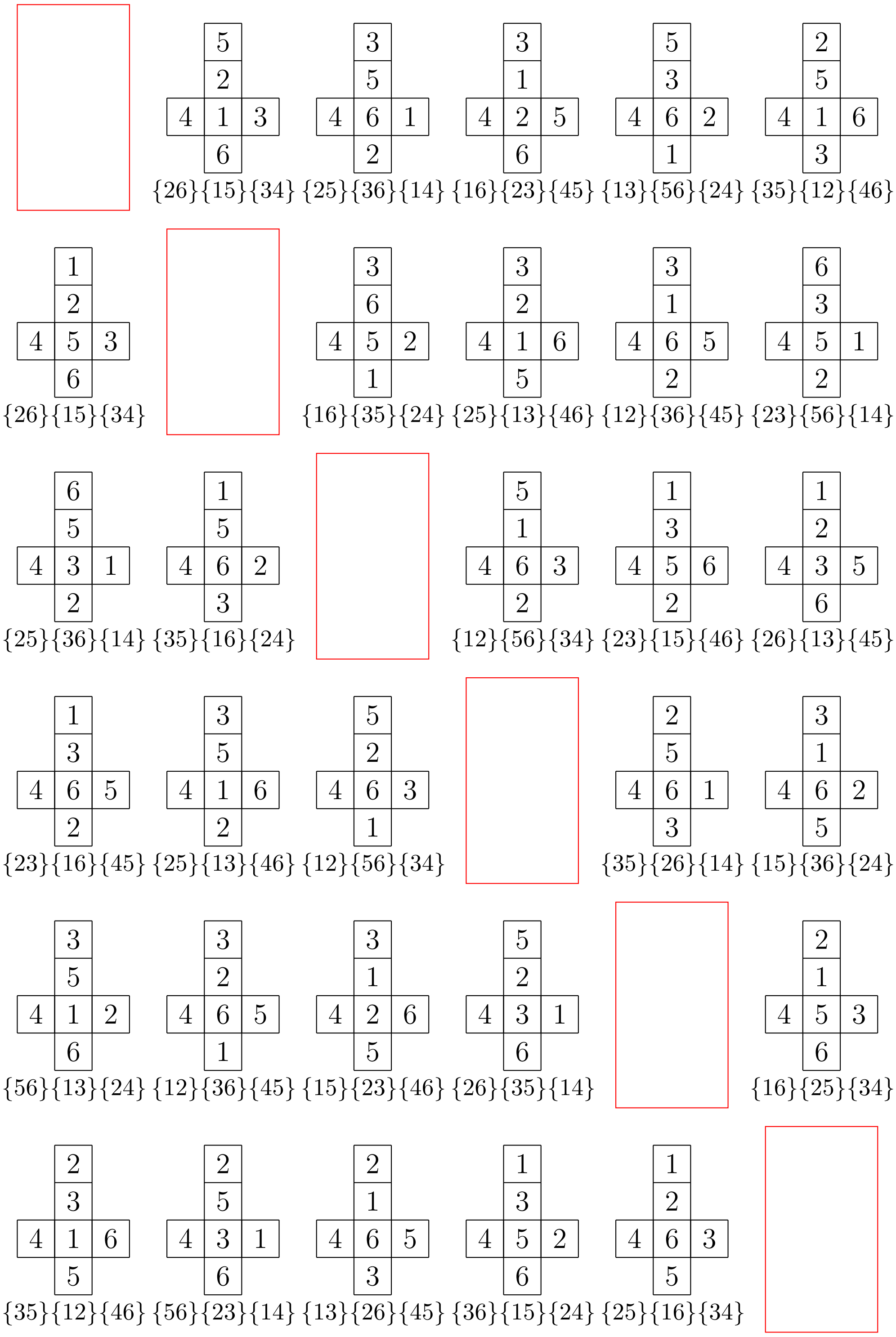
Appendix: The Cube Tableau



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