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Watchman's walks of Steiner triple system block intersection graphs

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Abstract

A watchman's walk in a graph $G = (V, E)$ is a minimum closed dominating walk. In this paper, it is shown that the number of vertices in a watchman's walk on the block intersection graph of a Steiner triple system is between $\frac{v-3}{4}$ and $\frac{v-7}{2}$, for admissible $v \geq 15$. Included are constructions to build a design that achieves the minimum bound for any admissible v.

1 Introduction

The *watchman's walk problem* was first introduced in [11] as a variation of the domination number. In domination, we seek a set of vertices S in a graph G such that the closed neighbourhood, $N[S]$, contains all the vertices of G. Such a set is called a *dominating set*. Particularly, we search for the least such set. The order of the smallest possible set is the *domination number* of G, denoted $\gamma(G)$.

If we imagine the graph as the layout of a museum with the vertices corresponding to rooms and the edges to doors connecting those rooms, we can imagine the size of a minimum dominating set as the minimum number of watchmen needed to keep a constant watch on the museum, with the proviso that guards can look through doorways to adjacent rooms, watching them without entering them. However, in dire financial times, the museum may instead decide to make do with only a single guard who must instead patrol through the entire building. Certainly, such a guard will walk continuously; that is, the start and end position of her patrol must be the same, so that when one circuit ends, the next may begin. While every room may not now be visited, every room must be adjacent to a visited room. Finally, of all such patrols, the one of shortest length is preferred, as that will serve to reduce the time a room is unguarded.

In a graph G, this is equivalent to finding a walk that is closed and whose vertices dominate G, and of all such walks, finding the one of shortest length. We call such a walk a *minimum closed dominating walk* (MCDW), and denote its length by $w(G)$. Consider the number of vertices in each MCDW of G; the maximum of these numbers is denoted by $w'(G)$. Note that in general, $w(G)$ and $w'(G)$ are two very different parameters, as a minimum closed dominating walk may often revisit a vertex. (Consider a star with every edge subdivided.)

Finding a MCDW involves aspects of several other domination problems. Certainly, since any walk is connected, finding a MCDW may be considered a restriction on the problem of finding a *connected dominating set* [17]. Similarly, the idea of motion implicit in the watchman problem is also reminiscent of the *eternal domination* problem [3]. However, little is known about the length of minimum closed dominating walks. The decision problem is (unsurprisingly) NP-complete, but easily computed for trees [11]. Some results are known for Cartesian products of graphs, and for graphs with large circumference [12, 13]. More work has been done on another variant of the watchman's walk, where the amount of time a vertex can go unguarded is fixed, and the minimum number of watchmen necessary to meet that restriction is sought [1, 7, 9].

This paper examines the length of minimum closed dominating walks on the block intersection graphs of Steiner triple systems. Cycles in block intersection graphs are well-studied; they are known to be Hamiltonian and, moreover, pancyclic [14, 16]. More recently, block intersection graphs have been the setting for the cops and robbers problem [2]. The desirable structure of block intersection graphs allowing these results are favourable to discrete movement problems, including the watchman's walk problem.

A *Steiner triple system of order* v, denoted STS(v), is a pair $(V, \mathcal{B})$ of two sets: the *point set*, V, of order v and the *block set*, $\mathcal{B}$, of triples of elements of V, with the property that every pair of points in V occurs exactly once in some block of $\mathcal{B}$. The *repetition number*, r, of the STS(v) is the number of blocks in which each point of V occurs, and it is well-known that $r = \frac{v-1}{2}$. The number v is said to be *admissible* when $v \equiv 1 \text{ or } 3 \pmod 6$ and $v \geq 7$ as these are the necessary and sufficient conditions for the existence of an STS(v).

The *block intersection graph* of an STS(v), D, is a graph $G(D) = (V, E)$ with $V = \mathcal{B}$ and an edge between vertices u and v if and only if the blocks corresponding to u and v have a nonempty intersection. Throughout this paper, the terms blocks and vertices will be used interchangeably and a watchman's walk on a Steiner triple system will refer to a watchman's walk on the block intersection graph of that Steiner triple system.

By the structure of Steiner triple systems, we know a lot about the adjacency of vertices in the corresponding block intersection graphs.

Lemma 1.1 *For any* STS(v)*,* D*,* $G(D)$ *has the following properties:*

i. $|N(B)| = 3(r-1) = \frac{3v-9}{2}$*, for any vertex* B*;*

ii. $|N(B_1) \cap N(B_2)| = \frac{v+3}{2}$*, for any pair of adjacent vertices* B_1 *and* B_2*;*

ii. $|N(B_1) \cap N(B_2)| = 9$*, for any pair of nonadjacent vertices* B_1 *and* B_2*.*

As stated above, for graphs in general, a watchman's walk may be required to revisit a single vertex many times. This is not the case for the $G(D)$ of an STS(v), D.

Lemma 1.2 *In the block intersection graph of an* STS(v)*, there is a watchman's walk with no repeated blocks.*

Proof: Suppose to the contrary that there is a watchman's walk, $W = (B_1, B_2, \ldots, B_t, A, R, X, B_{t+1}, \ldots, B_w)$, with R a repeated block.

First, note the necessary conditions to use R in this place. We must use R (or an equivalent repeated block) if $A \cap X = \emptyset$ and all blocks in $N(A) \cap N(X)$ occur elsewhere in W. If this is not the case, either R can be removed or replaced by a suitable unused block from $N(A) \cap N(X)$.

Since R is necessary, all points in X occur in other blocks of the walk and thus X is redundant in terms of domination. Take a block B in $N(A) \cap N(B_{t+1})$ and replace R and X. That is, the walk $W_2 = (B_1, B_2, \ldots, B_t, A, B, B_{t+1}, \ldots, B_w)$ is a closed dominating walk shorter than W. This is a contradiction, since a watchman's walk is minimum. Thus such a watchman's walk has no repeated block. □

Corollary 1.3 *For any* $STS(v)$*,* D*,*

$$w'(G(D)) = \begin{cases} 1 & \text{if } w(G(D)) = 0, \\ w(G(D)) & \text{otherwise.} \end{cases}$$

Proof: If $w(G(D)) = 0$, then any watchman's walk has only one vertex, and thus $w'(G(D)) = 1$. If $w(G(D)) > 0$, then there is a watchman's walk with no repeated blocks by the above lemma. Since there are no repeated blocks, and the walk is closed, there must be the same number of edges and vertices; thus $w'(G(D)) = w(G(D))$. □

Unlike for graphs in general, the number of vertices in a watchman's walk of $G(D)$ for an STS(v), D, is the same length as that walk, when $w(G(D)) > 0$. The only Steiner triple system, D, for which $w(G(D)) = 0$, is the Fano plane, the unique STS(7). Thus, from this point forward, we concentrate on counting the number of vertices in a watchman's walk.

2 Basic Bounds

First we consider some basic bounds on the number of vertices in a watchman's walk. We will call a walk on a block intersection graph *pivotal* if all blocks in the walk share a common point. Note that in the block intersection graph this corresponds to the vertices of such a walk forming a clique.

Lemma 2.1 *For any* STS(v), D, *with* $v \geq 7$, $w'(G(D)) \leq r - 2 = \frac{v-5}{2}$.

Proof: Take the element $x \in V$. Consider a pivotal walk, W, through all but two blocks containing x. Let the two blocks containing x that are not in W be $B_1 = \{x, a, b\}$ and $B_2 = \{x, c, d\}$. Now every point other than a, b, c, and d are in W. This means the only blocks not dominated by W contain three of these four points. Since $\{a, b\} \subset B_1$ and $\{c, d\} \subset B_2$, neither of these pairs can occur in another block and thus no block exists with three of these four elements. So W is a dominating walk, and $w'(G(D)) \leq E(W) = r - 2$. □

Note that the above bound is tight for $v \in \{7, 9\}$, but for both STS(13)s, a clever selection of blocks gives $w'(G(D)) = r - 3$. Of the eighty STS(15)s, none has a watchman's walk with $w'(G(D)) = 5$. In that light, we improve the bound from Lemma 2.1 to $r - 3$ for larger v.

If R is the set of blocks through the point x in an STS(v), then $|R| = r$ and $|\mathcal{B} \setminus R| = b - r = \left(\frac{v-3}{3}\right) r$. When proving the upper bound, we removed 2 blocks from the set R, since there is no block with all three elements contained in those two blocks outside R. Our goal is now to show that 3 blocks can always be removed.

Let $C_3(X)$ be the set of 3-subsets of X. Define $f : \mathcal{B} \setminus R \mapsto C_3(R)$, such that $f(\{a, b, c\}) = S$, where S is the 3-subset of $C_3(R)$, containing the three blocks of R that contain a, b, and c. Note that $S \in C_3(R)$ and f is well-defined, since each point of a block in $\mathcal{B} \setminus R$ occurs in exactly one block of R. Notice that $|\mathrm{Im} f| \leq |\mathcal{B} \setminus R| = \left(\frac{v-3}{3}\right) r$, since f is well-defined. Thus we can always remove 3 blocks from R and maintain dominance if there is a set of three blocks in $C_3(R)$ to which f maps no block. That is, if $|\mathrm{Im} f| < |C_3(R)|$. It is easy to show that $|C_3(R)| = \binom{r}{3} > |\mathcal{B} \setminus R| \geq |\mathrm{Im} f|$, for $v \geq 15$. This proves the following lemma.

Lemma 2.2 *For any* STS(v), D, *with* $v \geq 15$, $w'(G(D)) \leq r - 3 = \frac{v-7}{2}$.

Now we will look to find a basic lower bound. This is a simple counting bound taking nothing into account other than the basic properties of an STS(v) and a watchman's walk.

Lemma 2.3 *For an* STS(v), D, *with* $v \geq 9$, $w'(G(D)) \geq \frac{v+3}{6}$.

Proof: Take a watchman's walk $W = (B_1, B_2, \ldots, B_{w'})$ and consider how many new blocks are dominated by each block as the walk progresses. The block B_1 dominates $1 + 3(r - 1) = \frac{3v-7}{2}$ new blocks. Now consider how many new blocks

B_{i+1} dominates with respect to B_i, possibly over-counting with respect to actual new blocks as B_{i+1} may also intersect with other, previous blocks. This number is $2(r-3) = 2\left(\frac{v-1}{2} - 3\right) = v - 7$. Now counting this over all blocks: $|\mathcal{B}| = \frac{v(v-1)}{6} \leq \frac{3v-7}{2} + (w'-1)(v-7)$. Rearranging, we get $w' \geq \frac{v+3}{6}$. □

One approach to constructing designs with a desired dominating walk is to find a design with an embedded design with desirable properties. The following is a result of what is known as the Doyen-Wilson Theorem given in [8].

Theorem 2.4 *Given an* STS(v)*,* D*, there exists an* STS(u) *that contains* D *if and only if* $u \geq 2v + 1$*.*

Using the Doyen-Wilson Theorem for $u = 2v + 1$, we can now create a larger design from a smaller one and prescribe a dominating walk.

Theorem 2.5 *For* v *admissible, there is an* STS$(2v+1)$*,* D'*, with* $w'(G(D')) \leq \frac{v-1}{2}$*.*

Proof: We will construct a dominating walk in an STS(v), D, and show it is also a dominating walk in the STS$(2v + 1)$, D', which we construct. Suppose $V(D) = \{1, 2, \ldots, v\}$ and let W be a pivotal walk in $G(D)$, containing all blocks through the point 1. Clearly, W dominates all blocks in $\mathcal{B}(D)$.

In the construction of D', add $v+1$ new points, following the $2n+1$ construction of [15]. Note that each block not in D has one point in $V(D)$. Thus W is a dominating walk of D'. Since $|W| = \frac{v-1}{2}$, the result holds. □

The walk obtained in Theorem 2.5 has length $\frac{u-3}{4}$, where u is the number of points. This is not particularly close to the bound of $\frac{u+3}{6}$ given in Lemma 2.3. The goal is now to find the best lower bound.

3 Best lower bound

In an STS(v), D, a *complete arc* is a maximal set of points of which no three are contained in the same block. Define $Z(D) \subseteq V(D)$ to be the complete arc of maximum size in D. That is, $Z(D)$ is the maximum set of points in $V(D)$ such that for all $B \in \mathcal{B}$, $B \not\subseteq Z(D)$. Sauer and Schönheim give the upper bound and constructions for such sets in [18]. The proof of the upper bound is a simple counting argument.

Lemma 3.1 *If* D *is an* STS(v)*,*

$$|Z(D)| \leq \begin{cases} \frac{v+1}{2} & \text{if } v \equiv 3, 7 \pmod{12}, \\ \frac{v-1}{2} & \text{if } v \equiv 1, 9 \pmod{12}. \end{cases}$$

Note that this upper bound is achieved by the set of new points added, N, in the proof of Theorem 2.5 and by the construction N is a complete arc. Using this upper bound on the size of a complete arc, we can find a minimum bound on the size of a watchman's walk.

Theorem 3.2 *If D is an* STS(v)*, then* $w'(G(D)) \geq \frac{v-3}{4}$.

Proof: Let W be a watchman's walk in $G(D)$ on $w'(G(D))$ blocks of D. Let M be the set of points of D used by W. Each block in W (after the first) contributes at most two points not already in M, while the first contributes 3. Thus $2(w'(G(D))-1)+3 \geq |M|$, or $w'(G(D)) \geq \frac{|M|-1}{2}$.

Now consider the set $Z = V \setminus M$. Certainly, there can be no block made up entirely of points of Z, since such a block would not be dominated by the blocks of W. Thus, $|Z| \leq |Z(D)| \leq \frac{v+1}{2}$, by Lemma 3.1. This means $|M| = |V| - |Z| \geq |V| - \frac{v+1}{2} = \frac{v-1}{2}$ and hence $w'(G(D)) \geq \frac{v-3}{4}$. □

Due to the denominator of 4 in the bound, this can only be achieved for $v \equiv 3, 7 \pmod{12}$. Adjusting this for other admissible values we obtain the following corollary.

Corollary 3.3 *If D is an* STS(v) *and* $v \equiv 1, 9 \pmod{12}$*, then* $w'(G(D)) \geq \frac{v-1}{4}$.

A similar argument to that of Theorem 3.2 can be made for domination number. In that case each block in the domination set may contribute three points of D, as the blocks need not be connected.

Lemma 3.4 *If D is an* STS(v)*, then* $\gamma(G(D)) \geq \frac{v-1}{6}$.

An *independent set* of a graph is a set of nonadjacent vertices. For any graph G, the *independence number*, $\beta(G)$, denotes the size of the maximum independent set in G. For any STS(v), D, simple counting shows $\beta(G(D)) \leq \frac{v}{3}$ and, clearly, $\beta(G(D))$ is an upper bound on $\gamma(G(D))$.

Theorem 3.5 *If $G(D)$ is a block intersection graph on an* STS(v)*, D, with* $v \geq 15$*, then*

$$\frac{v-3}{4} \leq w'(G(D)) \leq \frac{v-7}{2}$$

and

$$\frac{v-1}{6} \leq \gamma(G(D)) \leq \frac{v}{3}.$$

4 Constructing Designs to Meet the Lower Bound

The goal of this section is to construct an STS(v), D, such that there is a watchman's walk on $G(D)$ that achieves the minimum bound. Define

$$\omega(v) = \begin{cases} \frac{v-3}{4} & \text{if } v \equiv 3, 7 \pmod{12} \\ \frac{v-1}{4} & \text{if } v \equiv 1, 9 \pmod{12}. \end{cases}$$

We say an STS(v), D, achieves the minimum bound if $w'(G(D)) = \omega(v)$. We say that any D that achieves this bound has a minimal watchman's walk. Note that the bound in Lemma 3.1 is equal to $v - (2\omega(v) + 1)$. First we will consider the structure of a minimal watchman's walk.

Lemma 4.1 *Suppose D is an* STS(v). *If $w'(G(D)) = \omega(v)$ then any watchman's walk of D is a pivotal walk.*

Proof: Let W be a watchman's walk of D containing $\omega(v)$ blocks. The blocks of W must contain at least $2\omega(v) + 1$ points, otherwise D would contain a complete arc larger than $v - (2\omega(v) + 1)$; a contradiction. By a counting argument similar to the one made in the proof of Theorem 3.2, there are at most $2\omega(v) + 1$ points in the blocks of W. Thus, the $\omega(v)$ blocks of W contain exactly $2\omega(v) + 1$ points. Due to the adjacency of consecutive blocks in W, deviating and returning to the point common to the first two blocks will use at most $2\omega(v)$ points. Thus, W is a pivotal watchman's walk. □.

For the small values of v, an STS(v) with a minimal watchman's walk is given in the Appendix, including those in the following lemma.

Lemma 4.2 *For $v = 7$, $v = 9$, and $v = 13$, there is an* STS(v) *that has a minimal watchman's walk.*

The construction for the cases 3 and 7 (mod 12) is given in Theorem 2.5. We need a new construction for 1 and 9 (mod 12). To do this, we will use 3-GDDs. A 3-GDD *of type* $g_1^{u_1} g_2^{u_2} \dots g_n^{u_n}$ is a set of points V, a set of groups $\mathcal{G}$ (with u_i groups of size g_i) that partition V ($|\mathcal{G}| > 1$), and a set of blocks $\mathcal{B}$ of size 3, such that every pair of distinct elements of V occurs in exactly one block or one group, but not both. A *Latin square* of side g is a 3-GDD of type g^3 and the three groups are referred to as rows, columns, and elements. By results from Zhu [19] and Colbourn, Hoffman, and Rees [6], we get the following lemma.

Lemma 4.3 *For $\delta \in \{0, 1\}$ and $u \geq 3$, there exists a* 3-GDD *of type* $4^\delta 6^u$.

The idea, as in many triple system constructions, is to use the correct 3-GDD and Steiner triple systems of small size to create the design of the desired size which, in this case, has a minimal watchman's walk.

Theorem 4.4 *For $v \equiv 1, 9 \pmod{12}$ and $v \geq 37$, there exists an* STS(v), *D, that has a pivotal minimal watchman's walk. That is, $w'(G(D)) = \omega(v) = \frac{v-1}{4}$.*

Proof: This proof is by construction. If $v = 12n + 1$, let $\mathcal{A}$ be a 3-GDD of type 6^n and if $v = 12n + 9$, let $\mathcal{A}$ be a 3-GDD of type $4^1 6^n$. Create a second copy of this 3-GDD called $\mathcal{A}'$. For any block $\{a', b', c'\}$ in $\mathcal{A}'$, replace it by $\{a, b', c'\}$, $\{a', b, c'\}$, and $\{a', b', c\}$, where $\{a, b, c\}$ is the corresponding block in $\mathcal{A}$. The only points remaining without pairs covered by these blocks and the blocks of A are those within the same group or in corresponding groups. Let A_i and A'_i be corresponding groups of size 6 for $1 \leq i \leq 3n + 1$, and let A and A' be the corresponding blocks of size 4, if they exist. Create the point ∞ and on the 13 points $A_i \cup A'_i \cup \{\infty\}$, for each i, put an STS(13) with the 3 blocks from a watchman's walk, through the point ∞ and covering the points of A_i. If necessary, for $A \cup A' \cup \{\infty\}$ place an STS(9) with the 2 blocks from a watchman's walk, through ∞, covering the points of A. This is a $STS(12n + 8\delta + 1)$. There are $3n + 2\delta = \frac{12n+8\delta-1}{4}$ blocks through ∞ that dominate all the blocks, thus we have a pivotal watchman's walk of the desired size through ∞. $\square$

There are now some small cases, $v < 37$, for which a design that has a minimal watchman's walk needs to be constructed. The explicit statement of some are given in the Appendix, and others can be crafted using the constructions that follow. See Table 1 to determine the construction of each small case.

Theorem 4.5 *For $v \equiv 1, 3 \pmod 6$ and $n \geq 0$, if there exists an* STS$(4n + v)$ *with a minimal pivotal watchman's walk through a point of a sub-*STS(v)*, which when restricted to the sub-*STS(v) *gives a minimal pivotal watchman's walk on that design, then there exists an* STS$(12n + v)$ *with a minimal pivotal watchman's walk.*

Proof: There are two situations that arise: $v \leq 3$ and $v > 3$. If $v = 1$, then the sub-STS(v) has no blocks and if $v = 3$, then it is a single block. For $v > 3$, let D be an STS(v) embedded in an STS$(4n + v)$, D', for some non-negative integer n. Moreover, suppose there is a watchman's walk, W, on D' through a single point in D, that achieves the minimum bound and restricting W to D gives a set of blocks, $W_{|D}$ that contain a watchman's walk of D. Note that if $v = 1$ or $v = 3$, $|W_{|D}|$ can be taken to be zero.

Let $X = \{1, 2, 3, \ldots, 4n\}$, $V(D) = \{a_1, a_2, \ldots, a_v\}$, and $V(D') = V(D) \cup X$. Without loss of generality, assume that every block of W contains the point a_1. Consider the Latin square, L, of side $4n$ on the points X, with entries arranged as follows:

X_1	X_2
X_2	X_1

where $X_1 = \{1, 2, \ldots, 2n\}$ and $X_2 = X \setminus X_1$. Let $\mathcal{R}_i = \{R_x | x \in X_i\}$ and $\mathcal{C}_i = \{C_x | x \in X_i\}$, for $i \in \{1, 2\}$, and $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ and $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$.

Note $\omega(4n + v) = n + \omega(v)$. Since $W_{|D}$ is a watchman's walk that achieves the minimal bound, it has length $\omega(v)$. Thus, the remaining n blocks of W must contain the vertex a_1 and exactly $2n$ points of $V(D') \setminus V(D)$. Let these $2n$ points be X_1. Create two more copies of D', call them D'_R and D'_C, using the points $V(D) \cup \mathcal{R}$ and $V(D) \cup \mathcal{C}$, respectively. Denote W_R and W_C as the walks in D'_R and D'_C corresponding to W. Place $\mathcal{R}_1$ as the first $2n$ rows, $\mathcal{R}_2$ as the last $2n$ rows, $\mathcal{C}_1$ as the first $2n$ columns, and $\mathcal{C}_2$ as the last $2n$ columns of L.

Let $\mathcal{B}(L)$ be the row-column-entry triples formed by L in the usual way.

We claim $\mathcal{B} = \mathcal{B}(D') \cup \mathcal{B}(D'_R) \cup \mathcal{B}(D'_C) \cup \mathcal{B}(L)$ is an STS$(12n + v)$ on the points $V(D) \cup X \cup \mathcal{R} \cup \mathcal{C}$. (We consider $\mathcal{B}$ a set and not a multiset, as the blocks of D occur in each of $\mathcal{B}(D') \cup \mathcal{B}(D'_R) \cup \mathcal{B}(D'_C)$.) Certainly, any edge in X, $\mathcal{C}$, or $\mathcal{R}$ occurs exactly once in the corresponding block from $\mathcal{B}(D')$, $\mathcal{B}(D'_C)$, or $\mathcal{B}(D'_R)$. Any other edge must occur exactly once in a block of $\mathcal{B}(L)$.

Now consider the blocks of $W^\star = W \cup W_R \cup W_C$. Certainly, blocks from D', D'_R, and D'_C are all dominated by blocks from the corresponding W, W_R, and W_C. By the structure of L, every block in $\mathcal{B}(L)$ has at least one point from $\mathcal{R}_1$, $\mathcal{C}_1$, or X_1. Thus every block is dominated by $W^\star$. Since all blocks in $W^\star$ contain the vertex a_1, we know there is a pivotal dominating walk through $W^\star$.

Finally, the number of blocks in $W^\star$ is $3n + w'(D)$. So,

$$|\mathcal{B}(W^\star)| = \begin{cases} 3n + \frac{v-3}{4} = \frac{(12n+v)-3}{4} & \text{if } v \equiv 3, 7 \pmod{12} \\ 3n + \frac{v-1}{4} = \frac{(12n+v)-1}{4} & \text{if } v \equiv 1, 9 \pmod{12}, \end{cases}$$

achieving the bounds of Theorem 3.2 and Corollary 3.3. Thus, $W^\star$ is a pivotal watchman's walk for the STS$(12n + v)$. □

Theorem 4.6 *For $v \equiv 3$ or $7 \pmod{12}$, if D is an* STS(v) *that has a minimal pivotal watchman's walk, then there exists an* STS$(3v)$ *that has a minimal pivotal watchman's walk.*

Proof: Suppose $V(D) = \{1, 2, 3, \ldots, v\}$ and there is a pivotal watchman's walk, W, of length $\frac{v-3}{4}$ through the point 1. These blocks contain $\frac{v-3}{2}$ points other than 1, say $\{2, 3, 4, \ldots, \frac{v-1}{2}\}$. By [10], there exists a Latin square of side v, on the set $\{1, 2, \ldots, v\}$, with a subsquare of side $\frac{v-1}{2}$ embedded in the upper left hand corner, on the set $\{1, 2, \ldots, \frac{v-1}{2}\}$. Treating the Latin square as a 3-GDD with groups of size v, place a copy of D on the groups represented by the rows, $\{R_1, R_2, \ldots, R_v\}$, and columns, $\{C_1, C_2, \ldots, C_v\}$, such that the last $\frac{v-1}{2}$ rows and columns correspond to the points $\{1, 2, 3, \ldots, \frac{v-1}{2}\}$. Let T be the set of blocks through 1 that contain $\{R_i, C_j\}$, with $i, j > \frac{v-1}{2}$. Consider $W \cup T$. There are $\frac{v+1}{2} + \frac{v-3}{4} = \frac{3v-1}{4}$ blocks in this set containing 1. The blocks from the upper left hand corner of the Latin square are dominated by the blocks of W. The remaining blocks of the Latin square are dominated by blocks of T. The blocks on the groups corresponding to the rows and columns all contain one point corresponding to the last $\frac{v-1}{2}$ rows or columns; each of these points occurs in a block of T. The blocks of D are clearly dominated by W. Since these blocks make an STS$(3v)$ and $3v \equiv 1, 9 \pmod{12}$, the new design has a minimal pivotal watchman's walk through the point 1. □

Lemma 4.7 *There is an* STS(33) *that has a pivotal minimal watchman's walk.*

Proof: Consider a 3-GDD $\mathcal{A}$ of type 4^4 [4]. Create a second copy, $\mathcal{A}'$, and for any block $\{a', b', c'\}$ in $\mathcal{A}'$, replace it by $\{a, b', c'\}$, $\{a', b, c'\}$, and $\{a', b', c\}$, where $\{a, b, c\}$ is the corresponding block in $\mathcal{A}$. The only points remaining without pairs are those in corresponding groups, A_i and A'_i, $1 \le i \le 4$. Create the point ∞ and on the 9 points $A_i \cup A'_i \cup \{\infty\}$, for each i, put an STS(9) with the 2 blocks from a watchman's walk, through the point ∞ and covering the points of A_i. This is an STS(33). There are $2(4) = 8 = \frac{33-1}{4}$ blocks through ∞ that dominate all the blocks, thus we have a pivotal watchman's walk of the desired size. □

Order	MCDW Length	Construction	Ingredients
7	1	Direct (see Appendix)	
9	2	Direct (see Appendix)	
13	3	Direct (see Appendix)	
15	3	Theorem 2.5	STS(7)
19	4	Theorem 2.5	STS(9)
21	5	Theorem 4.6	STS(7) and LS(7)
25	6	Theorem 4.5	STS(9) and LS(8)
27	6	Theorem 2.5	STS(13)
31	7	Theorem 2.5	STS(15)
33	8	Lemma 4.7	STS(9) and 3-GDD of type 4^4

Table 1: Construction Method for Small Orders

Theorem 4.8 *For every admissible* v*, there exists an* STS(v)*,* D*, that has a minimal pivotal watchman's walk. That is,* $w'(G(D)) = \omega(v)$*.*

Proof: For $v \equiv 3, 7 \pmod{12}$, the construction follows from Theorem 2.5. For $v \equiv 1, 9 \pmod{12}$ and $v \geq 37$, the conditions in the construction of the proof of Theorem 4.4 are met by Lemma 4.3. Thus for all admissible values of $v \geq 37$, there exists a construction of an STS(v) that has a minimal watchman's walk. The table in Table 1 gives the construction for all other admissible values. □

5 Directions of future studies

Having analyzed the case of the block intersection graphs of Steiner triple systems that achieve the lower bound, several natural avenues of investigation present themselves. First, we might consider what is the spectrum of values which can be achieved for a watchman's walk on the block intersection graphs of Steiner triple systems of fixed order. This would involve characterizing complete arcs in individual Steiner triple systems [5].

Further, while Steiner triple systems are a natural design to first consider, there is an obvious generalization to consider block intersection graphs of other designs. Particularly, the methods of Theorem 3.2 could be applied to such designs.

Theorem 3.5 points out that γ and w' are more closely related in block intersection graphs than graphs in general. This motivates further investigation of this and other variations on the domination problem in the context of block intersection graphs.

Finally, there are two problems on the watchman's walk that remain open: the relationship between the length of a walk and the time an individual vertex goes unguarded; and how the use of multiple watchmen alters the structure of the individual walks.

Appendix

The blocks of each design are written vertically. A minimum watchman's walk is contained in the blocks on the left for each design. Note that each given walk is pivotal.

<u>$v = 7$:</u> $\omega(7) = 1$

0	001122
1	353434
2	465665

<u>$v = 9$:</u> $\omega(9) = 2$

00	0011122236
13	4534534547
26	8787676858

<u>$v = 13$:</u> $\omega(13) = 3$

000	00011111222223334445556
135	79b3469a3467867868a7897
246	8ac578bc95acbbacc9bbac9

000	00011111222223334445556
135	79b3469a3467867868a7897
246	8ac578bc95abcbcac9babc9

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