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Flat graphs based on affine and affine-symplectic spaces

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Abstract

Let $AG(n,\mathbb{F}_q)$ be the n-dimensional affine space over the finite field $\mathbb{F}_q$. For $0 \leq m \leq n-1$, define a graph $G^{(m)}$ whose vertex set is the set of all m-flats of $AG(n,\mathbb{F}_q)$, such that two vertices F_1 and F_2 are adjacent if $\dim(F_1 \vee F_2) = m+1$, where $F_1 \vee F_2$ is the minimum flat containing both F_1 and F_2. Let $ASG(2\nu,\mathbb{F}_q)$ be the 2ν-dimensional affine-symplectic space over $\mathbb{F}_q$. Define a graph $S^{(\nu)}$ whose vertex set is the set of all maximal totally isotropic flats of $ASG(2\nu,\mathbb{F}_q)$ such that two vertices F_1 and F_2 are adjacent if $\dim(F_1 \vee F_2) = \nu+1$. In this paper we study structures of the maximal cliques for the graph $S^{(\nu)}$ and present several bounds on the size of error-correcting codes for the graphs $G^{(m)}$ and $S^{(\nu)}$.

1 Introduction

Let $\mathbb{F}_q$ be the finite field with q elements, where q is a prime power. Let $\mathbb{F}_q^n$ be the n-dimensional row vector space over $\mathbb{F}_q$. For an m-dimensional subspace P of $\mathbb{F}_q^n$, we mean by a matrix representation of P an $m \times n$ matrix whose rows form a basis of P, denoted by the same symbol P. For $1 \leq i \leq n$, we use e_i to denote the n-dimensional row vector whose ith component is 1 and the other components are 0. For any $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{F}_q^n$, denote by $\langle \alpha_1, \alpha_2, \ldots, \alpha_m \rangle$ the subspace of $\mathbb{F}_q^n$ generated by $\alpha_1, \alpha_2, \ldots, \alpha_m$. The *general linear group* of degree n over $\mathbb{F}_q$, denoted by $GL_n(\mathbb{F}_q)$, consists of all $n \times n$ nonsingular matrices over $\mathbb{F}_q$. There is an action of $GL_n(\mathbb{F}_q)$ on $\mathbb{F}_q^n$ as follows:

$$\begin{array}{rcl} \mathbb{F}_q^n \times GL_n(\mathbb{F}_q) & \to & \mathbb{F}_q^n \\ (x,T) & \mapsto & xT. \end{array}$$

Then the set of all m-dimensional subspaces of $\mathbb{F}_q^n$ forms an orbit under $GL_n(\mathbb{F}_q)$, denoted by $\left[{[n] \atop m}\right]$. Suppose that P is an m-dimensional subspace of $\mathbb{F}_q^n$. Then a coset

$P+x$ of $\mathbb{F}_q^n$ relative to P is called an *m-flat*. The *dimension* of the m-flat $P+x$ is m, denoted by $\dim(P+x)$. A flat F_1 is said to be *incident* with a flat F_2, if F_1 contains or is contained in F_2. The point set $\mathbb{F}_q^n$ with all the flats and the incidence relation among them defined above is said to be the n-dimensional *affine space*, denoted by $AG(n,\mathbb{F}_q)$. Denote by $F_1 \cap F_2$ the intersection of the flats F_1 and F_2, and by $F_1 \vee F_2$ the minimum flat containing both F_1 and F_2.

Lemma 1.1 *[6, 12] Let $F_1 = V_1+x_1$ and $F_2 = V_2+x_2$ be any two flats of $AG(n,\mathbb{F}_q)$, where V_1 and V_2 are two subspaces of $\mathbb{F}_q^n$, and $x_1, x_2 \in \mathbb{F}_q^n$. Then*

(i) *$F_1 \cap F_2 \neq \emptyset$ if and only if $x_2 - x_1 \in V_1 + V_2$.*

(ii) *If $F_1 \cap F_2 \neq \emptyset$, then $F_1 \cap F_2 = V_1 \cap V_2 + x$, where $x \in F_1 \cap F_2$.*

(iii) *$F_1 \vee F_2 = V_1 + V_2 + \langle x_2 - x_1 \rangle + x_1$. In particular,*

$$\dim(F_1 \vee F_2) = \begin{cases} \dim F_1 + \dim F_2 - \dim(F_1 \cap F_2), & \text{if } F_1 \cap F_2 \neq \emptyset, \\ \dim F_1 + \dim F_2 - \dim(V_1 \cap V_2) + 1, & \text{if } F_1 \cap F_2 = \emptyset. \end{cases}$$

For $0 \le m \le n-1$, define a graph $G^{(m)}$ whose vertex set is the set of all m-flats of $AG(n,\mathbb{F}_q)$; two vertices F_1 and F_2 are adjacent if $\dim(F_1 \vee F_2) = m+1$. The graph $G^{(m)}$ is called the *m-flat graph* in $AG(n,\mathbb{F}_q)$. Note that $G^{(0)}$ is a clique with q^n vertices and $G^{(n-1)}$ is a clique with $q(q^n-1)/(q-1)$ vertices. So we assume $1 \le m \le n-2$ in the rest of this paper. Li [9, 10] determined the distance function and the maximal cliques for the graph $G^{(m)}$. As an application, we study error-correcting codes for the graph $G^{(m)}$ and present several bounds on the size of codes in Section 2.

Let K be a $2\nu \times 2\nu$ nonsingular alternate matrix over $\mathbb{F}_q$. A $2\nu \times 2\nu$ matrix T over $\mathbb{F}_q$ is called a *symplectic matrix* with respect to K if $TKT^t = K$, where T^t is the transpose of T. The *symplectic group* of degree 2ν with respect to K over $\mathbb{F}_q$, denoted by $Sp_{2\nu}(\mathbb{F}_q, K)$, consists of all $2\nu \times 2\nu$ symplectic matrices with respect to K over $\mathbb{F}_q$. Let K and K' be two $2\nu \times 2\nu$ nonsingular alternate matrices over $\mathbb{F}_q$. Then there is a $2\nu \times 2\nu$ nonsingular matrix Q over $\mathbb{F}_q$ such that $QKQ^t = K'$, which implies that $T \in Sp_{2\nu}(\mathbb{F}_q, K)$ if and only if $QTQ^{-1} \in Sp_{2\nu}(\mathbb{F}_q, K')$, and therefore $Sp_{2\nu}(\mathbb{F}_q, K)$ is isomorphic to $Sp_{2\nu}(\mathbb{F}_q, K')$. Thus, in discussing symplectic groups, we can choose any particular $2\nu \times 2\nu$ nonsingular alternate matrix K and study $Sp_{2\nu}(\mathbb{F}_q, K)$.

From now on let us take

$$K = \begin{pmatrix} 0 & I^{(\nu)} \\ -I^{(\nu)} & 0 \end{pmatrix}.$$

The *symplectic group* of degree 2ν over $\mathbb{F}_q$, denoted by $Sp_{2\nu}(\mathbb{F}_q)$, consists of all $2\nu \times 2\nu$ matrices T over $\mathbb{F}_q$ satisfying $TKT^t = K$. The vector space $\mathbb{F}_q^{2\nu}$ together with the right multiplication action of $Sp_{2\nu}(\mathbb{F}_q)$ is called the 2ν-dimensional *symplectic space* over $\mathbb{F}_q$. An m-dimensional subspace P in 2ν-dimensional symplectic space is said to be of *type* (m, s) if PKP^t is of rank $2s$. In particular, subspaces of type $(m, 0)$ are called m-dimensional *totally isotropic subspaces*, and ν-dimensional totally isotropic subspaces are called *maximal totally isotropic subspaces*. By [12], subspaces of type (m,s) exist if and only if $2s \leq m \leq \nu + s$. Suppose that P is a subspace of type (m,s) in $\mathbb{F}_q^{2\nu}$. Then a coset $P+x$ of $\mathbb{F}_q^{2\nu}$ relative to P is called an (m,s)-*flat*. The *dimension* of the (m,s)-flat $P+x$ is m, denoted by $\dim(P+x)$. The point set $\mathbb{F}_q^{2\nu}$ with all the flats and the incidence relation among them defined above is said to be the 2ν-dimensional *affine-symplectic space*, denoted by $ASG(2\nu,\mathbb{F}_q)$.

For $1 \leq \nu$, define a graph $S^{(\nu)}$ whose vertex set is the set of all maximal totally isotropic flats of $ASG(2\nu,\mathbb{F}_q)$; two vertices F_1 and F_2 are adjacent if $\dim(F_1 \vee F_2) = \nu+1$. The graph $S^{(\nu)}$ is called the *maximal totally isotropic flat graph* in $ASG(2\nu,\mathbb{F}_q)$. In Section 3 we determine the distance function and the maximal cliques, study error-correcting codes, and present several bounds on the size of codes for the graph $S^{(\nu)}$.

The Grassmann graphs and dual polar graphs are important distance-regular graphs. So the study of their features is of interest to many mathematicians; see [1]. Applying the matrix method, Wan, Dai, Feng and Yang [13] computed all parameters of the Grassmann graphs and dual polar graphs. Wang, Li and Huo [11, 15, 16] gave in matrix form the structure of all subconstituents of the dual polar graphs. The reason we study the graphs $G^{(m)}$ and $S^{(\nu)}$ is that they contain as subgraphs Grassmann graphs and dual polar graphs.

The author and Gao [5, 7] discussed the maximal totally isotropic flat graphs based on affine-unitary spaces and affine-orthogonal spaces, and determined their distance functions and maximal cliques. It seems to be interesting to discuss the error-correcting codes in these graphs. Affine polar spaces were first introduced by Cohen and Shult in [3]. The idea of that new concept resembles the affine reducing of a projective space. Start with a polar space and simply delete a fixed hyperplane in it. It seems to be interesting to discuss the general flat graphs based on affine polar spaces.

2 The affine case

The *Grassmann graph* $J_q(n,m)$ has the vertex set $\begin{bmatrix} [n] \\ m \end{bmatrix}$, and two vertices are adjacent if their intersection has dimension $m-1$. Two vertices of $J_q(n,m)$ are at distance i if and only if their intersection has dimension $m-i$. It is well known that the Grassmann graph $J_q(n,m)$ is a distance-regular graph of diameter $\min\{m, n-m\}$, see [1].

By Chapters 5 and 6 in [2], we may obtain the following lemma.

Lemma 2.1 *For a given $x \in \mathbb{F}_q^n$, let $G^{(m)}(x)$ be the subgraph of $G^{(m)}$ induced by all m-flats of $AG(n,\mathbb{F}_q)$ containing x. Then $G^{(m)}(x)$ is isomorphic to the Grassmann graph $J_q(n,m)$.*

Since the graph $G^{(m)}(x)$ is a subgraph of $G^{(m)}$, the graph $G^{(m)}$ is a generalization of the Grassmann graph $J_q(n,m)$.

The set of matrices of the form

$$\begin{pmatrix} T & 0 \\ v & 1 \end{pmatrix},$$

where $T \in GL_n(\mathbb{F}_q)$ and $v \in \mathbb{F}_q^n$, forms a group under matrix multiplication, which is denoted by $AGL_n(\mathbb{F}_q)$ and called the *affine group* of degree n over $\mathbb{F}_q$. Define the action of $AGL_n(\mathbb{F}_q)$ on $AG(n, \mathbb{F}_q)$ as follows:

$$\begin{array}{rcl} AG(n,\mathbb{F}_q) \times AGL_n(\mathbb{F}_q) & \to & AG(n,\mathbb{F}_q) \\ \left(x, \begin{pmatrix} T & 0 \\ v & 1 \end{pmatrix}\right) & \mapsto & xT + v. \end{array}$$

The above action induces an action on the set of flats of $AG(n, \mathbb{F}_q)$, i.e., a flat $P + x$ is carried by

$$\begin{pmatrix} T & 0 \\ v & 1 \end{pmatrix} \in AGL_n(\mathbb{F}_q)$$

to the flat $PT + (xT + v)$. By Theorem 1.21 in [12] $AGL_n(\mathbb{F}_q)$ is transitive on the set of m-flats in $AG(n, \mathbb{F}_q)$ for a given $0 \le m \le n$.

Lemma 2.2 *For any*

$$\begin{pmatrix} T & 0 \\ v & 1 \end{pmatrix} \in AGL_n(\mathbb{F}_q),$$

where $T \in GL_n(\mathbb{F}_q)$ *and* $v \in \mathbb{F}_q^n$, *define*

$$\begin{array}{rcl} \sigma_{(T,v)} : G^{(m)} & \to & G^{(m)} \\ P + x & \mapsto & PT + (xT + v). \end{array}$$

Then $\sigma_{(T,v)} \in \mathrm{Aut}\,(G^{(m)})$.

Proof. Note that $P + x = Q + y$ if and only if $P = Q$ and $y - x \in P$, if and only if $PT = QT$ and $(y - x)T \in PT$, if and only if $PT + (xT + v) = QT + (yT + v)$. It follows that $\sigma_{(T,v)}$ is a bijection. For any $P + x, Q + y \in G^{(m)}$, we have that $\dim((P + x) \vee (Q + y)) = m + 1$ if and only if either $\dim(P + Q) = m + 1$ and $y - x \in P + Q$, or $P = Q$ and $y - x \notin P$, if and only if $\dim(\sigma_{(T,v)}(P+x) \vee \sigma_{(T,v)}(Q+y)) = m+1$. It follows that $P+x$ and $Q+y$ are adjacent if and only if $\sigma_{(T,v)}(P+x)$ and $\sigma_{(T,v)}(Q+y)$ are adjacent. Hence, $\sigma_{(T,v)} \in \mathrm{Aut}\,(G^{(m)})$. □

Li [9] determined the distance function of the graph $G^{(m)}$ and obtained the following result.

Theorem 2.3 *[9] For any two vertices* F_1, F_2 *of* $G^{(m)}$, *let* $d(F_1, F_2)$ *be the distance between* F_1 *and* F_2. *Then* $\dim(F_1 \vee F_2) = m + r$ *if and only if* $d(F_1, F_2) = r$. *In particular,* $G^{(m)}$ *is a vertex transitive graph of diameter* $\min\{m + 1, n - m\}$ *with* $|G^{(m)}| = q^{n-m} \left[{n \atop m}\right]_q$.

Lemma 2.4 *[1] Suppose* $\max\{0, r+s-n\} \leq i \leq \min\{r,s\}$. *Let* P_0 *be an* r-*dimensional subspace of* $\mathbb{F}_q^n$. *Then the number of* s-*dimensional subspaces* Q *of* $\mathbb{F}_q^n$ *satisfying* $\dim(P_0 \cap Q) = i$ *is* $q^{(r-i)(s-i)} \begin{bmatrix} r \\ i \end{bmatrix}_q \begin{bmatrix} n-r \\ s-i \end{bmatrix}_q$.

For a given vertex F in $G^{(m)}$ and a given integer r with $0 \leq r \leq \min\{m+1, n-m\}$, let $G_r^{(m)}(F)$ denote the set of vertices F' in $G^{(m)}$ satisfying $d(F, F') = r$.

Lemma 2.5 *Let* r *be an integer with* $0 \leq r \leq \min\{m+1, n-m\}$ *and* F *be a given vertex in* $G^{(m)}$. *Then*

$$|G_r^{(m)}(F)| = (q^{m-r+1}-1)q^{r(r-1)} \begin{bmatrix} m \\ r-1 \end{bmatrix}_q \begin{bmatrix} n-m \\ r-1 \end{bmatrix}_q + q^{r(r+1)} \begin{bmatrix} m \\ r \end{bmatrix}_q \begin{bmatrix} n-m \\ r \end{bmatrix}_q.$$

Proof. Since $G^{(m)}$ is vertex transitive, the size of $G_r^{(m)}(F)$ is independent of the special choice for F. Without loss of generality, assume that F is a fixed m-dimensional subspace of $\mathbb{F}_q^n$. Let $F' = U + x$ be any element of $G_r^{(m)}(F)$. From Theorem 2.3 we deduce that $\dim(F \vee F') = m + r$, which implies that either $\dim(F \cap U) = m - r + 1$ and $x \notin F + U$, or $\dim(F \cap U) = m - r$ and $x \in F + U$. If $\dim(F \cap U) = m - r + 1$ and $x \notin F + U$, by Lemma 2.4 there are $q^{(r-1)^2} \begin{bmatrix} m \\ r-1 \end{bmatrix}_q \begin{bmatrix} n-m \\ r-1 \end{bmatrix}_q$ choices for U. For a given U with $\dim(F \cap U) = m - r + 1$, by Lemma 1.1 there are q^{r-1} choices for $U + x$ satisfying $(U + x) \cap F \neq \emptyset$, which implies that there are $q^m - q^{r-1}$ choices for $U + x$ satisfying $(U + x) \cap F = \emptyset$. Therefore there are $(q^{m-r+1} - 1)q^{r(r-1)} \begin{bmatrix} m \\ r-1 \end{bmatrix}_q \begin{bmatrix} n-m \\ r-1 \end{bmatrix}_q$ choices for $U + x$. If $\dim(F \cap U) = m - r$ and $x \in F + U$, similar to the above discussion we obtain that there are $q^{r(r+1)} \begin{bmatrix} m \\ r \end{bmatrix}_q \begin{bmatrix} n-m \\ r \end{bmatrix}_q$ choices for $U + x$. Hence the desired result follows. □

Li [10] determined the structures of the maximal cliques of the graph $G^{(m)}$ and obtained the following result.

Theorem 2.6 *[10] Any maximal clique in* $G^{(m)}$ *is isomorphic to*

$$\Omega_1 = \{F \in G^{(m)} \mid F \subseteq \langle e_1, e_2, \ldots, e_{m+1} \rangle\},$$

$$\Omega_2 = \{F \in G^{(m)} \mid F \supseteq \langle e_1, e_2, \ldots, e_{m-1} \rangle\},$$

or

$$\Omega_3 = \{\langle e_1, e_2, \ldots, e_m \rangle + x \mid x \in \mathbb{F}_q^n\}.$$

The size of the maximal clique in $G^{(m)}$ *is* $q(q^{m+1}-1)/(q-1), (q^{n-m+1}-1)/(q-1)$ *or* q^{n-m}.

Coding in the projective space has received recently a lot of attention due to its application in network coding. The determination of bounds on the size of codes with given minimum distance is the main problem in the context of coding theory. Bounds on the size of codes in the projective space are considered in recent years: see [8] for the Sphere-packing bound, [14] for the Wang-Xing-Safavi-Naini bound, [17] for the Johnson bound and [4] for the Gilbert-Varshamov bound. Next, we study the error-correcting codes and present several bounds on the size of codes in the graph $G^{(m)}$.

We say that nonempty subset $\mathbb{C}$ of the vertex set of $G^{(m)}$ is an (n, M, d, m) *code* in $G^{(m)}$ if $|\mathbb{C}| = M$ and $d(\mathbb{C}) \geq d$, where $d(\mathbb{C}) = \min\{d(U+x, W+y) \mid U+x, W+y \in \mathbb{C}, U+x \neq W+y\}$. Let $\mathcal{A}(n, d, m)$ denote the maximum number of codewords in an (n, M, d, m) code. An (n, M, d, m) code is called *optimal* if it has $\mathcal{A}(n, d, m)$ codewords.

The *sphere* of radius t centered at a vertex F is defined to be the set of all vertices whose distance from F is less than or equal to t, i.e., the set

$$S_t(F) = \{S \in G^{(m)} \mid d(F, S) \leq t\}.$$

By Lemma 2.5 one obtains that

$$\begin{aligned} |S_t(F)| &= \sum_{r=0}^{t} |G_r^{(m)}(F)| \\ &= \sum_{r=0}^{t} \left((q^{m-r+1} - 1) q^{r(r-1)} \begin{bmatrix} m \\ r-1 \end{bmatrix}_q \begin{bmatrix} n-m \\ r-1 \end{bmatrix}_q + q^{r(r+1)} \begin{bmatrix} m \\ r \end{bmatrix}_q \begin{bmatrix} n-m \\ r \end{bmatrix}_q \right). \quad (1) \end{aligned}$$

Theorem 2.7 *(Sphere-packing bound) Let $t = \lfloor (d-1)/2 \rfloor$. Then*

$$\mathcal{A}(n, d, m) \leq \frac{q^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}_q}{\sum_{r=0}^{t} \left((q^{m-r+1} - 1) q^{r(r-1)} \begin{bmatrix} m \\ r-1 \end{bmatrix}_q \begin{bmatrix} n-m \\ r-1 \end{bmatrix}_q + q^{r(r+1)} \begin{bmatrix} m \\ r \end{bmatrix}_q \begin{bmatrix} n-m \\ r \end{bmatrix}_q \right)}.$$

Proof. Let $\mathbb{C}$ be an (n, M, d, m) code in $G^{(m)}$. Then the spheres of radius t about distinct codewords in $\mathbb{C}$ are disjoint. By (1) each of these spheres contains $\sum_{r=0}^{t} |G_r^{(m)}(F)|$ vertices in $G^{(m)}$. Since $M \sum_{r=0}^{t} |G_r^{(m)}(F)|$ cannot exceed the total number of vertices, which implies that $M \sum_{r=0}^{t} |G_r^{(m)}(F)| \leq q^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}_q$. Therefore the desired result follows. □

The following result is an analog of the Wang-Xing-Safavi-Naini bound [14] in $G^{(m)}$.

Theorem 2.8 *(Wang-Xing-Safavi-Naini bound) Let $d \leq \min\{m+1, n-m\}$. Then*

$$\mathcal{A}(n, d, m) \leq \frac{q^{n-m} \begin{bmatrix} n \\ m-d+1 \end{bmatrix}_q}{\begin{bmatrix} m \\ m-d+1 \end{bmatrix}_q}.$$

Proof. Let $\mathbb{C}$ be an (n, M, d, m) code in $G^{(m)}$. Then each codeword of $\mathbb{C}$ contains exactly $q^{d-1}\left[{m \atop m-d+1}\right]_q$ many $(m-d+1)$-flats. On the other hand, any given $(m-d+1)$-flat of $AG(n, \mathbb{F}_q)$ cannot be contained in two distinct codewords of $\mathbb{C}$. In fact, suppose that F and S are two distinct codewords of $\mathbb{C}$ with $\dim(F \cap S) \geq m - d + 1$. Then $F \cap S \neq \emptyset$ by $m - d + 1 \geq 0$. By Theorem 2.3 we have that

$$d(F, S) = \dim(F \vee S) - m = m - \dim(F \cap S) \leq m - (m - d + 1) = d - 1,$$

a contradiction. Therefore $Mq^{d-1}\left[{m \atop m-d+1}\right]_q$ cannot exceed the total number of $(m - d + 1)$-flats, which implies that $Mq^{d-1}\left[{m \atop m-d+1}\right]_q \leq q^{n+d-m-1}\left[{n \atop m-d+1}\right]_q$. Therefore the desired result follows. □

The following result is an analog of the Johnson bound [17] in $G^{(m)}$.

Theorem 2.9 *(Johnson bound) Let $m \leq n - 1$. Then*

$$\mathcal{A}(n, d, m) \leq \frac{q(q^n - 1)}{q^{n-m} - 1}\mathcal{A}(n - 1, d, m).$$

Proof. Let $\mathbb{C}$ be an (n, M, d, m) code in $G^{(m)}$. For each $(n-1)$-flat S of $AG(n, \mathbb{F}_q)$, define

$$\mathbb{C}_S = \{F \in \mathbb{C} \mid F \subseteq S\}.$$

Then $\mathbb{C}_S$ is an $(n - 1, M_S, d', m)$ code with $d' \geq d$. For any given m-flat F of $AG(n, \mathbb{F}_q)$, there are $(q^{n-m} - 1)/(q - 1)$ many $(n - 1)$-flats of $AG(n, \mathbb{F}_q)$ containing F. It follows that each codeword of $\mathbb{C}$ belongs to $(q^{n-m} - 1)/(q - 1)$ distinct codes $\mathbb{C}_S$, and therefore

$$\sum_S |\mathbb{C}_S| = M\frac{q^{n-m} - 1}{q - 1}.$$

Since the number of $(n-1)$-flats is $q(q^n-1)/(q-1)$, there exists at least one $(n-1)$-flat S such that $|\mathbb{C}_S| \geq M\frac{q^{n-m}-1}{q-1}/\frac{q(q^n-1)}{q-1}$. Since $\mathcal{A}(n - 1, d, m) \geq |\mathbb{C}_S|$, we have

$$\mathcal{A}(n, d, m) \leq \frac{q(q^n - 1)\mathcal{A}(n - 1, d, m)}{q^{n-m} - 1},$$

as desired. □

The following result is an analog of the Gilbert-Varshamov bound [4] in $G^{(m)}$.

Theorem 2.10 *(Gilbert-Varshamov bound) Let $d \leq \min\{m + 1, n - m\}$. Then*

$$\mathcal{A}(n, d, m) \geq \frac{q^{n-m}\left[{n \atop m}\right]_q}{\sum\limits_{r=0}^{d-1}\left((q^{m-r+1} - 1)q^{r(r-1)}\left[{m \atop r-1}\right]_q\left[{n-m \atop r-1}\right]_q + q^{r(r+1)}\left[{m \atop r}\right]_q\left[{n-m \atop r}\right]_q\right)}.$$

Proof. Let $\mathbb{C}$ be an (n, M, d, m) code in $G^{(m)}$ with $M = \mathcal{A}(n, d, m)$. Then there is no vertex S in $G^{(m)}$ such that $d(F, S) \geq d$ for all $F \in \mathbb{C}$. Therefore for any vertex S in $G^{(m)}$, there exists a sphere of radius $d-1$ centered at some $F \in \mathbb{C}$ such that $S \in S_{d-1}(F)$, which implies that

$$\sum_{F\in\mathbb{C}} |S_{d-1}(F)| \geq q^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}_q .$$

By the transitivity of $AGL_n(\mathbb{F}_q)$, we have

$$\sum_{F\in\mathbb{C}} |S_{d-1}(F)| = M|S_{d-1}(F)|,$$

and therefore $M \geq q^{n-m}\left[{n \atop m}\right]_q / |S_{d-1}(F)|$, as desired. □

Remark. Let $n = 4$. The bounds listed in Theorems 2.7, 2.8 and 2.10 are given in the following table:

Name	$(d, m) = (1, 2)$	$(d, m) = (1, 1)$	$(d, m) = (2, 2)$
Theorem 2.7	$q^2(q^2 + 1)(q^2 + q + 1)$	$q^3(q + 1)(q^2 + 1)$	$q^2(q^2 + 1)(q^2 + q + 1)$
Theorem 2.8	$q^2(q^2 + 1)(q^2 + q + 1)$	$q^3(q + 1)(q^2 + 1)$	$q^2(q^2 + 1)$
Theorem 2.10	$q^2(q^2 + 1)(q^2 + q + 1)$	$q^3(q + 1)(q^2 + 1)$	$\frac{(q^2+1)(q^2+q+1)}{q^2+2q+2}$

The above table tells us that the $(4, q^2(q^2 + 1)(q^2 + q + 1), 1, 2)$ code and $(4, q^3(q + 1)(q^2 + 1), 1, 1)$ code are optimal.

3 The affine-symplectic case

For a given $x \in \mathbb{F}_q^{2\nu}$, let $\mathcal{M}(x)$ be the set of all maximal totally isotropic flats of $ASG(2\nu, \mathbb{F}_q)$ containing x. The *dual polar graph* $C_\nu(q)$ has the vertex set $\mathcal{M}(0)$, and two vertices are adjacent if their intersection has dimension $\nu - 1$. Two vertices of $C_\nu(q)$ are at distance i if and only if their intersection has dimension $\nu - i$. It is well known that $C_\nu(q)$ is a distance-regular graph of diameter ν with the following parameters:

$$b_i = q^{i+1} \begin{bmatrix} \nu - i \\ 1 \end{bmatrix}_q (0 \leq i \leq \nu - 1), \quad c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_q (1 \leq i \leq \nu). \tag{2}$$

Lemma 3.1 *For a given $x \in \mathbb{F}_q^{2\nu}$, let $S^{(\nu)}(x)$ be the subgraph of $S^{(\nu)}$ induced by $\mathcal{M}(x)$. Then $S^{(\nu)}(x)$ is isomorphic to the dual polar graph $C_\nu(q)$.*

Since the graph $S^{(\nu)}(x)$ is a subgraph of $S^{(\nu)}$, the graph $S^{(\nu)}$ is a generalization of the dual polar graph $C_\nu(q)$.

The set of matrices of the form

$$\begin{pmatrix} T & 0 \\ v & 1 \end{pmatrix},$$

where $T \in Sp_{2\nu}(\mathbb{F}_q)$ and $v \in \mathbb{F}_q^{2\nu}$, forms a group under matrix multiplication, which is denoted by $ASp_{2\nu}(\mathbb{F}_q)$ and called the *affine-symplectic group* of degree 2ν over $\mathbb{F}_q$. Define the action of $ASp_{2\nu}(\mathbb{F}_q)$ on $ASG(2\nu, \mathbb{F}_q)$ as follows:

$$\begin{array}{rcl} ASG(2\nu,\mathbb{F}_q) \times ASp_{2\nu}(\mathbb{F}_q) & \to & ASG(2\nu,\mathbb{F}_q) \\ \left(x, \begin{pmatrix} T & 0 \\ v & 1 \end{pmatrix}\right) & \mapsto & xT + v. \end{array}$$

The above action induces an action on the set of flats of $ASG(2\nu, \mathbb{F}_q)$ and $ASp_{2\nu}(\mathbb{F}_q)$ is transitive on the set of (m, s)-flats in $ASG(2\nu, \mathbb{F}_q)$ for a given (m, s).

Lemma 3.2 *For any*

$$\begin{pmatrix} T & 0 \\ v & 1 \end{pmatrix} \in ASp_{2\nu}(\mathbb{F}_q),$$

where $T \in Sp_{2\nu}(\mathbb{F}_q)$ and $v \in \mathbb{F}_q^{2\nu}$, define

$$\begin{array}{rcl} \sigma_{(T,v)} : S^{(\nu)} & \to & S^{(\nu)} \\ P + x & \mapsto & PT + (xT + v). \end{array}$$

Then $\sigma_{(T,v)} \in \mathrm{Aut}\,(S^{(\nu)})$.

Theorem 3.3 *For any two vertices F_1, F_2 of $S^{(\nu)}$, let $d(F_1, F_2)$ be the distance between F_1 and F_2. Then $\dim(F_1 \vee F_2) = \nu + r$ if and only if $d(F_1, F_2) = r$. In particular, $S^{(\nu)}$ is a vertex transitive graph of diameter ν with $|S^{(\nu)}| = q^\nu \prod_{i=1}^{\nu}(q^i + 1)$.*

Proof. First, we prove that $\dim(F_1 \vee F_2) \le \nu + d(F_1, F_2)$ by induction on $d(F_1, F_2)$. The case $d(F_1, F_2) = 1$ is trivial. Suppose $d(F_1, F_2) = r$. Then there exists some $F \in S^{(\nu)}$ such that $d(F_1, F) = r - 1$ and $d(F, F_2) = 1$. By the induction hypothesis, $\dim(F_1 \vee F) \le \nu + r - 1$ and $\dim(F \vee F_2) = \nu + 1$. By Lemma 1.1 and $F \subseteq (F_1 \vee F) \cap (F \vee F_2)$, we obtain that

$$\begin{array}{rcl} \dim(F_1 \vee F_2) & \le & \dim((F_1 \vee F) \vee (F \vee F_2)) \\ & = & \dim(F_1 \vee F) + \dim(F \vee F_2) - \dim((F_1 \vee F) \cap (F \vee F_2)) \\ & \le & 2\nu + r - \dim F \\ & = & \nu + r \\ & = & \nu + d(F_1, F_2). \end{array}$$

Next, we prove that $\dim(F_1 \vee F_2) \ge \nu + d(F_1, F_2)$. If $F_1 \cap F_2 \ne \emptyset$, then there exists some $x \in F_1 \cap F_2$ such that $F_1, F_2 \in S^{(\nu)}(x)$. Since $S^{(\nu)}(x)$ is isomorphic to $C_\nu(q)$, we obtain $d(F_1, F_2) \le \dim(F_1 \vee F_2) - \nu$. If $F_1 \cap F_2 = \emptyset$, then we can write $F_1 = V_1 + x, F_2 = V_2 + y$, where V_1 and V_2 are two maximal totally isotropic subspaces of $\mathbb{F}_q^{2\nu}$, $x, y \in \mathbb{F}_q^{2\nu}$ and $y - x \notin V_1 + V_2$. Pick $F = V_2 + x$. Then $d(F, F_2) = 1$ and $d(F_1, F_2) \le d(F_1, F) + d(F, F_2) = d(F_1, F) + 1$. From $F_1, F \in S^{(\nu)}(x)$ we deduce that

$d(F_1, F) \leq \dim(F_1 \vee F) - \nu$, which implies that $d(F_1, F_2) \leq \dim(F_1 \vee F) - \nu + 1 = \dim(F_1 \vee F_2) - \nu$.

By Corollary 3.19 in [12], the number of maximal totally isotropic subspaces of the symplectic space $\mathbb{F}_q^{2\nu}$ is $\prod_{i=1}^{\nu}(q^i + 1)$. So $|S^{(\nu)}| = q^\nu \prod_{i=1}^{\nu}(q^i + 1)$. □

For a given vertex F in $S^{(\nu)}$ and a given integer r with $0 \leq r \leq \nu$, let $S_r^{(\nu)}(F)$ denote the set of vertices F' in $S^{(\nu)}$ satisfying $d(F, F') = r$.

Lemma 3.4 *Let r be an integer with $0 \leq r \leq \nu$ and F be a given vertex in $S^{(\nu)}$. Then*

$$|S_r^{(\nu)}(F)| = (q^{\nu-r+1} - 1)q^{(r-1)(r+2)/2} \begin{bmatrix} \nu \\ r-1 \end{bmatrix}_q + q^{r(r+3)/2} \begin{bmatrix} \nu \\ r \end{bmatrix}_q.$$

Proof. Since $S^{(\nu)}$ is vertex transitive, without loss of generality assume that $F = \langle e_1, \ldots, e_\nu \rangle$. By [13] there are $q^{i(i+1)/2} \begin{bmatrix} \nu \\ i \end{bmatrix}_q$ maximal totally isotropic subspaces P satisfying $\dim(F \cap P) = \nu - i$. Let $U + x$ be any element of $S_r^{(\nu)}(F)$, where U is a maximal totally isotropic subspace of $\mathbb{F}_q^{2\nu}$ and $x \in \mathbb{F}_q^{2\nu}$. Then either $\dim(F \cap U) = \nu - r + 1$ and $x \notin F + U$, or $\dim(F \cap U) = \nu - r$ and $x \in F + U$. If $\dim(F \cap U) = \nu - r + 1$ and $x \notin F + U$, then there are $q^{(r-1)r/2} \begin{bmatrix} \nu \\ r-1 \end{bmatrix}_q$ choices for U. For a given U, there are q^{r-1} maximal totally isotropic flats $U + x$ such that $(U + x) \cap F \neq \emptyset$, which implies that there are $q^\nu - q^{r-1}$ maximal totally isotropic flats $U + x$ such that $(U + x) \cap F = \emptyset$. Therefore there are $(q^\nu - q^{r-1})q^{(r-1)r/2} \begin{bmatrix} \nu \\ r-1 \end{bmatrix}_q$ choices for $U + x$. Similarly, if $\dim(F \cap U) = \nu - r$ and $x \in F + U$, then there are $q^r q^{r(r+1)/2} \begin{bmatrix} \nu \\ r \end{bmatrix}_q$ choices for $U + x$. Therefore the desired result follows. □

Theorem 3.5 *Any maximal clique in $S^{(\nu)}$ is isomorphic to*

$$\Omega_1 = \{F \in S^{(\nu)} \mid F \subseteq \langle e_1, e_2, \ldots, e_{\nu+1} \rangle\}$$

or

$$\Omega_2 = \{\langle e_1, e_2, \ldots, e_\nu \rangle + x \mid x \in \mathbb{F}_q^{2\nu}\}.$$

The size of the maximal clique in $S^{(\nu)}$ is $(q + 1)q$ or q^ν.

Proof. Let Ω be any maximal clique of $S^{(\nu)}$. We prove that Ω is isomorphic to Ω_1 or Ω_2. Let $V + x \in \Omega$, where V is a maximal totally isotropic subspace of $\mathbb{F}_q^{2\nu}$ and $x \in \mathbb{F}_q^{2\nu}$. Let $\Omega(x) = \{F' \in \Omega \mid x \in F'\}$. Then $\Omega(x)$ is a clique of $S^{(\nu)}(x)$ containing $V + x$. There are the following two cases to be considered.

Case 1: $|\Omega(x)| \geq 2$. Then there exists a $U + x \in \Omega(x)$ such that $\dim(U + V) = \nu + 1$. It follows that $U + V$ is a subspace of type $(\nu + 1, 1)$. Let

$$\Delta_1 = \{S \in \Omega \mid S \subseteq (U + V) + x\}.$$

Since $\langle e_1, e_2, \ldots, e_{\nu+1}\rangle$ is a $(\nu+1,1)$-flat in $ASG(2\nu, \mathbb{F}_q)$, Δ_1 is isomorphic to Ω_1. In order to prove Ω is isomorphic to Ω_1, we only need to show that $\Omega \subseteq \Delta_1$. Let $P+y$ be any element in Ω, where P is a maximal totally isotropic subspace of $\mathbb{F}_q^{2\nu}$ and $y \in \mathbb{F}_q^{2\nu}$. We prove $P+y \subseteq (U+V)+x$. We first prove that $P \subseteq U+V$. Assume that $V \neq P \neq U$. By Lemma 1.1 and the definition of $S^{(\nu)}$, we have that $U, V, P \in S^{(\nu)}(0)$ and $\dim(P+V) = \dim(P+U) = \nu+1$. It follows that P and V (respectively P and U) are adjacent in the graph $S^{(\nu)}(0)$. By (2) there are

$$b_0 - b_1 - c_1 = q\begin{bmatrix}\nu\\1\end{bmatrix}_q - q^2\begin{bmatrix}\nu-1\\1\end{bmatrix}_q - 1 = q-1$$

choices for P. Since the subspace $U+V$ is of type $(\nu+1,1)$ in $\mathbb{F}_q^{2\nu}$, by Theorem 3.27 in [12] the number of maximal totally isotropic subspaces contained in $U+V$ is $q+1$. It follows that the number of vertices of $S^{(\nu)}(0)$ contained in $U+V$ is $q+1 = (b_0 - b_1 - c_1)+2$, and therefore $P \subseteq U+V$. Next, we prove that $y \in (U+V)+x$. Assume that $y \notin (U+V)+x$, then $y-x \notin U+V$. If $P=V$ or U, without loss of generality assume that $P=V$, then $\dim((U+x)\vee(P+y)) = \dim(P+U+\langle y-x\rangle) = \nu+2$, a contradiction. If $V \neq P \neq U$, then $P+V = U+V = P+U$, which implies that $\dim((U+x)\vee(P+y)) = \dim(P+U+\langle y-x\rangle) = \nu+2$, a contradiction. Therefore we have $\Omega \subseteq \Delta_1$ and $|\Omega_1| = (q+1)q$.

Case 2: $|\Omega(x)| = 1$. Then $\Omega(x) = \{V+x\}$. Let $P+y$ be any element in $\Omega \setminus \{V+x\}$, where P is a maximal totally isotropic subspace of $\mathbb{F}_q^{2\nu}$ and $y \in \mathbb{F}_q^{2\nu}$. We show that $P=V$. Assume that $P \neq V$. Since Ω is a maximal clique, we have $\dim((V+x)\vee(P+y)) = \dim(P+V+\langle y-x\rangle) = \dim(P+V) = \nu+1$, which implies that $y-x \in P+V$. It follows that $P+y \subseteq (P+V)+y = (P+V)+x$. Since $\{S \in S^{(\nu)} \mid S \subseteq (P+V)+x\}$ is a clique containing $V+x$ and $P+y$ and Ω is a maximal clique containing $V+x$ and $P+y$, one obtains that $\{S \in S^{(\nu)} \mid S \subseteq (P+V)+x\} \subseteq \Omega$, which implies that $P+x \in \Omega$, a contradiction. Let $\Delta_2 = \{V+y \mid y \in \mathbb{F}_q^{2\nu}\}$. Then $\Omega \subseteq \Delta_2$. Since Δ_2 is a clique and Ω is a maximal clique, $\Omega = \Delta_2$. Therefore Ω is isomorphic to Ω_2 and $|\Omega_2| = q^\nu$. □

We say that a nonempty subset $\mathbb{C}$ of the vertex set of $S^{(\nu)}$ is a $(2\nu, M, d, \nu)$ *code* in $S^{(\nu)}$ if $|\mathbb{C}| = M$ and $d(\mathbb{C}) \geq d$, where $d(\mathbb{C}) = \min\{d(U+x, W+y) \mid U+x, W+y \in \mathbb{C}, U+x \neq W+y\}$. Let $\mathcal{A}(2\nu, d, \nu)$ denote the maximum number of codewords in a $(2\nu, M, d, \nu)$ code. A $(2\nu, M, d, \nu)$ code is called *optimal* if it has $\mathcal{A}(2\nu, d, \nu)$ codewords.

The *sphere* of radius t centered at a vertex F is defined to be the set of all vertices whose distance from F is less than or equal to t, i.e., the set

$$B_t(F) = \{S \in S^{(\nu)} \mid d(F,S) \leq t\}.$$

By Lemma 3.4 one obtains that

$$|B_t(F)| = \sum_{r=0}^{t} |S_r^{(\nu)}(F)| = \sum_{r=0}^{t}\left((q^{\nu-r+1}-1)q^{(r-1)(r+2)/2}\begin{bmatrix}\nu\\r-1\end{bmatrix}_q + q^{r(r+3)/2}\begin{bmatrix}\nu\\r\end{bmatrix}_q\right).$$

Theorem 3.6 *(Sphere-packing bound) Let $t = \lfloor (d-1)/2 \rfloor$. Then*

$$\mathcal{A}(2\nu, d, \nu) \leq \frac{q^\nu \prod_{i=1}^{\nu}(q^i+1)}{\sum_{r=0}^{t}\left((q^{\nu-r+1}-1)q^{(r-1)(r+2)/2}{\nu \brack r-1}_q + q^{r(r+3)/2}{\nu \brack r}_q\right)}.$$

Proof. The proof is similar to that of Theorem 2.7, and is omitted. □

The following result is an analog of the Wang-Xing-Safavi-Naini bound [14] in $S^{(\nu)}$.

Theorem 3.7 *(Wang-Xing-Safavi-Naini bound) Let $d \leq \nu$. Then*

$$\mathcal{A}(2\nu, d, \nu) \leq q^\nu \prod_{i=d}^{\nu}(q^i+1).$$

Proof. Let $\mathbb{C}$ be a $(2\nu, M, d, \nu)$ code in $S^{(\nu)}$. Similar to the proof of Theorem 2.8, we have that $Mq^{d-1}{\nu \brack \nu-d+1}_q$ cannot exceed the total number of $(\nu-d+1)$-dimensional totally isotropic flats. By Corollary 3.19 in [12], the number of $(\nu-d+1)$-dimensional totally isotropic subspaces is ${\nu \brack \nu-d+1}_q \prod_{i=d}^{\nu}(q^i+1)$, which implies that $Mq^{d-1}{\nu \brack \nu-d+1}_q \leq q^{\nu+d-1}{\nu \brack \nu-d+1}_q \prod_{i=d}^{\nu}(q^i+1)$. Therefore the desired result follows. □

The following result is an analog of the Gilbert-Varshamov bound [4] in $S^{(\nu)}$.

Theorem 3.8 *(Gilbert-Varshamov bound) Let $d \leq \nu$. Then*

$$\mathcal{A}(2\nu, d, \nu) \geq \frac{q^\nu \prod_{i=1}^{\nu}(q^i+1)}{\sum_{r=0}^{d-1}\left((q^{\nu-r+1}-1)q^{(r-1)(r+2)/2}{\nu \brack r-1}_q + q^{r(r+3)/2}{\nu \brack r}_q\right)}.$$

Proof. The proof is similar to that of Theorem 2.10, and is omitted. □

Remark. Let $\nu = 2$. The bounds listed in Theorems 3.6, 3.7 and 3.8 are given in the following table:

Name	$d=1$	$d=2$
Theorem 3.6	$q^2(q+1)(q^2+1)$	$q^2(q+1)(q^2+1)$
Theorem 3.7	$q^2(q+1)(q^2+1)$	$q^2(q^2+1)$
Theorem 3.8	$q^2(q+1)(q^2+1)$	$\frac{(q+1)(q^2+1)}{q+2}$

The above table tells us that the $(4, q^2(q+1)(q^2+1), 1, 2)$ code is optimal.

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