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The crossing number of the strong product of two paths

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Abstract

Let $P_m \boxtimes P_n$ be the strong product of two paths P_m and P_n. In 2013, Klešč et al. conjectured that the crossing number of $P_m \boxtimes P_n$ is equal to $(m-1)(n-1)-4$ for $m \geq 4$ and $n \geq 4$. In this paper we show that the above conjecture is true except when $m=4$ and $n=4$, and that the crossing number of $P_4 \boxtimes P_4$ is four.

1 Introduction

Let G and H be two disjoint graphs. The strong product $G \boxtimes H$ of G and H is the graph with vertex set $V(G) \times V(H)$ and edge set $\{(u,v)(x,y) : u = x \text{ and } vy \in E(H), \text{ or } v = y \text{ and } ux \in E(G), \text{ or } ux \in E(G) \text{ and } vy \in E(H)\}$.

Suppose that j is a positive integer. Let P_j be a path with j vertices. In 2013, Klešč et al. [1] firstly studied the crossing number of the strong product of two graphs. They showed that the crossing number of $P_3 \boxtimes P_n$ is equal to $n-3$ if $n \geq 3$, and they established

Lemma 1.1 *[1] The crossing number of $P_m \boxtimes P_n$ is at most $(m-1)(n-1)-4$ for $m \geq 4$ and $n \geq 4$.*

Subsequently, Klešč et al. conjectured that the crossing number of $P_m \boxtimes P_n$ is equal to $(m-1)(n-1)-4$ for $m \geq 4$ and $n \geq 4$. In this paper we shall show that the above conjecture is true except for $m=4$ and $n=4$, and that the crossing number of $P_4 \boxtimes P_4$ is four.

The arrangement of the paper is as follows. In Section 2, we give some lemmas and show that the crossing number of a subgraph of $P_4 \boxtimes P_5$ is at least eight. In

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Section 3, we first show using induction on n that the crossing number of a subgraph $W_{4,n}$ of $P_4 \boxtimes P_n$ is equal to $3(n-1)-4$ for $n \geq 5$. Then we prove using induction on m that the crossing number of a subgraph $W_{m,n}$ of $P_m \boxtimes P_n$ is equal to $(m-1)(n-1)-4$ for $m \geq 4$ and $n \geq 4$ and $(m,n) \neq (4,4)$. Subsequently, we determine the crossing number of $P_m \boxtimes P_n$ when $m \geq 4$ and $n \geq 4$ and $(m,n) \neq (4,4)$. In Section 4 we show that the crossing number of $P_4 \boxtimes P_4$ is four.

The rest of this section is contributed to some terminology for crossing numbers and graph theory.

Let G be a graph. By a drawing of G, we mean a drawing of G in the plane in which: no edge has a vertex as an interior point; no two adjacent edges cross each other; no two edges cross each other more than once; and no three edges cross in a common point.

Suppose that Φ is a drawing of a graph. The number of edge crossings in Φ is denoted by $\mathrm{cr}(\Phi)$. The *crossing number* of a graph G, denoted by $\mathrm{cr}(G)$, is the minimum number of edges crossings over all drawings of G. A drawing Ψ of G is *optimal* if $\mathrm{cr}(\Psi) = \mathrm{cr}(G)$.

A graph G' is a *subdivision* of G if G' is isomorphic to G or G' can be obtained from G by inserting vertices of degree two in some edges. Obviously, $\mathrm{cr}(G') = \mathrm{cr}(G)$ if G' is a subdivision of G. A graph H is a *minor* of G if H is isomorphic to a graph obtained from a subgraph F of G by contracting some edges in $E(F)$. A vertex of G is called a *branch vertex* if its degree is at least three in G. The complete graph with n vertices is denoted by K_n.

By Kuratowski's theorem [2], a graph is planar if and only if it contains no subdivision of either the complete graph K_5 or the complete bipartite graph $K_{3,3}$. Hence, if a graph G has a subgraph isomorphic to a subdivision of K_5 or $K_{3,3}$, then $\mathrm{cr}(G) \geq 1$.

2 Basic lemmas and the crossing number of the graph in Figure 10

Let $P_m = u_1u_2 \ldots u_m$ and $P_n = z_1z_2 \ldots z_n$ be two paths, where $m \geq 4$ and $n \geq 4$. For brevity, the vertex (u_i, z_j) in $P_m \boxtimes P_n$ is labeled by $w_{i,j}$. It is easy to find that there are many induced subgraphs in $P_m \boxtimes P_n$ such that each is isomorphic to K_4. For example, the graph $P_4 \boxtimes P_5$ has twelve induced subgraphs in which each is isomorphic to K_4. But K_4 is a planar graph. In order to give a lower bound for the crossing number of $P_m \boxtimes P_n$, we need a nonplanar graph which contains K_4 as subgraph. We shall define this graph in next paragraph. The drawing of $P_4 \boxtimes P_5$ shown in Figure 1 can be generalized to obtain a drawing of $P_m \boxtimes P_n$ with $(m-1)(n-1)-4$ crossings.

Let T_6 be the graph shown in Figure 2. Obviously, T_6 has a subgraph isomorphic to $K_{3,3}$. So $\mathrm{cr}(T_6) \geq 1$. It is easy to find that there are many subgraphs in $P_m \boxtimes P_n$ such that each is isomorphic to a subdivision of T_6.

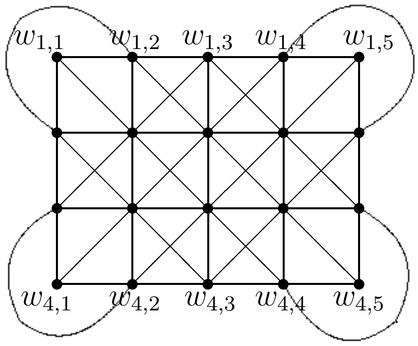

Figure 1 The graph $P_4 \boxtimes P_5$

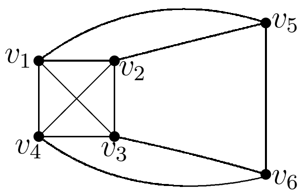

Figure 2 The graph T_6

Lemma 2.1 *Let F be a graph isomorphic to a subdivision of T_6. Let Q be a subgraph of F which is isomorphic to a subdivision of K_4. Then at least one edge of Q is crossed in any drawing of F.*

Proof Suppose that Φ is a drawing of F. Since F is isomorphic to a subdivision of T_6, Φ has at least one crossing.

Suppose that v_1, v_2, v_3 and v_4 are the four branch vertices of Q. If some edge of Q is crossed by some other edge of Q, then we have the desired result. Otherwise, the drawing of Φ restricted in Q divides the plane into four regions in which one is unbounded and its closure contains three branch vertices of Q. Also, the boundary of each of the other three regions contains three branch vertices of Q. Let F' be the graph obtained from F by deleting all edges in $E(Q)$. Then F' is a connected graph. Suppose that v_5 and v_6 are the other two branch vertices of F. By the Jordan Curve Theorem, no matter which regions v_5 and v_6 are in, at least one edge of Q is crossed by some edge in F'. □

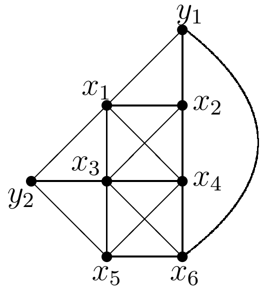

Figure 3 The graph H_1

Lemma 2.2 *Let H_1 be the graph shown in Figure 3. Then $cr(H_1) = 2$.*

Proof Let Q be the subgraph of H_1 induced by the four vertices x_1, x_2, x_3 and x_4. Then Q is isomorphic to K_4. It is easy to find that H_1 has a subgraph isomorphic to T_6 which contains Q. By Lemma 2.1, some edge in Q has at least one crossing in any drawing of H_1. For any edge e in Q, it can be checked that $H_1 - e$ has a minor isomorphic to K_5. So $\mathrm{cr}(H_1) \geq \mathrm{cr}(K_5) + 1 \geq 2$. Also, Figure 3 exhibits a drawing of H_1 with two crossings. Hence $\mathrm{cr}(H_1) = 2$. □

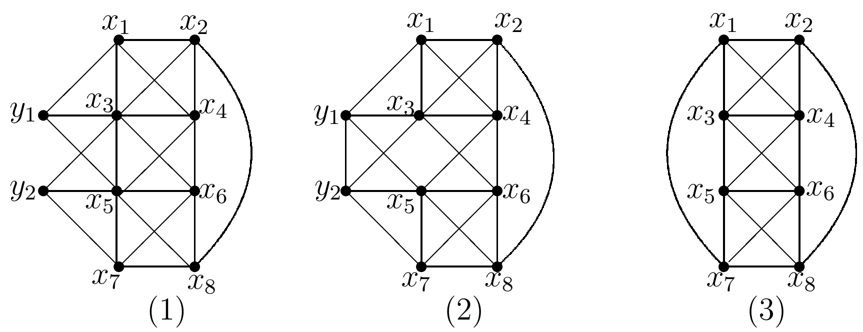

Figure 4 Three graphs H_2, H_3 and H_4

Lemma 2.3 *Let H_2 be the graph shown in Figure 4(1). Then $cr(H_2) \geq 3$.*

Proof Suppose that Φ is an optimal drawing of H_2. Let Q be the subgraph of H_2 induced by the four vertices x_1, x_2, x_3, and x_4. Clearly, Q is isomorphic to K_4, and H_2 has a subgraph isomorphic to a subdivision of T_6 which contains Q. By Lemma 2.1, some edge e in Q has at least one crossing in Φ. If e is some edge in $E(Q)\backslash\{x_2x_3, x_2x_4, x_3x_4\}$, it is easy to find that $H_2 - e$ contains a subgraph isomorphic to H_1. If e is x_2x_3 or x_2x_4, then $H_2 - e$ contains the path $x_1x_2x_8$ and the path $x_3x_1x_4$. So $H_2 - e$ has a subgraph isomorphic to a subdivision of H_1. If e is the edge x_3x_4, then $H_2 - e$ contains the path $x_3x_1x_4$ and the path $x_3x_2x_4$. Hence $H_2 - e$ has a subgraph isomorphic to a subdivision of H_1. So $\mathrm{cr}(H_2) \geq \mathrm{cr}(H_1) + 1 \geq 3$. □

Lemma 2.4 *Let H_3 be the graph shown in Figure 4(2). Then $cr(H_3) \geq 3$.*

Proof Let Q be the subgraph of H_3 induced by the four vertices x_1, x_2, x_3, and x_4. Proceeding the similar argument to that in the proof of Lemma 2.3, one can show that $\mathrm{cr}(H_3) \geq 3$. □

Lemma 2.5 *Let H_4 be the graph shown in Figure 4(3). Then $cr(H_4) \geq 3$.*

Proof Suppose that $\mathrm{cr}(H_4) = k$, and that Φ is an optimal drawing of H_4.

For $i = 1, 2, 3$, let Q_i be the subgraph of H_4 induced by the four vertices x_{2i-1}, x_{2i}, x_{2i+1} and x_{2i+2}. Then each Q_i is isomorphic to K_4. It is easy to find that H_4 contains a subgraph isomorphic to a subdivision of T_6 which contains Q_1. By Lemma 2.1, some edge e_1 in Q_1 has at least one crossing in Φ. Let Φ'_1 be the drawing obtained from Φ by deleting e_1. Then $\mathrm{cr}(\Phi'_1) \leq k - 1$. It is easy to find that $H_4 - e_1$ contains a subgraph which is isomorphic to a subdivision of T_6 which contains Q_3. By Lemma 2.1, some edge e_2 in Q_3 has at least one crossing in Φ'_1. Let Φ'_2 be the drawing obtained from Φ'_1 by deleting e_2. Then $\mathrm{cr}(\Phi'_2) \leq k - 2$.

We observe that $H_4 - e_1 - e_2$ has the following properties.

(1) It contains the path $x_3x_1x_4$ or $x_3x_2x_4$.

(2) It contains the path $x_5x_7x_6$ or $x_5x_8x_6$.

Without loss of generality, suppose that $H_4 - e_1 - e_2$ contains the path $x_3x_1x_4$. If $H_4 - e_1 - e_2$ contains $x_5x_7x_6$, then $H_4 - e_1 - e_2$ contains a subgraph isomorphic to $K_{3,3}$ if the cycle $x_3x_5x_4x_6x_3$ is considered. Otherwise, $H_4 - e_1 - e_2$ contains $x_5x_8x_6$. Moreover, $H_4 - e_1 - e_2$ contains x_7x_8. In this case $H_4 - e_1 - e_2$ contains a subgraph isomorphic to a subdivision of $K_{3,3}$ if the cycle $x_3x_5x_4x_6x_3$ is considered. So $H_4 - e_1 - e_2$ is nonplanar. Thus, $k \geq 3$. □

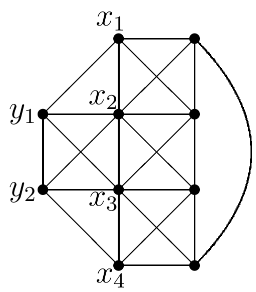

Figure 5 The graph H_5

Lemma 2.6 *Let H_5 be the graph shown in Figure 5. Then $cr(H_5) = 4$.*

Proof Suppose that $\mathrm{cr}(H_5) = k$, and suppose that Φ is an optimal drawing of H_5.

Let Q be the induced subgraph of H_5 by the four vertices y_1, y_2, x_2 and x_3. Obviously, Q is isomorphic to K_4, and H_5 has a subgraph isomorphic to a subdivision of T_6 which contains Q. By Lemma 2.1, some edge e in Q has at least one crossing in Φ. Let Φ' be the drawing obtained from Φ by deleting e. If e is some edge in $\{y_1x_2, y_1x_3, y_2x_2, y_2x_3\}$, then $H_5 - e$ has a subgraph isomorphic to a subdivision of H_4 defined in Lemma 2.5. Thus, we have that $k \geq 4$ in this case. Otherwise, we consider the edge y_1y_2. If it has at least one crossing in Φ, then we take it as e. So $H_5 - e$ is isomorphic to H_2 defined in Lemma 2.3. Then $k \geq 4$. If y_1y_2 has not any crossing in Φ, then e is exactly x_2x_3 in Φ. So $H_5 - e$ is isomorphic to H_3 defined in Lemma 2.4. So $k \geq 4$. Also, Figure 5 exhibits a drawing of H_5 with four crossings. Hence $\mathrm{cr}(H_5) = 4$. □

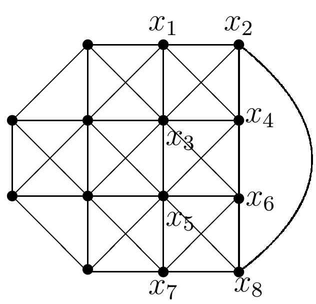

Figure 6 The graph G_1

Lemma 2.7 *Let G_1 be the graph shown in Figure 6. Then $cr(G_1) = 7$.*

Proof Suppose that $\mathrm{cr}(G_1) = k$, and that Ψ is an optimal drawing of G_1.

For $i = 1, 2, 3$, let Q_i be the induced subgraph of G_1 by the four vertices x_{2i-1}, x_{2i}, x_{2i+1} and x_{2i+2}. Then each Q_i is isomorphic to K_4. Moreover, G_1 has a subgraph isomorphic to a subdivision of T_6 which contains Q_1. By Lemma 2.1, some edge in Q_1 has at least one crossing in Ψ. We now apply the following operations.

(1) If some edge in $\{x_1x_2, x_1x_4, x_2x_3, x_2x_4\}$ has at least one crossing in Ψ, then it is deleted and turn to (4). Otherwise, turn to (2).

(2) Suppose that the edge x_1x_3 has at least one crossing in Ψ. Notice that x_2x_1 and x_2x_3 are not successive if the edges incident with x_2 are oriented in clockwise or anticlockwise in Ψ, otherwise, we redraw x_1x_3 near to the path $x_1x_2x_3$, obtaining a drawing of G_1 with at most $k-1$ crossings, a contradiction. Now x_1x_3 is redrawn near to $x_1x_2x_3$ such that it crosses exactly x_2x_4. Delete x_2x_4 and turn to (4). If x_1x_3 has not any crossing in Ψ, turn to (3).

(3) The edge x_3x_4 has at least one crossing in Ψ. A similar argument as the one used in (2) shows that we may redraw x_3x_4 near to the path $x_3x_2x_4$ such that it crosses exactly x_1x_2. Delete x_1x_2 and turn to (4).

(4) Let Ψ_1' be the obtained drawing, and let G_1' the obtained graph. Then $\mathrm{cr}(\Psi_1') \leq k-1$, and G_1' contains the edge x_1x_4 or the path $x_1x_2x_4$.

Since G_1' contains the edge x_2x_8, it is easy to find that G_1' has a subgraph isomorphic to a subdivision of T_6 which contains Q_3. By Lemma 2.1, some edge in Q_3 has at least one crossing in Ψ_1'. An argument similar to the one used for Q_1 shows that there is a drawing Ψ_2' which is obtained from Ψ_1' by deleting some edge e in $E(Q_3) \setminus \{x_5x_6, x_5x_7\}$ and $\mathrm{cr}(\Psi_2') \leq k-2$. Let G_1'' be the graph obtained from G_1' by deleting e. Then G_1'' contains the edge x_7x_6 or the path $x_7x_8x_6$.

Since G_1'' contains one of x_1x_4 and $x_1x_2x_4$ and one of x_7x_6 and $x_7x_8x_6$, G_1'' has a subgraph isomorphic to a subdivision of T_6 which contains Q_2. By Lemma 2.1, some edge in Q_2 has at least one crossing in Ψ_2'. We consider two cases.

Case 1: G_1'' contains one of x_1x_2 and x_7x_8. Without loss of generality, suppose that G_1'' contains x_1x_2. If some edge in $E(Q_2) \setminus \{x_3x_5\}$ has at least one crossing in Ψ_2', then it is deleted. Otherwise, x_3x_5 must has at least one crossing in Ψ_2'. We now redraw x_3x_5 near to $x_3x_4x_5$. Notice that each of x_4x_3 and x_4x_5 has not any crossing in Ψ_2'. But x_3x_5 may crosses some edges in $\{x_4x_1, x_4x_2, x_4x_6\}$. If this case occur, then we delete those edges. Thus, the new drawing of x_3x_5 has not any crossing. Let Ψ_3' be the obtained drawing in the above procedure, and let $\bar{G}_1$ be the obtained graph. Then $\mathrm{cr}(\Psi_3') \leq k-3$, and $\bar{G}_1$ contains one of $x_1x_2x_8x_7$ and $x_1x_2x_8x_6x_7$. Thus $\bar{G}_1$ has a subgraph isomorphic to a subdivision of the graph H_5 defined in Lemma 2.6. So $k-3 \geq 4$. Thus $k \geq 7$.

Case 2: G_1'' contains none of x_1x_2 and x_7x_8. In this case, G_1'' is isomorphic to the graph shown in Figure 7(1).

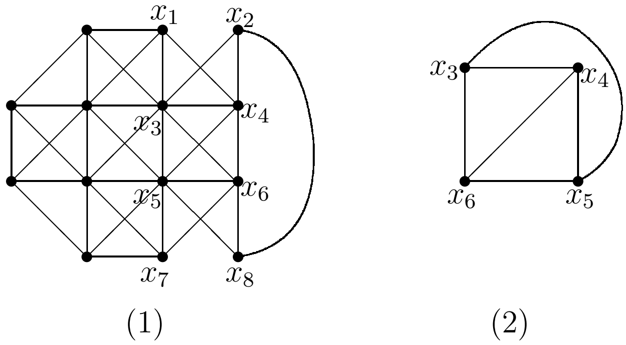

Figure 7 Two graphs defined in Case 2

If some edge in $E(Q_2) \setminus \{x_3x_5\}$ has at least one crossing in Ψ'_2, then it is deleted. The graph obtained has a subgraph isomorphic to a subdivision of H_5 defined in Lemma 2.6. Following a similar argument to that in Case 1, we have $k \geq 7$. If not, then x_3x_5 has at least one crossing in Ψ'_2. Since each edge in $E(Q_2) \setminus \{x_3x_5\}$ is not crossed, the drawing of Ψ'_2 restricted in $E(Q_2)$ is shown in Figure 7(2).

We observe that none of x_2 and x_8 is in the interior of the region whose boundary is the cycle $x_3x_4x_6x_3$. Otherwise, the existence of the path $x_2x_8x_5$ in G''_1, and Jordan Curve Theorem show that one edge of the cycle $x_3x_4x_6x_3$ must be crossed. Similarly, none of x_2 and x_8 is in the interior of the region whose boundary is the cycle $x_4x_5x_6x_4$. Let F_1 be the region whose boundary is the cycle $x_3x_4x_5x_3$, let F_2 the unbounded region in Figure 7(2). If x_2 and x_8 are in F_1 and F_2, respectively, then x_2x_8 must intersect x_3x_5. In this case, x_2x_8 is deleted. Clearly, the obtained drawing has at most $k-3$ crossings, and the obtained graph has a subgraph isomorphic to a subdivision of H_5 defined in Lemma 2.6. So $k \geq 7$. If x_2 and x_8 are in the same region, there are two cases to consider. If they are in F_1, then x_6x_8 must intersect x_3x_5. In this case, x_6x_8 is deleted. Then the obtained graph has a subgraph isomorphic to a subdivision of H_5. Thus, $k \geq 7$. If they are in F_2, then x_2x_4 must intersect x_3x_5. Similarly, we have that $k \geq 7$. Notice that Figure 6 exhibits a drawing of G_1 with four crossings. Hence $\mathrm{cr}(G_1) = 7$. □

Lemma 2.8 *Let G_2 be the graph shown in Figure 8. Then $cr(G_2) \geq 7$.*

Proof Suppose that $\mathrm{cr}(G_2) = k$, and that Ψ is an optimal drawing of G_2.

Let Q_1 be the subgraph of G_2 induced by the four vertices x_1, x_2, x_3 and x_4. Clearly, Q_1 is isomorphic to K_4, and G_2 has a subgraph isomorphic to a subdivision of T_6 which contains Q_1. By Lemma 2.1, some edge in $E(Q_1)$ has at least one crossing in Ψ. We now apply the operations to Q_1 which are similar to those in the proof of Lemma 2.7. If some edge in $E(Q_1) \setminus \{x_1x_3, x_3x_4\}$ has at least one crossing in Ψ, then it is deleted. Otherwise, one of x_1x_3 and x_3x_4 has at least one crossing in Ψ. In this case the edge is redrawn and some edge in $\{x_1x_2, x_2x_4\}$ is deleted if necessary such that at least one crossing is eliminated.

Let Ψ'_1 be the drawing so obtained, and let G'_2 the graph. Then $\mathrm{cr}(\Psi'_1) \leq k-1$, and G'_2 contains one of the following three subgraphs.

(a) The path $x_1x_2x_4$. (b) The path $x_1x_4x_2$. (c) $\{x_3x_2\} \cup \{x_1x_4\}$.

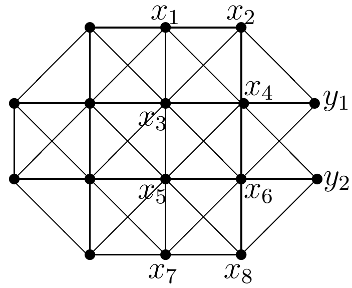

Figure 8 The graph G_2

Let Q_2 be the induced subgraph of G_2' by the four vertices x_5, x_6, x_7 and x_8. Obviously, Q_2 is isomorphic to K_4, and G_2' has a subgraph isomorphic to a subdivision of T_6 which contains Q_2. By Lemma 2.1, some edge in $E(Q_2)$ has at least one crossing in Ψ_1'. An argument similar to one used for Q_1 shows that there is a drawing Ψ_2' obtained form Ψ_1' by deleting some edge e in $E(Q_2)\setminus\{x_5x_6, x_5x_7\}$ and $\mathrm{cr}(\Psi_2') \leq k-2$. Let G_2'' be the graph obtained. We observe that G_2'' contains one of the following subgraphs.

(a) The path $x_7x_8x_6$. (b) The path $x_7x_6x_8$. (c) $\{x_5x_8\} \cup \{x_7x_6\}$.

Let Q_3 be the induced subgraph of G_2'' by the four vertices x_3, x_4, x_5 and x_6. Then Q_3 is isomorphic to K_4, and G_2'' has a subgraph isomorphic to a subdivision of T_6 which contains Q_3. By Lemma 2.1, some edge in $E(Q_3)$ has at least one crossing in Ψ_2'. We consider two cases.

Case 1: G_2'' contains one of x_1x_2 and x_7x_8. Without loss of generality, suppose that G_2'' contains the edge x_1x_2. If some edge e_1 in $E(Q_3)\setminus\{x_3x_5\}$ has at least one crossing in Ψ_2', then the drawing obtained from Ψ_2' by deleting e_1 has at most $k-3$ crossings. Moreover, $G_2'' - e_1$ contains a subdivision of the graph H_5 defined in Lemma 2.6, since there is a path $x_1x_2y_1x_6x_7$ or $x_1x_2y_1x_6x_8x_7$ in $G_2'' - e_1$. So $k \geq 7$. If not, then x_3x_5 has at least one crossing in Ψ_2'. We now delete edges x_4y_1 and x_4y_2. If x_2x_4 was not removed, then it is deleted. Next, x_3x_5 can be drawn near to the path $x_3x_4x_5$ such that it has at most one crossing. If x_3x_5 has one crossing, then it can be drawn such that it crosses exactly x_4x_6. Thus, the drawing obtained from Ψ_2' by deleting x_4x_6 has at most $k-3$ crossings, and the obtained graph has a subgraph isomorphic to a subdivision of the graph H_5 defined in Lemma 2.6. So $k \geq 7$. If x_3x_5 has not any crossing, we also let Ψ_2' be the obtained drawing. Clearly, $\mathrm{cr}(\Psi_2') \leq k-3$. So $k \geq 7$.

Case 2: G_2'' contains none of x_1x_2 and x_7x_8. In this case, G_2'' is the graph shown in Figure 9.

If some edge in $E(Q_3)\setminus\{x_3x_5\}$ has at least one crossing in Ψ_2', then it is deleted. The obtained graph has a subgraph isomorphic to a subdivision of the graph H_5 defined in Lemma 2.6. Hence, $k \geq 7$. Otherwise, x_3x_5 has at least one crossing in Ψ_2'. The drawing of Ψ_2' restricted in Q_3 is as in Figure 7(2). Let F_1 and F_2 be the regions whose boundaries are $x_3x_4x_6x_3$ and $x_4x_5x_6x_4$, respectively. Proceeding the similar argument as x_2 and x_8 in Case (2) in the proof of Lemma 2.7, none of x_1 and

x_7 is in F_1 or F_2.

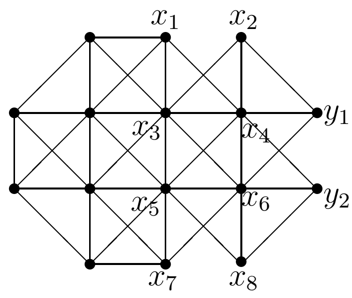

Figure 9 The graph defined in Case 2

Let J be the graph obtained from G_2'' by deleting x_2, x_3, x_4, x_5, x_6, x_8, y_1 and y_2. Then J is connected graph. It is easy to find that there are two internally disjoint paths P_1 and P_2 from x_1 to x_7 in J. Let F_3 be the region whose boundary is the cycle $x_3x_4x_5x_3$, and let F_4 the unbounded region in Figure 7(2).

If x_1 and x_7 are in F_3 and F_4, respectively, it can be found that x_3x_5 has at least two crossings if P_1 and P_2 are considered. We now delete x_3x_5. Then the obtained graph has a subgraph isomorphic to a subdivision of the graph H_3 defined in Lemma 2.4. So $k \geq 7$.

If x_1 and x_7 are in the same region, we consider two cases.

(a) Both x_1 and x_7 are in F_3. Then x_6x_7 must intersect x_3x_5 by Jordan Curve Theorem. If x_3x_5 has at least two crossings, then it is deleted. Considering that the obtained graph has a subgraph isomorphic to a subdivision of H_3, we have that $k \geq 7$. If x_3x_5 has exactly one crossing, then it is produced by x_3x_5 and x_6x_7. We now consider the vertex x_8. We claim that x_8 must be in F_2. For, if x_8 is in F_1, then x_8x_5 must cross some edge in the cycle $x_3x_4x_6x_3$ by Jordan Curve Theorem. If x_8 is in F_4, then the path $x_8y_2x_4$ must cross some edge in the cycle $x_3x_5x_6x_3$. If x_8 is in F_3, then the edge x_8x_6 must cross some edge in the cycle $x_3x_4x_5x_3$. So x_8 is in F_2.

We now discuss how many crossings are eliminated after x_7x_8 has been deleted to obtain G_2''. If there are at least two crossings being eliminated, then Ψ_2' has at most $k-3$ crossings. In this case, we delete x_3x_5. Considering that the obtained graph has a subgraph isomorphic to a subdivision of H_3 defined in Lemma 2.4, we have that $k \geq 7$. If there is exactly one crossing being eliminated, then the crossing must be produced by x_7x_8 and x_4x_5. Now x_7x_8 is added back in the primitive way. Next, x_7x_6 is newly drawn such that it is near to the path $x_6x_8x_7$. We now delete all edges incident with x_8 other than x_8x_6 and x_8x_7 in the interior of the region F_2. Then x_6x_7 has exactly one crossing which is produced by x_6x_7 and x_4x_5. Notice that x_4x_5 has at least two crossings in this case. Next, x_4x_5 is deleted. The obtained drawing has at most $k-3$ crossings, and the obtained graph has a subgraph isomorphic to a subdivision of H_5 defined in Lemma 2.6. Hence, we have $k \geq 7$.

(b) Both x_1 and x_7 are in F_4. Then x_1x_4 must intersect x_3x_5. Next, we proceed the similar argument as x_6x_7. The difference are that x_8 is replaced by x_2, that x_7x_8 is replaced by x_1x_2, and that F_2 is replaced by F_1. So $k \geq 7$. □

Lemma 2.9 *Let G_3 be the graph shown in Figure 10. Then $cr(G) \geq 8$.*

Proof Suppose that $\mathrm{cr}(G_3) = k$, and that Ψ is an optimal drawing of G_3.

Let Q be the subgraph of G_3 induced by the four vertices x_1, x_2, x_3 and x_4. Obviously, Q is isomorphic to K_4, and G_3 has a subgraph isomorphic to a subdivision of T_6 which contains Q. By Lemma 2.1, some edge in $E(Q)$ has at least one crossing in Ψ.

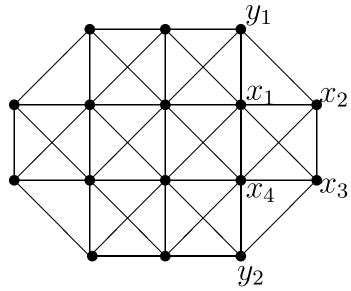

Figure 10 The graph G_3

If some edge in $\{x_1x_2, x_1x_3, x_2x_4, x_3x_4\}$ has at least one crossing in Ψ, then the edge is deleted. The obtained graph has a subgraph isomorphic to a subdivision of G_1 defined in Lemma 2.7. So $k \geq 8$. Otherwise, we consider x_2x_3. If x_2x_3 has at least one crossing in Ψ, then it is deleted. The obtained graph is isomorphic to G_2 defined in Lemma 2.8. Thus, we have that $k \geq 8$. If not, then x_1x_4 has at least one crossing in Ψ. If we can redraw x_1x_4 so that it is not crossed, then we obtain a drawing of G_3 with less crossings that in Ψ, a contradiction. Otherwise, x_1x_4 can be redrawn such that it crosses exactly x_2x_3. Since the graph obtained from G_3 by deleting x_2x_3 is isomorphic to G_2, we have $k \geq 8$. □

3 The crossing number of $P_m \boxtimes P_n$ for $m \geq 4$, $n \geq 4$ and $(m, n) \neq (4, 4)$

Let $W_{m,n}$ be the graph obtained from $P_m \boxtimes P_n$ by deleting the four vertices $w_{1,1}$, $w_{1,n}$, $w_{m,1}$ and $w_{m,n}$.

Lemma 3.1 *$cr(W_{4,n}) \geq 3(n-1) - 4$ for $n \geq 5$.*

Proof We use the induction on n. If $n = 5$, then $\mathrm{cr}(W_{4,n}) \geq 8$, since $W_{4,n}$ is isomorphic to the graph defined in Lemma 2.9.

Assume that $\mathrm{cr}(W_{4,t}) \geq 3(t-1) - 4$, where $t \geq 5$. Suppose that $\mathrm{cr}(W_{4,t+1}) = k$, and that Π is an optimal drawing of $W_{4,t+1}$. Let Q_1 be the subgraph of $W_{4,t+1}$ induced by the four vertices $w_{1,t-1}$, $w_{2,t-1}$, $w_{1,t}$ and $w_{2,t}$. Obviously, Q_1 is isomorphic to K_4, and $W_{4,t+1}$ has a subgraph isomorphic to a subdivision of T_6 which contains Q_1. By Lemma 2.1, some edge in Q_1 has at least one crossing in Π. We now apply the following operations which are similar to that in the proof of Lemma 2.7.

(1) If some edge in $E(Q_1) \setminus \{w_{1,t-1}w_{2,t-1}, w_{2,t-1}w_{2,t}\}$ has at least one crossing in Π,

then it is deleted and turn to (4). Otherwise, turn to (2).

(2) Suppose that the edge $w_{1,t-1}w_{2,t-1}$ has at least one crossing in Π. We can redraw $w_{1,t-1}w_{2,t-1}$ near to the path $w_{1,t-1}w_{1,t}w_{2,t-1}$ such that it has at most one crossing. If the new drawing of $w_{1,t-1}w_{2,t-1}$ has no crossing, then a drawing of $W_{4,t+1}$ with at most $k-1$ crossings is obtained, a contradiction. Otherwise,$w_{1,t-1}w_{2,t-1}$ can be redrawn near to $w_{1,t-1}w_{2,t-1}$ such that it crosses exactly $w_{1,t}w_{2,t}$. Delete $w_{1,t}w_{2,t}$ and turn to (4). If $w_{1,t-1}w_{2,t-1}$ has not any crossing in Π, turn to (3).

(3) The edge $w_{2,t-1}w_{2,t}$ has at least one crossing in Ψ. A similar argument as the one used in (2) shows that we may redraw $w_{2,t-1}w_{2,t}$ near to the path $w_{2,t-1}w_{1,t}w_{2,t}$ such that it crosses exactly $w_{1,t-1}w_{1,t}$. Delete $w_{1,t-1}w_{1,t}$ and turn to (4).

(4) Let Π'_1 be the obtained drawing, and let $W^{(1)}_{4,t+1}$ the obtained graph.

Then $\mathrm{cr}(\Pi'_1) \leq k-1$, and $W^{(1)}_{4,t+1}$ contains the edge $w_{1,t-1}w_{2,t}$ or the path $w_{1,t-1}w_{1,t}w_{2,t}$.

Let Q_2 be the subgraph of $W^{(1)}_{4,t+1}$ induced by the four vertices $w_{3,t-1}$, $w_{3,t}$, $w_{4,t-1}$ and $w_{4,t}$. Next, we proceed the similar argument as Q_1. Let Π'_2 be the drawing so obtained, and let $W^{(2)}_{4,t+1}$ be the graph corresponding to Π'_2. Then $\mathrm{cr}(\Pi'_2) \leq \mathrm{cr}(\Pi'_1) - 1 \leq k-2$ and $W^{(2)}_{4,t+1}$ contains the edge $w_{4,t-1}w_{3,t}$ or the path $w_{4,t-1}w_{4,t}w_{3,t}$.

Let Q_3 be the induced subgraph of $W^{(2)}_{4,t+1}$ by the four vertices $w_{2,t}$, $w_{2,t+1}$, $w_{3,t}$ and $w_{3,t+1}$. Obviously, Q_3 is isomorphic to K_4, and $W^{(2)}_{4,t+1}$ has a subgraph isomorphic to a subdivision of T_6 which contains Q_3. By Lemma 2.1, some edge in $E(Q_3)$ has at least one crossing in Π'_2. We now apply the following operations.

(a) If some edge in $E(Q_3) \setminus \{w_{2,t}w_{3,t}\}$ has at least one crossing in Π'_2, then it is deleted and turn to (c). Otherwise, turn to (b).

(b) The edge $w_{2,t}w_{3,t}$ is crossed in Π'_2. We redraw this edge near to the path $w_{2,t}w_{2,t+1}w_{3,t}$. If in such new drawing $w_{2,t}w_{3,t}$ is not crossed, then turn to (c). Otherwise, $w_{2,t}w_{3,t}$ can be drawn such that it crosses exactly the edge $w_{2,t+1}w_{3,t+1}$. Now the edge $w_{2,t+1}w_{3,t+1}$ is deleted, and turn to (c).

(c) Let Π'_3 be the obtained drawing, and let $W^{(3)}_{4,t+1}$ be the obtained graph.

Then $\mathrm{cr}(W^{(3)}_{4,t+1}) \leq \mathrm{cr}(W^{(2)}_{4,t+1}) - 1 \leq k-3$, and $W^{(3)}_{4,t+1}$ has a subgraph isomorphic to a subdivision of $W_{4,t}$. By the inductive assumption, $\mathrm{cr}(W_{4,t}) \geq 3(t-1)-4$. This implies that $k \geq 3t-4$. So $\mathrm{cr}(W_{4,t+1}) \geq 3t-4$. Therefore, $\mathrm{cr}(W_{4,n}) \geq 3(n-1)-4$. $\square$

Lemma 3.2 *$cr(W_{m,n}) \geq (m-1)(n-1)-4$ for $m \geq 4$ and $n \geq 5$.*

Proof We use the induction on m. By Lemma 3.1, $\mathrm{cr}(W_{m,n}) \geq (m-1)(n-1)-4$ if $m=4$. Assume that $\mathrm{cr}(W_{m,n}) \geq (m-1)(n-1)-4$ if $m=q$. Suppose that $\mathrm{cr}(W_{q+1,n}) = k$, and that Π is an optimal drawing of $W_{q+1,n}$.

Let Q'_1 (Q'_{n-1}, respectively) be the induced subgraph of $W_{q+1,n}$ by the four vertices $w_{q-1,1}$, $w_{q-1,2}$, $w_{q,1}$ and $w_{q,2}$ ($w_{q-1,n-1}$, $w_{q-1,n}$, $w_{q,n-1}$ and $w_{q,n}$, respectively). For $i = 1, 2, \ldots, n-3$, let Q'_{i+1} be the induced subgraph of $W_{q+1,n}$ by the four vertices

$w_{q,i+1}$, $w_{q,i+2}$, $w_{q+1,i+1}$ and $w_{q+1,i+2}$. It is obvious that Q'_j is isomorphic to K_4 for $j = 1, 2, \ldots, n-1$.

We start with Q'_1 and deal with it as Q_1 in the proof of Lemma 3.1. If some edge in $E(Q'_1) \setminus \{w_{q-1,1}w_{q-1,2}, w_{q-1,2}w_{q,2}\}$ has at least one crossing in Π, then it is deleted. Otherwise, one of $w_{q-1,1}w_{q-1,2}$ and $w_{q-1,2}w_{q,2}$ has at least one crossing. In this case the edge is redrawn and some edge in $\{w_{q-1,1}w_{q,1}, w_{q,1}w_{q,2}\}$ is deleted if necessary such that at least one crossing is eliminated.

After Q'_1 has been dealt with, the obtained graph has a subgraph isomorphic to a subdivision of T_6 which contains Q'_2. By Lemma 2.1, some edge in Q'_2 has at least one crossing in the present drawing. We now deal with Q'_2 in the similar way to that of Q_1 in the proof of Lemma 3.1. If some edge in $E(Q'_2) \setminus \{w_{q,2}w_{q,3}, w_{q,3}w_{q+1,3}\}$ has at least one crossing in the present drawing, then it is deleted. Otherwise, one of $w_{q,2}w_{q,3}$ and $w_{q,3}w_{q+1,3}$ has at least one crossing in the present drawing. In this case the edge is redrawn and some edge in $\{w_{q,2}w_{q+1,2}, w_{q+1,2}w_{q+1,3}\}$ is deleted if necessary such that at least one crossing is eliminated.

For $i = 3, \ldots, n-2$, Q'_i is dealt with as Q'_2. At last, Q'_{n-1} is dealt with in the similar way to Q_3 in the proof of Lemma 3.1. Let G be the obtained graph after removing at least one crossing for each of Q_i, $i \in \{1, 2, \ldots, n-1\}$. Then G has a subgraph isomorphic to a subdivision of the graph $W_{q,n}$. Thus, $\mathrm{cr}(G) \leq k-(n-1)$. By the inductive assumption, $\mathrm{cr}(G) \geq (q-1)(n-1)-4$. This implies that $k \geq q(n-1)-4$. So $\mathrm{cr}(W_{q+1,n}) \geq q(n-1)-4$. Therefore, $\mathrm{cr}(W_{m,n}) \geq (m-1)(n-1)-4$. □

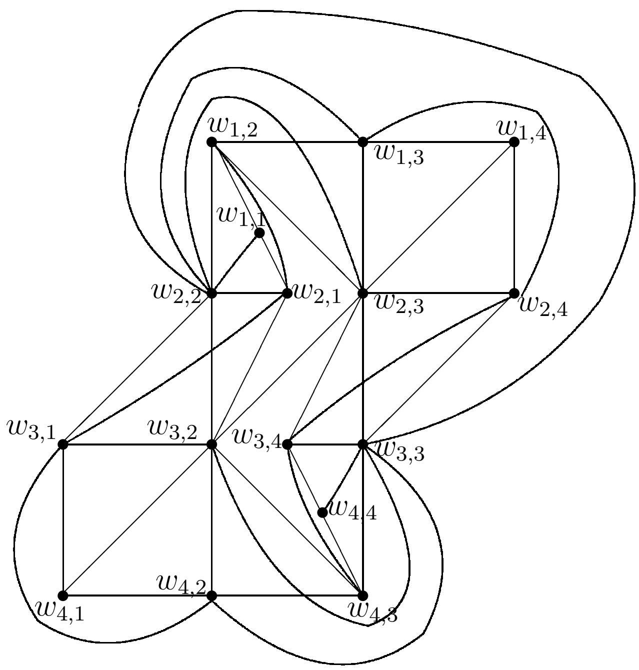

Figure 11 A drawing of $P_4 \boxtimes P_4$

Since $P_m \boxtimes P_n$ is isomorphic to $P_n \boxtimes P_m$, we have that $\mathrm{cr}(P_m \boxtimes P_n) = \mathrm{cr}(P_n \boxtimes P_m)$.

Theorem 3.3 $cr(P_m \boxtimes P_n) = (m-1)(n-1) - 4$ *for* $m \geq 4$, $n \geq 4$ *and* $(m,n) \neq (4,4)$.

Proof The theorem follows from Lemmas 1.1 and 3.2. $\square$

4 The crossing number of $P_4 \boxtimes P_4$

Theorem 4.1 $cr(P_4 \boxtimes P_4) = 4$.

Proof The drawing of $P_4 \boxtimes P_4$ shown in Figure 11 implies that $\mathrm{cr}(P_4 \boxtimes P_4) \leq 4$. Since $P_4 \boxtimes P_4$ has a subgraph isomorphic to a subdivision of H_5 defined in Lemma 2.6, $\mathrm{cr}(P_4 \boxtimes P_4) \geq 4$. Hence $\mathrm{cr}(P_4 \boxtimes P_4) = 4$. $\square$

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