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Characterization of quasi-symmetric designs with eigenvalues of their block graphs

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Abstract

A quasi-symmetric design (QSD) is a (v, k, λ) design with two intersection numbers x, y, where $0 \leq x < y < k$. The block graph of a QSD is a strongly regular graph (SRG), whereas the converse is not true. Using Neumaier's classification of SRGs related to the smallest eigenvalue, a complete parametric classification of QSDs whose block graph is an SRG with smallest eigenvalue -3, or second largest eigenvalue 2, is obtained.

1 Introduction

Let X be a finite set of v elements called points, and β be a set of k-element subsets of X called blocks, such that each pair of points occurs in λ blocks. Then the pair

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$\mathbf{D} = (X, \beta)$ is called 2-(v, k, λ) *design*. For a 2-(v, k, λ) design $\mathbf{D}$, the number of blocks containing α in X is r, which is independent of α. The number of blocks in $\mathbf{D}$ is denoted by b. A number x, $0 \leq x < k$, is called an *intersection number* of $\mathbf{D}$ if there exist $B, B' \in \beta$ such that $|B \cap B'| = x$. Symmetric designs have exactly one intersection number.

A 2-design with two intersection numbers is said to be a *quasi-symmetric design* (QSD). Denote these intersection numbers by x and y, where $0 \leq x < y < k$. The parameters $(v, b, r, k, \lambda; x, y)$ are called the *standard parameters* of a QSD. The standard parameters are called *feasible* if they satisfy all necessary conditions. The *block graph* Γ of a QSD $\mathbf{D}$ has vertices that are blocks of $\mathbf{D}$, where two distinct blocks B, B' are adjacent if and only if $|B \cap B'| = y$. It was shown in [15] and [4] that a connected Γ is a *strongly regular graph* (SRG) with parameters (b, a, c, d). Here b is the number of vertices of Γ, i.e. the number of blocks of the design $\mathbf{D}$, a its valency, any two adjacent vertices have exactly c common neighbors, and any two non-adjacent vertices have exactly d common neighbors. We assume, as is customary, that an SRG is neither the null graph nor the complete graph. The block graph of $\mathbf{D}$ and block graph of $\overline{\mathbf{D}}$, the complement of the design $\mathbf{D}$, are isomorphic.

It is well-known that a connected block graph of a QSD is an SRG with three eigenvalues, the smallest of which is $-m = -\frac{k-x}{y-x}$. In Theorem 6 and Remark 7 of [11], a technique was developed to prove the non-existence of many classes of QSDs with prescribed block graphs. Let Γ be a (b, a, c, d) SRG. To find feasible parameters of a QSD whose block graph is Γ, steps given in the Remark 7 of [11] are followed. In Appendix II, using these steps, we give the Mathematica code which finds the feasible parameters of QSDs associated with an SRG having parameters (b, a, c, d). We have assigned $(15, 6, 1, 3)$ to (b, a, c, d) and executed the code. The code with output is shown.

Theorem 1 ([11], Theorem 6)**.** *Let* $\mathbf{D}$ *be a QSD with parameters* $(v, b, r, k, \lambda; x, y)$ *and* Γ *be the strongly regular block graph with parameters* (b, a, c, d) *of* $\mathbf{D}$. *Let* $y = z + x$, $k = mz + x$ *and* $r = nz + \lambda$, *for positive integers* m *and* n, *where* $m \leq n$. *Then the following conditions hold:*

(i) $n = \dfrac{m^2 - 2m + a - c}{m - 1}$, $c - d = n - 2m$ *and* $a - d = m(n - m)$;

(ii) $m = \dfrac{1}{2}\left(d - c + \sqrt{(d-c)^2 + 4(a-d)}\right)$;

(iii) $z = \dfrac{(-a + c - d + m + bm)\,(b - s)\,s}{b\,(c - d + 2m)\,(-a - m + bm)}$ *for some positive integer* s;

(iv) $0 \leq b^2 - 4q$, *where*

$$q = \frac{b\,(c - d + 2m)\,(-a - m + bm)}{\gcd(b\,(c - d + 2m)\,(-a - m + bm)\,, -a + c - d + m + bm)}.$$

Using Neumaier's classification of SRGs with smallest eigenvalue $-m$ (see Theorem 2), we prove Theorem 4 which gives a complete parametric classification of QSDs having feasible block graph parameters and with least eigenvalue -3. Theorem 6 deals with the case where the second eigenvalue is 2.

Theorem 2 ([7], Theorem 5.1)**.** *Let Γ be an SRG with smallest eigenvalue $-m$, where m is an integer with $m \geq 2$. Then Γ is one of:*

1. *the complete multipartite graphs with s classes of size m, with parameters*

$$(ms, m(s-1), m(s-2), m(s-1));$$

2. *the Latin square graphs $LS_m(n)$, with parameters*

$$\left(n^2, m(n-1), n+m^2-3m, m(m-1)\right);$$

3. *the Steiner graphs $S_m(n)$, with parameters*

$$\left(\frac{(m+n(m-1))(n+1)}{m}, mn, n+m^2-2m, m^2\right);$$

4. *finitely many other* exceptional *graphs.*

Theorem 2 generalizes Seidel's classification result ([12], Theorem 14) about strongly regular graphs with least eigenvalue -2.

The Mathematica code given in Appendix I, is written, using inequalities and Krein conditions given in [7], to find exceptional feasible parameters of SRGs with smallest eigenvalue -3. To find SRGs with smallest eigenvalue $-k$, the same code may be executed by assigning $m = k$. As a QSD and its complement have isomorphic block graphs, we list a set of parameters of a QSD, where $v \geq 2k$.

We refer to [2] for existence results of SRGs and to [5, 6, 8, 9, 14] for QSDs. For relations between parameters of QSD and SRG we refer to the sections "**Preliminaries**" of [10] and [11]. For basics in design theory we refer to [1], and for quasi-symmetric designs to [13].

The Computer Algebra Systems **Mathematica 5** [16] and **WxMaxima** [17] are used to simplify symbolic calculations given in this paper.

The following inequalities are necessary conditions for existence of QSD which are used in the Mathematica code given in Appendix II.

Lemma 3 ([3])**.** *Let* **D** *be a QSD with standard parameter set $(v, b, r, k, \lambda; x, y)$. Then the following inequalities hold:*

$$\begin{aligned}0 \leq\ & k(v-6)(v-3)(v-k)(2k-x-y)^2 \\ & -2k(v-3)(v-k)(2k(v-k)-3v)(2k-x-y) \\ & +(6-v)(v-3)(v-1)(k-x)(k-y)(2k-x-y) \\ & +k(v-k)(5v+3k(v-k)(k(v-k)-2(v-1))-3) \\ & +(v-3)(k(v-k)(3v+2)-6(v-1)v)(k-x)(k-y);\end{aligned}$$

$$\begin{aligned}0 \leq\ & k(v-k)(k(v-k)-1)+(v-2)(v-1)(k-x)(k-y) \\ & -k(v-2)(v-k)(2k-x-y).\end{aligned}$$

2 QSDs with $m = 3$ or $n = m + 2$

In this section we give a complete parametric classification of QSDs having feasible block graph parameters with $m = 3$ and QSDs with $n = m + 2$, which are complements of QSDs with $m = 3$. We rely on Neumaier's Theorem 2.

Theorem 4. *Let Γ be an SRG with smallest eigenvalue -3 (i.e. $m = 3$) and having feasible parameters of the block graph of a QSD* $\mathbf{D}$. *Then one of the cases below occurs:*

1. *Γ is the complete multi-partite graph with s classes of size 3, with parameters, $(3s, 3(s-1), 3(s-2), 3(s-1))$ and $\mathbf{D}$ is a 2-$(9(1+2u), 6(1+2u), 5+12u)$ design with intersection numbers $3(1+2u)$ and $4(1+2u)$ or the complement of this design.*

2. *Γ is the Steiner graph $S_3(n)$, with parameters $\left(\frac{(3+2n)(n+1)}{3}, 3n, n+3, 9\right)$ and D has parameters $v = 3+2n$, $k = 3$, $z = \lambda = 1$, if $x = 0$ and $v = \frac{9(z-1)z+6xz+x^2}{x}$, $k = 3z + x$ and $\lambda = \frac{(x+3z-1)(x+3z)}{6}$, if $x \neq 0$, or its complement.*

3. *Γ and D* (up to complementation) *are one of the cases given in Table 1.*

Proof. Using Theorem 2 leads to one of: the complete multipartite graph with s classes of size 3; the Latin square graph $LS_3(n)$; the Steiner graph $S_3(n)$; or finitely many other exceptional graphs. The conclusions 1 and 2 follow by using, respectively, Theorem 24, Proposition 23, and Theorem 25 from [10].

It now remains to find the parameters of the finitely many exceptional graphs arising in Theorem 2. We use the Mathematica code given in Appendix I to find exceptional feasible parameters of SRGs with eigenvalue -3. For each set (b, a, c, d) of exceptional feasible parameters of SRGs obtained above, the code given in Appendix II is run to find feasible parameters of QSDs whose block graph parameters

Table 1: Feasible parameters of QSDs related to known exceptional SRGs with eigenvalue -3.

Strongly Regular Graph						Quasi-symmetric Design					
Sr.No.	b	a	c	d	$\exists$	v	k	λ	x	y	$\exists$
1)	15	6	1	3	Y	10	4	2	1	2	Y
2)	56	45	36	36	Y	21	6	4	0	2	Y
3)	69	20	7	5	Y	24	8	7	2	4	N
4)	77	60	47	45	Y	22	6	5	0	2	Y
5)	85	14	3	2	?	35	7	3	1	3	N
						35	14	13	5	8	?
6)	120	77	52	44	Y	21	7	12	1	3	Y
7)	176	105	68	54	Y	22	7	16	1	3	Y
8)	231	30	9	3	Y	56	16	18	4	8	?
9)	253	140	87	65	Y	23	7	21	1	3	Y

are (b, a, c, d), and the outcome is given in Table 1. During this process, the following parameters of SRGs were found:

$$(49, 32, 21, 20),\quad (57, 42, 31, 30),\quad (76, 54, 39, 36),\quad (96, 57, 36, 30),$$
$$(209, 16, 3, 1),\quad (841, 200, 87, 35),\quad (1344, 221, 88, 26),\quad (1911, 270, 105, 27).$$

However, by [2], these SRGs do not exist. We thus arrive at conclusion 3 of the theorem. □

If **D** is a QSD with block graph Γ, then the second largest eigenvalue of Γ is $n - m$. We now use the following theorem to classify QSDs with $n - m = 2$.

Theorem 5. *If Γ is an SRG with second largest eigenvalue 2, then one of the following assertions holds:*

1. *Γ is a pseudo-geometric graph for $pg(t-1, t+1, t-2)$ or the complement of a pseudo Latin square graph $L_3(t+1)$.*

2. *Γ is a pseudo-geometric graph corresponding to $pg(3t-2, 2t+1, 2(t-1))$, complement of a Steiner graph $S_3(3t)$ or complement of a Steiner graph $S_3(3t-1)$.*

3. *Γ is a pseudo-geometric graph corresponding to $pg(t+1, s+1, s-2)$ where (s, t) belongs to:* $(3, 3), (3, 5), (3, 9), (4, 1), (4, 7), (4, 9), (4, 12), (4, 17), (4, 27), (5, 1), (5, 7), (5, 9), (5, 12), (5, 17), (5, 27), (6, 18), (7, 25), (8, 3), (8, 5), (8, 15), (8, 21), (9, 42), (14, 2), (14, 4), (14, 32), (32, 5)$.

4. *Γ has one of the parameter sets:*

$(26, 15, 8, 9)$, $(36, 14, 4, 6)$, $(56, 10, 0, 2)$, $(76, 30, 8, 14)$,
$(77, 16, 0, 4)$, $(81, 20, 1, 6)$, $(99, 56, 28, 36)$, $(100, 22, 0, 6)$,
$(105, 52, 21, 30)$, $(105, 32, 4, 12)$, $(120, 42, 8, 18)$, $(126, 100, 80, 84)$,
$(126, 50, 13, 24)$, $(154, 72, 26, 40)$, $(162, 56, 10, 24)$, $(162, 92, 46, 60)$,
$(176, 70, 18, 34)$, $(225, 128, 64, 84)$, $(232, 154, 96, 114)$, $(243, 110, 37, 60)$,
$(253, 112, 36, 60)$, $(300, 182, 100, 126)$, $(351, 210, 113, 144)$,
$(375, 272, 190, 216)$, $(405, 272, 172, 204)$, $(441, 352, 276, 300)$,
$(476, 342, 236, 270)$, $(540, 392, 274, 312)$, $(703, 520, 372, 420)$.

Proof. If the degree of Γ is greater than 2 and Γ has the eigenvalues a, 2 and $-m$, then the complementary graph $\overline{\Gamma}$ has the eigenvalues $b-a-1$, $m-1$ and -3. Now use the classification of SRGs corresponding to smallest eigenvalue -3 given in Theorem 2 to complete the proof. Observe that the complement of SRGs given in Case 1 of Theorem 4 are not connected and also do not have second largest eigenvalue 2. □

Theorem 6. *If* $\mathbf{D}$ *is a QSD whose block graph is the SRG with second largest eigenvalue* 2*, then* $\mathbf{D}$ *is one of the designs mentioned in Table 2 or its complement.*

Table 2: Feasible parameters of QSDs related to SRGs with eigenvalue 2.

Strongly Regular Graph							Quasi-symmetric Design					
Sr.No.	$(s,t)*$	b	a	c	d	$\exists$	v	k	λ	x	y	$\exists$
1)	$(4,1)$	15	8	4	4	Y	6	2	1	0	1	Y
2)	$(4,9)$	95	40	12	20	?	76	36	21	16	18	?
3)	$(5,12)$	126	65	28	39	Y	105	40	18	14	16	?
4)	$(8,5)$	69	48	32	36	?	46	6	1	0	1	N
							46	16	8	4	6	?
5)	$(8,21)$	261	176	112	132	?	232	56	15	12	14	?
6)	$(14,4)$	85	70	57	60	?	51	21	14	6	9	N
7)	–	26	15	8	9	Y	13	3	1	0	1	Y
8)	–	77	16	0	4	Y	56	16	6	4	6	Y

*SRGs obtained from $pg(t+1, s+1, s-2)$

Proof. The block graph of $\mathbf{D}$ is one of the graphs mentioned in Cases 1–4 of Theorem 5. For Cases 1 and 2, we rule out the possibility of QSDs, whose block graphs are given below, by observing that $\Delta = b^2 - 4q < 0$, using part (iv) of Theorem 1.

Theorem 26 of [10] rules out the possibility of a QSD whose block graph is the complement of the pseudo Latin square graph $L_3(t+1)$ for $t \geq 3$.

Let Γ be the complement of a Steiner graph $S_3(3t)$ for $t \geq 2$ with parameters $((1+2t)(1+3t), 2t(3t-2), 8-13t+6t^2, 2(t-1)(3t-2))$. Using part (iii) of Theorem 6 in [11], observe that $z = \dfrac{s\,(-1+2t)\,(1-s+5t+6t^2)}{3t(1+2t)^2\,(-2+3t)}$, $(-1+2t)$ and $t(1+2t)^2\,(-2+3t)$ are relatively prime. Write

$$s\left(1-s+5t+6t^2\right) = pt(1+2t)^2\,(-2+3t)$$

for positive integer p, and observe that the discriminant

$$\Delta = -(1+2t)^2\left(-1+(-6-8p)\,t+(-9+12p)\,t^2\right)$$

of this quadratic in s is negative for $t \geq 5$.

Observe that, for $t = 2$, $\Delta = -25(-49+32p)$ is negative for $p \geq 2$ and for $p = 1$, z is not an integer; and for $t = 3$, $\Delta = -196(-25+21p)$ is negative for $p \geq 2$ and for $p = 1$, z is not an integer. Similarly the possibility of $t = 4$ may be ruled out.

Let Γ be the complement of a Steiner graph $S_3(3t-1)$ for $t \geq 2$, with parameters $(t(1+6t), 2(-1+t)(-1+3t), 13-17t+6t^2, 2(-1+t)(-4+3t))$. Observe that $z = \dfrac{s\,(-5+6t)\,(-s+t+6t^2)}{(-1+t)\,(-1+3t)\,(1+6t)^2}$, with $(-5+6t)$ and $(-1+t)((-1+3t)(1+6t)^2$ being relatively prime. Hence $s(-s+t+6t^2) = p(-1+t)(-1+3t)(1+6t)^2$. The discriminant of this quadratic in s is $\Delta = -(1+6t)^2\left(-t^2+p\,(4-16t+12t^2)\right)$. Since $\Delta < 0$, this case is ruled out.

For each set of parameters given in Cases 3 and 4 we run the Mathematica code given in Appendix II to get parameters of QSDs mentioned in Table 2. □

Appendix I

flag:=1
m:=3
a:=$d+m(n-m)$
c:=$d+n-2m$
b:=$\frac{(d+(m-1)(n-m))(d+m(n+1-m))}{d}$
f:=$\frac{(m-1)(d+m(n-m))(d+m(n+1-m))}{dn}$
d1:=$\frac{m(m-1)(n+1-m)(n-m)}{d}$
n1:=$\frac{m(m-1)(m^3(2m-3)+1)+2m-2}{2}+1$
k1:=Max[0, $2m-n$]
k2:=Max[0, $n-2m+2$]
For[$d=1, d<m^3(2m-3)+1$,
For[$n=m+1, n<$ n1,
If[IntegerQ[b]&&IntegerQ[f]
&&IntegerQ[d1]&&$d(n-m(m-1)) \leq (m-1)(n-m)(n+m(m-1))$
&&$b \leq f(f+3)/2$&&$d \neq m^2$&&$d \neq m(m-1)$&&IntegerQ[$\sqrt{(c-d)^2+4(a-d)}$]
&&$d \geq$ k1&&d1 $\geq$ k2&&$2(n-m+1) \leq m(m-1)(d+1)$,
Print[flag++, " : ", b, "&", a, "&", c, "&", d, "& \\\\ "]]; n++]; d++]

Appendix II

b:=15
a:=6
c:=1
d:=3
ANS:="No"
k:=$mz + x$
r:=$nz + \lambda$
y:=$z + x$
m:=$(d - c + \sqrt{\text{Factor}[(d-c)\hat{}2 + 4(a-d)]})/2$
λ:=$\frac{bx+az+mz-nxz-mnz^2}{x+mz}$
x:=$\frac{-az-mz+bmz-msz}{s}$
n:=$c - d + 2m$
z:=$\frac{(-a+c-d+m+bm)(b-s)s}{b(c-d+2m)(-a-m+bm)}$
q:=$b(c-d+2m)(-a-m+bm)/\text{GCD}[b(c-d+2m)(-a-m+bm),$
$(-a+c-d+m+bm)]$
s:=$\frac{1}{2}(b + \sqrt{b^2 - 4pq})$
v:=bk/r
P:=$k(v-6)(v-3)(v-k)(2k-x-y)\hat{}2 - 2k(v-3)(v-k)(2k(v-k)-3v)(2k-x-y)$
$+(6-v)(v-3)(v-1)(k-x)(k-y)(2k-x-y) + k(v-k)(5v+3k(v-k)(k(v-k)$
$-2(v-1))-3) + (v-3)(k(v-k)(3v+2) - 6(v-1)v)(k-x)(k-y)$
Q:=$k(v-k)(k(v-k)-1) + (v-2)(v-1)(k-x)(k-y) - k(v-2)(v-k)(2k-x-y)$
For[$p = 1, p \leq$ Max[1, Quotient[$b\hat{}2, 4q$]],
If[$b^2 - 4pq \geq 0$&&IntegerQ[s]&&IntegerQ[z]&&IntegerQ[x]&&IntegerQ[λ]&&IntegerQ[v]
&&$P \geq 0$&&$Q \geq 0$,
{Print[b, "&", a, "&", c, "&", d, "&", v, "&", k, "&", λ, "&", x, "&", y, "& \\\\ "],
ANS = "YES"}]; p++]
Print["Does Design parameters exist? ", ANS]

Output:
15&6&1&3&10&4&2&1&2& \\

Does Design parameters exist? YES

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