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A geometric approach to orbital recognition in Chevalley-type coherent configurations and association schemes*

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Dedicated to the memory of Dan Archdeacon (1954–2015)

Abstract

We say that a coherent configuration is of Chevalley type if its basis relations correspond to orbitals of a geometric action (G, Ω), where G is a finite Chevalley group. If the action is transitive, the resulting configuration is an association scheme. In this paper, we seek to provide an effective universal strategy for orbital recognition in coherent configurations of Chevalley type. This must be preceded by a universal orbital characterization strategy, which we also propose. There are a number of ways to represent the objects in Ω to enable orbital recognition but, invariably, these strategies depend on the structure of the ambient group. On the other hand, a universal strategy does exist, which involves a canonical embedding of Ω in the flag geometry $\mathfrak{F}(\mathcal{G})$ of G, however this strategy alone does not facilitate fast orbital recognition. In this paper, we further identify these embedded objects in $\mathfrak{F}(\mathcal{G})$ with certain distinguished elements in the upper Borel subalgebra $\mathfrak{L}^U$ of the Lie algebra for G. This reduces orbital recognition to an investigation of systems of equations that arise from vanishing of the Lie product in $\mathfrak{L}^U$. In many instances these systems of equations turn out to be linear.

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1 Introduction

Let G be a finite Chevalley group. The main objective of this paper is to formulate a general procedure for orbital recognition in coherent configurations that arise from geometric actions (G, Ω). One of the desired features of such a recognition scheme is that it be universal, that is, independent of the type of Chevalley group under consideration. Another is that it be computationally effectual.

Any orbital recognition scheme must be preceded by a characterization of the orbitals under investigation. Thus we require a universal orbital characterization scheme as well. It should be apparent that the effectiveness of the recognition scheme depends crucially on what choices are made in formulating the orbital characterization strategy.

We start with a canonical embedding of the set Ω into the flag geometry $\mathfrak{F}(\mathcal{G})$ of G. From here we move progressively, from an initial orbital characterization in terms of incidence chains in the Lie geometry, to one in terms of chains in the Weyl geometry, and finally to a recognition scheme in terms of disinguished elements of the upper Borel subalgebra $\mathfrak{L}^U$ of the Lie algebra $\mathfrak{L}$ of G. This recognition scheme enables a transformation of incidences occurring in chains to an investigation of systems of equations in the field over which G is defined.

The universal geometric action is given by $(G, \mathcal{F}(G))$, where $\mathcal{F}(G)$ denotes the set of objects (flags) in $\mathfrak{F}(\mathcal{G})$. From here, one may define a geometric action to be any action (G, Ω) where Ω is a G-invariant subset of $\mathcal{F}(G)$.

Of fundamental importance is the fact that $\mathcal{F}(G)$ is in bijective correspondence with $\bigcup_J (G : P_J)$, where $\{P_J\}_{J \subseteq I}$ is the set of all (standard) parabolic subgroups of G, and $(G : P_J)$ is the coset space of P_J in G.

The set of orbitals of $(G, \mathcal{F}(G))$ forms a coherent configuration in the sense of D. Higman [14], as do the orbitals of any geometric action (G, Ω). When (G, Ω) is transitive, the resulting configuration is an association scheme. We provide below some important examples of geometric actions.

(1) $(G, \mathcal{F}(G))$, which is the action discussed above. This case is intimately related to the study of Tits buildings, see [19]. The action is intransitive.

(2) $(G, \gamma(G))$, where $\gamma(G) = \bigcup_{i \in I} (G : P_i)$. Here $\{P_i\}_{i \in I}$ is the set of all maximal parabolic subgroups of G. Objects in $\gamma(G)$ may be identified with minimal flags in $\mathfrak{F}(\mathcal{G})$. This action, which is commonly referred to as the Lie geometry of G, is intransitive.

(3) $(G, (G : P_i))$, that is, the restriction of the action of $(G, \gamma(G))$ to cosets of a fixed maximal parabolic subgroup P_i of G. This is a transitive action.

(4) $(G, (G : B))$, where B is a Borel subgroup of G. Here $(G : B)$ may be identified with the set of all maximal flags in $\mathcal{F}(G)$. The corresponding Hecke algebra (double coset alegebra) is a specialization of the generic algebra introduced by J. Tits (see [6]). This Hecke algebra is very important in the theory of representations of Chevalley groups (see the survey [9]). This is a transitive action.

We can further extend the notion of a geometric action to the Weyl group of G, as will be explained in Subsec. 6.4.

Note that only in cases (3) and (4) are the orbitals of the indicated geometric action forming the basis relations of an association scheme. But even were our interest confined solely to association schemes, we would still require the fuller picture that the flag geometry provides. Indeed, much like the manner in which one extends a group to a group ring to gain additional structural information about the group, we pursue a similar methodological approach here by extending a transitive geometric action (G, Ω) to the flag geometry $\mathfrak{F}(\mathcal{G})$ to enhance our understanding of the structural properties of association schemes of Chevalley type.

Since our ultimate goal is to provide fast orbital recognition in terms of systems of equations, the notion of a metric association scheme is especially important to us. This is because orbital recognition in metric schemes guarantees that the aforementioned systems be linear.

Restricting our attention to geometric actions of the kind (3) above, we obtain metric association schemes only for certain values of i (see [6]). For example, when G is a Chevalley group of classical type B_n, C_n or D_n, the resulting schemes are metric only when i corresponds to an end-node of the Dynkin diagram for G. In sharp contrast to this, any choice of i results in a metric scheme for G a Chevalley group of classical type A_n.

In fact, there are even instances in non-metric association schemes where particular orbitals yield linear systems. These are the so-called "short" orbitals, by which we mean orbitals whose key data correspond to incidence chains of length at most 2, see Subsec. 8.2.

The balance of our paper is organized as follows.

Sections 2–6 are in a sense preliminary. While we believe it adequate for certain readers to safely skip these sections, we encourage those who are less familiar with the subject matter to give Sections 2–6 a fairly comprehensive reading. These sections cover the following general topics: group actions (Section 2), coherent configurations (Section 3), groups of Lie type (Section 4), Lie algebras (Section 5), and group geometries (Section 6).

Section 7 deals with double coset algebras and their relation to orbitals. We emphasize the critical role played by the isomorphism between the double coset algebras of $\mathfrak{F}(\mathcal{G})$ and $\mathfrak{F}(\mathcal{W})$, where $\mathfrak{F}(\mathcal{W})$ is the flag geometry of the Weyl group W of G. This is a quite remarkable property that only occurs in groups of Lie type, and it plays a major role in the sequel.

In Section 8, we begin the proper treatment of orbital characterization by introducing the concepts of an incidence chain and an orbital key. We later put forth the notion of a Coxeter trace, which provides a means by which the role of incident chains in the Lie geometry $\mathcal{G}$ may be articulated in terms of so-called trace chains in the Weyl geometry $\mathcal{W}$. Two orbital recognition schemes are proposed, the first based on incidence chains in the Lie geometry and the second based on the Coxeter trace concept. Both proposed schemes rely on the orbital key construct.

In a strong sense, Section 9 is the cornerstone of the paper. It explains how geometric actions (G, Ω) may be embedded in the upper Borel subalgebra $\mathfrak{L}^U$ of the Lie algebra $\mathfrak{L}$ of G. This embedding transforms incidence chains to systems of equations that arise from vanishing of the Lie product in $\mathfrak{L}^U$. The role of the underlying group, so intrinsically evident in the structure of its geometric actions, is completely hidden from view at this stage. One need only follow simple rules of computation in $\mathfrak{L}^U$ to derive equations that unearth the orbital structure of (G, Ω).

In Section 10 we describe all invariant objects that comprise chains in the Weyl geometries of type A_n, B_n, C_n and D_n. These objects are essential to the formulation of optimal characterization and recognition strategies for Chevalley groups of classical type. We further elaborate by means of an example, the specific steps involved in obtaining an orbital characterization scheme for the group $A_2(q)$.

Finally, in Section 11 we give explicit characterization and recognition schemes for the classical Chevalley groups of type A_n. A flowchart depicting the recognition scheme is also provided.

2 Group actions

Let G be a group with identity element e, and let Ω be a nonempty set. A (*left*) *group action* of G on Ω is a mapping $\varphi : G \times \Omega \to \Omega$ which satisfies (i) $\varphi(gh, x) = \varphi(g, \varphi(h, x))$ and (ii) $\varphi(e, x) = x$, for all $x \in \Omega$ and $g, h \in G$. A right group action is defined similarly.

From (i) and (ii) above, it follows that $\varphi(g, \cdot\,) : \Omega \to \Omega$ induces a permutation of Ω for every $g \in G$. Let us write $g{\cdot}x$ in place of $\varphi(g, x)$. We also denote by $G{\cdot}x$ the set $\{g{\cdot}x \mid g \in G\}$, and call it a *$G$-orbit* of Ω. Clearly, the G-orbits form a partition of Ω.

Given an action of G on Ω, the kernel of the action is defined by $K = \{g \in G \mid g{\cdot}x = x, \forall x \in \Omega\}$. Clearly K is a normal subgroup of G, and the factor group G/K embeds in the symmetric group $\mathrm{Sym}(\Omega)$. If the action is faithful, i.e., if $K = \{e\}$, then we may identify G with a subgroup of $\mathrm{Sym}(\Omega)$. In this case we express the identified subgroup as (G, Ω) and refer to it as a (faithful) permutation group.

We call (G, Ω) *transitive* if $G \cdot x = \Omega$ for some (and hence every) $x \in \Omega$. Otherwise, we call (G, Ω) *intransitive*. Analogously, we say that G *acts transitively*, or *intransitively*, *on* Ω.

Finally, a transitive permutation group (G, Ω) is said to be *regular* if $G_x = \{e\}$ for some (and hence all) $x \in \Omega$. Here $G_x := \{g \in G \mid g{\cdot}x = x\}$ is referred to as the *stabilizer* of $x \in \Omega$. In this case we also say that G *acts regularly on* Ω.

2.1 Coset actions

Let G be a group with subgroup H, and denote by $(G : H)$ the set of left cosets of H in G. Then $(G, (G : H))$ is a permutation group with respect to the action $g \cdot (kH) := gkH$, $g, k \in G$. We call this a *coset action*. Note that $(G, (G : H))$ is a transitive permutation group.

In fact, all transitive permutation groups arise in this manner. Indeed, if G acts transitively on Ω then (G, Ω) is isomorphic to $(G, (G : G_x))$ as permutation groups, where G_x is the stabilizer of $x \in \Omega$.

Now let (G, Ω) be an arbitrary permutation group with orbits $\Omega_1, \ldots, \Omega_s$, and consider the transitive constituents (G, Ω_i), $1 \leq i \leq s$. Here we may identify Ω_i with the coset space $(G\!:\!G_i)$, where G_i is the stabilizer of an element of Ω_i. In this manner Ω becomes identified with the union of cosets $\bigcup_{i=1}^{s}(G\!:\!G_i)$, from which we conclude that all group actions are coset actions, up to isomorphism.

2.2 Actions that preserve incidence

According to definition, it is not essential that a set Ω on which a group acts have any structure. However, structureless sets are of minimal interest to the investigator. Most often, Ω carries a natural structure (such as the vertex set of a graph), while at other times it is desirable to impart some extrinsic structure to Ω in order to fill a specific investigative need. One way to accomplish this is to define an incidence relation $\mathcal{I}$ on Ω, that is, a subset of $\mathcal{I} \subseteq \Omega \times \Omega$. We customarily write $x \,\mathcal{I}\, y$ to indicate that $(x, y) \in \mathcal{I}$. We further say that "x is incident to y" (and conversely, if the incidence relation $\mathcal{I}$ is symmetric).

A special case of the above corresponds to the classical notion of an *incidence structure*, which is a triple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ where $\mathcal{P}$ and $\mathcal{L}$ are nonempty sets and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$. Indeed, by setting $\Omega = \mathcal{P} \cup \mathcal{L}$, one may view $\mathcal{I}$ as a relation on Ω. The same applies to higher rank analogues of incidence structures, usually called *geometries.*

Given a geometry $\mathcal{G} = (\Omega, \mathcal{I})$, let us denote by $(G, \mathcal{G})$ the set of permutations $g \in (G, \Omega)$ that preserve incidence in $\mathcal{G}$, i.e., $(g\cdot x)\,\mathcal{I}\,(g\cdot y)$ iff $x \,\mathcal{I}\, y$. Note that $(G, \mathcal{G})$ is a subgroup of the full automorphism group of $\mathcal{G}$, that is, $(G, \mathcal{G}) \leq (\mathrm{Sym}(\Omega), \mathcal{G})$.

Our main attention will be focused on the case in which $\mathcal{G}$ is a geometry of either a finite Chevalley group or its corresponding Weyl group. This will be discussed later on.

3 Coherent configurations

We begin with the concept of a *color graph* Γ. Let Ω be a nonempty set, and let $\mathcal{R} = \{R_0, R_1, \ldots R_d\}$ be a partition of $\Omega \times \Omega$. We define Γ to be the directed graph with vertex set Ω and arc set $\bigcup_{i=0}^{d} R_i$. Thus Γ is a complete directed graph with a single loop at each vertex. To complete the picture, we regard the cells R_i as distinct color classes, that is, we define the color set $C := \{0, 1, \ldots, d\}$, and we color the arcs of Γ so that arc (x, y) receives color i if and only if $(x, y) \in R_i$.

A *coherent configuration* is a color graph $\Gamma = \Gamma(\Omega, \mathcal{R})$ for which the following hold:

Axiom 1. There is a subset H of C for which $\Delta_\Omega := \{(x, x) \mid x \in \Omega\}$ is the union of color classes R_h, $h \in H$.

Axiom 2. For each $i \in C$ there exists $j \in C$ such that ${R_i}^t = R_j$, where ${R_i}^t = \{(y, x) \mid (x, y) \in R_i\}$.

Axiom 3. For any $i, j, k \in C$ and $(x, y) \in R_k$, the number $c_{i,j}^{k}$ of elements $z \in \Omega$ for which $(x, z) \in R_i$ and $(z, y) \in R_j$ is a constant depending only on i, j, k. Hence it is independent of the choice of $(x, y) \in R_k$.

In the above, we refer to each $R_i \in \mathcal{R}$ as a basis relation, to Δ_Ω as the diagonal relation, to R_i^t as the transpose relation of R_i, and to $c_{i,j}^k$ as the structure constants of the configuration $(\Omega, \mathcal{R})$. The subset H in Axiom 1 provides a partition of Ω into its subsets Ω_h such that $R_h = \{(x, x) \mid x \in \Omega_h\}$, $h \in H$. We refer to the subsets Ω_h as *fibers* of $(\Omega, \mathcal{R})$.

3.1 The homogeneous case

A coherent configuration is said to be *homogeneous* provided it satsfies one additional requirement. We formulate this as a strengthening of Axiom 1 to the following:

Axiom 1*. The set $\Delta = \{(x, x) \mid x \in \Omega\}$ is a single color class, conventionally denoted by R_0.

Thus any color graph that satisfies Axioms 1*, 2 and 3 is said to be a *homogeneous coherent configuration.* Equivalently, a homogeneous configuration is a configuration with just one fiber. Another terminology for homogeneous coherent configuration is *association scheme.*

We stress that throughout our paper, association schemes are not assumed to be symmetric (as they are defined in [3]) or commutative (as defined in [10]). Rather, our definition follows closely that of Zieschang, see [25].

Association schemes first arose in the late 1930s as a purely statistical device in the theory of experimental design. The concept, however, would not attract the attention of mathematicians until the late 1950s, when R. C. Bose and D. M. Mesner recast the theory in purely algebraic terms, see [3]. Nevertheless, one could legitimately point to the 1973 thesis [10] of P. Delsarte as the birth of modern association scheme theory.

3.2 Schurian configurations

Let (G, Ω) be a permutation group, and consider the induced action of G on $\Omega \times \Omega$ defined by $g \cdot (x, y) = (g \cdot x, g \cdot y)$, where $g \in G$ and $x, y \in \Omega$. This defines a permutation group $(G,\, \Omega \times \Omega)$. Following established convention, we refer to the orbits of $(G,\, \Omega \times \Omega)$ as *orbitals* of (G, Ω).

If we take $\mathcal{R}$ to be the set of orbitals of (G, Ω), then it is easy to check that $(\Omega, \mathcal{R})$ is a coherent configuration. Such a configuration is said to be *schurian* (in honor of I. Schur). The reader may verify that $(\Omega, \mathcal{R})$ is an association scheme precisely when the permutation group (G, Ω) is transitive. In such case, we call $(\Omega, \mathcal{R})$ a *schurian association scheme.*

We mention that all coherent configurations and association schemes considered in this paper are schurian.

3.3 Metric schemes

Let Γ be an undirected graph with vertex set Ω, and let $\mathrm{d}_\Gamma : \Omega \times \Omega \to \mathbb{N}$ denote the distance metric on Γ. We define the *distance-i* graph Γ_i of Γ as follows:

(i) the vertex set of Γ_i is Ω,

(ii) vertices $x, y \in \Omega$ are adjacent in Γ_i precisely when $\mathrm{d}_\Gamma(x, y) = i$.

Assume now that $(\Omega, \mathcal{R})$ is a *symmetric* association scheme, by which we mean $R_i^t = R_i$ for all $1 \leq i \leq d$. In this case, the color graph $\Gamma = \Gamma(\Omega, \mathcal{R})$ (minus its loops) is a simple undirected complete graph. Let us refer to any subgraph of Γ as *monochromatic* if all of its edges have the same color.

Now suppose there exists a subgraph Γ_1 of Γ for which the following hold:

(i) Γ_1 has diameter d,

(ii) Γ_i is monochromatic for every $1 \leq i \leq d$,

(iii) Γ_i and Γ_j are differently colored whenever $i \neq j$.

Then we call Γ a *distance-regular graph.*

In such case, we refer to $(\Omega, \mathcal{R})$ as a *metric association scheme* due to the fact that its basis relations respect the graph metric d_Γ. Metric association schemes have important application to coding theory, graph theory, and the theory of block designs, e.g., see [1, 6] .

3.4 Fusion schemes and subschemes

There are two basic procedures by which one can manufacture new association schemes from a given one. We start with the scheme $(\Omega, \mathcal{R})$, where $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$.

Suppose there exists another association scheme defined over the same set Ω, let us say, $(\Omega, \mathcal{S})$ where $\mathcal{S} = \{S_0, S_1, \ldots, S_r\}$. We call $(\Omega, \mathcal{S})$ a *fusion scheme* of $(\Omega, \mathcal{R})$ provided each basis relation in $\mathcal{S}$ is a union (or merging) of suitable basis relations from $\mathcal{R}$. However, we caution the reader that a random merging of basis relations rarely results in a fusion scheme.

The second procedure begins by choosing a nonempty subset Ω' of Ω. Each relation $R_i \in \mathcal{R}$ is then reduced to $R_i' = R_i \cap (\Omega' \times \Omega')$. Now define $\mathcal{R}' = \{R_i' \mid R_i' \neq \emptyset,\ 0 \leq i \leq d\}$. If $(\Omega', \mathcal{R}')$ is an association scheme, we call it a *subscheme* of $(\Omega, \mathcal{R})$.

These two procedures often lead to very interesting association schemes. For example, the schemes of Hemmeter and those of Ustimenko can both be realized as fusion schemes, while the Hermitean forms schemes, the q-analogue of the Hamming schemes, and the schemes of Egawa can all be realized as subschemes of appropriate association schemes (see [6] for details).

Finally, we mention that these procedures may be applied more generally to coherent configurations. In fact, an interesting area of investigation involves the construction of association schemes as fusion schemes of non-homogeneous coherent configurations, e.g., see [17].

REMARK. Our definition of subscheme follows Delsarte [10]. However, we alert the reader that some authors apply the term "subscheme" to what we are calling a fusion scheme. The source of this ambiguous terminology stems from the fact that the adjacency algebra of a fusion scheme $(\Omega, \mathcal{S})$ is a *sub*algebra of the adjacency algebra of $(\Omega, \mathcal{R})$. In fact, the term "fusion scheme" was coined by the author AW in [23] with the express intent of eliminating this ambiguity. A similar attempt was made by Brouwer, Cohen and Neumaier, who in their classic text [6] adopt the term "merging".

4 Groups of Lie type

Complex Lie groups play a ubiquitous role in differential geometry and physics, where they arise as groups of automorphisms of complex Lie algebras, as well as closed compact manifolds. However, in this paper we shall only be interested in the finite analogues of these groups. These make up the great majority of the finite simple groups, and have widespread application to various areas of discrete mathematics [2, 11, 16, 18].

The theory of finite groups of Lie type was clarified by the theory of algebraic groups and the seminal work of C. Chevalley [8] on Lie algebras, by which means the Chevalley group concept was isolated. Namely, Chevalley constructed an integral basis for the universal enveloping algebra of every complex simple Lie algebra, nowadays referred to as the Chevalley basis in his honor. The Chevalley basis allows one to define the corresponding algebraic groups over $\mathbb{Z}$, which in turn enables their range of definition to be extended to any finite field. This is precisely how finite groups of Lie type are constructed.

In fact, somewhat surprisingly, there exists a small number of families of finite groups of Lie type which are not strict analogues of complex Lie groups. Their construction depends on the existence of field automorphisms that interchange roots of different Euclidean length, hence they place very rigid restrictions on the fields over which they are defined.

Finite groups of Lie type (and, more generally, simple algebraic groups over arbitrary fields) act on simplicial complexes called spherical buildings. This is all encapsulated in the theory of buildings and BN-pairs due to J. Tits [19], but we shall not need this level of explanation here. The crucial point for us is that this action may be articulated in the manner of Subsec. 2.1.

4.1 Chevalley groups

Finite groups of Lie type can be categorized in various ways, but for our purposes a broad dichotomy works best. Thus a finite group of Lie type is either "twisted" or "untwisted" [7]. In another terminology, untwisted groups of Lie type are called *Chevalley groups*.

The Chevalley groups are further divided into nine classes according to type: A_n, B_n, C_n, D_n (classical Chevalley types) and E_6, E_7, E_8, F_4, G_2 (exceptional Chevalley types). In each case, the subscript indicates the rank of the group, which coincides with the number of nodes in the corresponding Dynkin diagram (see Fig. 1). Each such group is defined over a finite field, and this is indicated notationally. For example, if G is a Chevalley group of classical type A_n defined over $GF(q)$, then we express G as $A_n(q)$.

We mention that the reader is most likely already familiar with the groups $A_n(q)$, $B_n(q)$, $C_n(q)$, $D_n(q)$ by different names. Roughly speaking, they are the linear group

(A_n), two types of orthogonal group (B_n and D_n), and the symplectic group (C_n).

Finally, we stress that there is some ambiguity built into the notion of a finite group of Chevalley type which stems from the case of algebraic groups defined over fields of characteristic zero. As an example, the notation $A_n(q)$ may refer to the simple group $PSL(n+1, q)$ but it may also refer to one of the non-simple relatives of this group, for example $PGL(n+1, q)$. In all instances in which a distinction is either necessary or desirable, we will provide additional clarification.

4.2 Internal structure

Let $G = G_n(q)$ be a finite group of Lie type of rank n defined over the field $GF(q)$, where $q = p^\alpha$. We refer to p as the characteristic of G.

Let now U be a fixed Sylow p-subgroup of G, and denote by B the normalizer $N_G(U)$ of U in G. We refer to the group U as a *unipotent subgroup* (or *unipotent radical*) of G, and to B as a *Borel subgroup* of G.

As G has rank n, there are precisely $2^n - 1$ proper subgroups of G that contain a fixed Borel subgroup B (including B itself). We refer to these as *standard parabolic subgroups* of G. Of these parabolics exactly n are maximal in G. We refer to these as *maximal* (*standard*) *parabolic subgroups* of G, and denote them by $P_1, P_2, \dots, P_n$.

In fact, the complete picture is quite more attractive. Every parabolic subgroup is the intersection of maximal parabolics, so that the residual group lattice of G (based at B) is isomorphic to the inverted lattice of all subsets of $I := \{1, 2, \dots, n\}$. For example, to any subset $J \subseteq I$ there corresponds the standard parabolic subgroup $P_J = \bigcap_{i \in J} P_i$. In particular, the entire set I corresponds to B while the empty set corresponds (vacuously) to G. We do not consider G to be a parabolic subgroup.

4.3 The Weyl group

Every group of Lie type contains a section W, called its *Weyl group*. Indeed, a Borel subgroup B of $G = G_n(q)$ is a split extension (semidirect product) of a unipotent subgroup U by a subgroup T called a maximal torus. The great relevance of T resides in the fact that $N_G(T)/T$ is isomorphic to W.

In fact, in this context one actually has $T = \bigcap_{w \in W} wBw^{-1}$. Note that W is not a subgroup of G, so to correctly interpret this expression one needs to identify each $w \in W$ with one of its pre-images in $N_G(T)$. Nonetheless, this abuse of notation is standard and we shall use it throughout.

Methodologically, one constructs W from its associated root system Φ, that is, a set of vectors in Euclidean space that is subject to very strong structural requirements, see [4]. As G has rank n, Φ spans an n-dimensional vector space. Subject to how the roots of Φ are ordered, one can identify a special basis $\Pi \subset \Phi$ which we call a *fundamental basis* of Φ. Let us write $\Pi = \{r_1, r_2, \dots, r_n\}$. Then to each $r_i \in \Pi$ there exists an element $w_i \in W$ which acts as a reflection in the hyperplane orthogonal to r_i. The Weyl group W is generated by these fundamental reflections, i.e., $W = \langle w_i \mid 1 \leq i \leq n \rangle$. Observe that W is a subgroup of the full isometry group of Φ.

The Weyl group W may be viewed as a skeletal version of the group G, in the sense that it encapsulates with surprising economy much of the structural information about G. Indeed, W also contains $2^n - 1$ special proper subgroups, also called parabolics, of which n are maximal. We describe these $2^n - 1$ Weyl parabolics as follows: For $J \subseteq I$, set $W_J = \langle w_i \mid i \in I - J \rangle$, where by $I - J$ we mean the complement of J in I. When $J = \{i\}$, we obtain the maximal parabolic subgroups $W_{\{i\}}$ which we simplify notationally to W_i. (Note that some authors exclude $W_I = 1$ from the list of parabolic subgroups of W but we shall not.) This gives a bijective correspondence between the parabolics of W and those of G, specifically $P_J = BW_JB$. This is reminiscent of the Bruhat decomposition $G = BWB = \bigcup_{w \in W} BwB$, which is a defining property of BN-pairs.

As is the case with geometric actions of G, one can also realize geometric actions of W in the manner of Subsec. 2.1. This is of essential importance to us, as the Weyl group is a crucial ingredient in achieving our main objective in this paper.

4.4 Dynkin diagrams

The fundamental importance of Dynkin diagrams is that they are used to classify semisimple Lie algebras over algebraically closed fields. The classification proceeds by first determining all possible root systems, which are codified by their Dynkin diagrams. Once a fixed choice of field is made, each diagram uniquely determines a number of algebraic objects, including a Lie algebra, a simple Lie group (and its associated algebraic groups), a Weyl group, etc. The nodes of the diagram have many interpretations, e.g., the fundamental roots of the root system, generators of the Weyl group via fundamental reflections, maximal (or minimal) parabolics of the Weyl group, maximal parabolics of the Lie group, and so on. The edges, which may be multiple and directed, determine the angle between fundamental roots, as well as the orders in the Weyl group of products of pairs of fundamental relections. Indeed, the Dynkin diagram is one of the most compact modes of description in all of mathematics.

In the interest of completeness, we provide in Fig. 1 the Dynkin diagrams for the nine families of Chevalley groups. We remark that each group of Lie type also has a corresponding Coxeter diagram, which is obtained from its Dynkin diagram by simply removing arrows (orientation of edges). There is a well developed theory of Coxeter systems that generalizes the theory of root systems, e.g., see [15]. Observe that B_n and C_n have the same Coxeter diagram.

Example 1. In order to facilitate a better understanding of the concepts introduced in this section, we illustrate the case of the group $G = A_n(q) \cong PGL(n+1, q)$ in detail. This is the group of projective general linear transformations of dimension $n+1$ over the field $GF(q)$. For convenience, we interpret G as a matrix group.

Recall that a unipotent subgroup U is a Sylow-p subgroup of G, where p is the characteristic of G. A most convenient choice for U here is the group of upper triangular matrices with ones along the diagonal. The Borel subgroup $B = N_G(U)$ in this case is simply the group of upper triangular matrices.

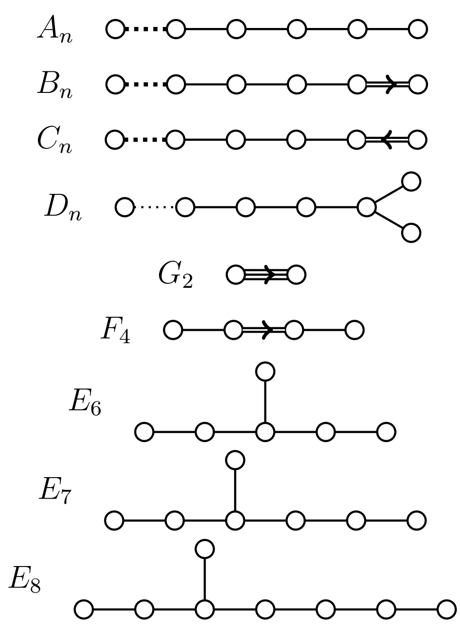

Figure 1: Dynkin diagrams of the Chevalley groups

Recall that B is a split extension of U by a maximal torus T. Here T is the group of diagonal matrices. Its normalizer $N_G(T)$ consists of all monomial matrices, that is, matrices with exactly one nonzero entry in each row and column.

Finally, recall that the Weyl group W arises as the factor group $N_G(T)/T$. Here W consists of all monomial matrices whose nonzero entries are all equal to one. This is the group of $(n+1)\times(n+1)$ permutation matrices, which is evidently isomorphic to the symmetric group S_{n+1} on $n+1$ letters. Thus the Weyl group of type A_n is nothing more than a symmetric group. □

5 Lie algebras

Lie algebras are inextricably linked to Lie groups. As previously mentioned, one may interpret a Lie group as a differentiable manifold, in which case its Lie algebra is canonically defined as the tangent space to the manifold at the identity. Conversely, any semisimple algebraic group defined over an algebraically closed field of prime characteristic is uniquely determined by its Lie algebra. We, however, shall only be interested in the algebraic structure of Lie algebras over finite fields, and their combinatorial and geomteric properties.

Abstractly, a *Lie algebra* $\mathfrak{L}$ is a vector space over some field $\mathbb{F}$, endowed with an operation $[\cdot,\cdot]:\mathfrak{L}\times\mathfrak{L}\to\mathfrak{L}$ called the *Lie product* (or *Lie bracket*, or *commutator operation*). The Lie product is bilinear and skew-symmetric, and satisfies the Jacobi identity:

$$[\alpha,[\beta,\gamma]]+[\beta,[\gamma,\alpha]]+[\gamma,[\alpha,\beta]]=0,\quad \alpha,\beta,\gamma\in\mathfrak{L}\,.$$

As such, $\mathfrak{L}$ is a non-associative algebra.

5.1 Internal structure

Let $G = G_n(q)$ be a finite simple group of Lie type, and let $\mathfrak{L} = \mathfrak{L}(G)$ be its corresponding Lie algebra, both defined over $GF(q)$. Let Φ be the root system corresponding to the Dynkin diagram of G, and let Π be a fundamental basis for Φ. Note that $|\Pi| = n$ because n is the rank of G.

First $\mathfrak{L}$ contains a unique maximal self-normalizing subalgebra $\mathfrak{H}$, called the *Cartan subalgebra* of $\mathfrak{L}$. The Cartan subalgebra is generated by the dual basis Π^* of Π, and hence $\mathfrak{H}$ is an n-dimensional vector subspace of $\mathfrak{L}$.

In addition, $\mathfrak{L}$ contains two other very relevant vector subspaces, denoted $\mathfrak{L}^+$ and $\mathfrak{L}^-$, commonly referred to as the *positive root space* and *negative root space* of $\mathfrak{L}$, respectively.

The above terms reflect a deep theoretical connection between roots and their corresponding root vectors. If $r \in \Phi$ is a root, we denote its corrsponding root vector by e_r. Now write $\Phi = \Phi^+ \bigcup \Phi^-$, where Φ^+ and Φ^- denote the sets of positive and negative roots of Φ, respectively. In fact, we have $\Phi^- = -\Phi^+$. Then

$$\mathfrak{L}^+ = \bigoplus_{r\in\Phi^+} \langle e_r \rangle, \qquad \text{and} \qquad \mathfrak{L}^- = \bigoplus_{-r\in\Phi^-} \langle e_{-r} \rangle \,.$$

Crucially, the one-dimensional root spaces $\langle e_r \rangle$, $r \in \Phi$, are $\mathfrak{H}$-invariant.

We now have all the ingredients to describe the *Cartan decomposition* of $\mathfrak{L}$. Namely,

$$\mathfrak{L} = \mathfrak{H} \bigoplus \mathfrak{L}^+ \bigoplus \mathfrak{L}^- \,.$$

From this decomposition one can visibly detect two additional subalgebras of $\mathfrak{L}$:

$$\mathfrak{L}^U = \mathfrak{H} \bigoplus \mathfrak{L}^+ \qquad \text{and} \qquad \mathfrak{L}^L = \mathfrak{H} \bigoplus \mathfrak{L}^- \,.$$

We refer to $\mathfrak{L}^U$ and $\mathfrak{L}^L$ as the *upper* and *lower Borel subalgebras* of $\mathfrak{L}$, respectively. The upper Borel algebra will be of special significance to us later on.

Example 2. Our main intention here is to make the abstract notions introduced above a bit more tangible to the reader. We do not pretend to give detailed explanations of all features of the example.

Let us consider the Lie algebra $\mathfrak{L}$ of the simple group $A_2(q)$ of Lie type, that is, the projective special linear group $PSL(3, q)$ with corresponding Weyl group isomorphic to the symmetric group S_3.

First, the root system of type A_2 is given by $\Phi = \Phi^+ \bigcup \Phi^-$, where

$$\Phi^+ = \{r_1, r_2, r_1 + r_2\} \quad \text{and} \quad \Phi^- = \{-r_1, -r_2, -r_1 - r_2\}.$$

A fundamental basis for Φ is given by $\Pi = \{r_1, r_2\}$. We denote its dual basis by $\Pi^* = \{r_1^*, r_2^*\}$. Thus the Cartan subalgebra $\mathfrak{H}$ is spanned by the dual roots r_1^*, r_2^*. Convenient matrix representations for these dual roots are as follows:

$$r_1^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad r_2^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

Hence $\mathfrak{H}$ consists of all 3×3 diagonal trace zero matrices over $GF(q)$. Similarly, we give matrix representations of the root vectors:

$$e_{r_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_{r_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_{r_1+r_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_{-r_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_{-r_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad e_{-r_1-r_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We thereby conclude that $\mathfrak{L}$ consists of all 3×3 trace zero matrices over $GF(q)$. This Lie algebra is traditionally designated as $\mathfrak{sl}_3(q)$.

Subject to the afforded matrix representation, the Lie product in $\mathfrak{L}$ is easily described as $[A, B] = AB - BA$, where juxtaposition indicates the usual matrix product. □

6 Group geometries

6.1 Coset geometries

Let G be a group with subgroups H_1 and H_2, and denote by $\mathcal{P} = (G : H_1)$ and $\mathcal{L} = (G : H_2)$ the corresponding coset spaces. Now define an incidence relation $\mathcal{I}$ as follows: For $\alpha \in \mathcal{P}$ and $\beta \in \mathcal{L}$, define $\alpha \mathcal{I} \beta$ precisely when $\alpha \cap \beta \neq \emptyset$. Then $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is an incidence structure, commonly referred to as a *rank-*2 *coset geometry.*

Rank-n coset geometries can be similarly constructed from rank n analogues of incidence structures. In this case one is working simultaneously with n coset spaces of the ambient group, but the incidence relation is the same, i.e., nonempty intersection of cosets.

Rank-2 coset geometries arise naturally from rank 2 groups of Lie type, where the roles of the subgroups H_1 and H_2 are played by the two standard maximal parabolics. Here the resulting incidence structures turn out to be the remarkable "generalized polygons", see [19]. Indeed, restricting our attention to the Chevalley case one obtains in this manner the generalized 3-gon (when $G = A_2(q)$), the generalized 4-gon (when $G = B_2(q)$ or $C_2(q)$) and the generalized 6-gon (when $G = G_2(q)$).

We mention in passing that the geometry of ${}^2F_4$ is the generalized 8-gon, however ${}^2F_4$ is a twisted group of Lie type, hence not a Chevalley group.

Lastly, we remark that the generalized 3-gon constructed from the group $A_2(q) \cong PGL(3, q)$ is none other than the classical projective plane $PG(2, q)$ of order q. (Note, however, that the full collineation group of $PG(2, q)$ is $P\Gamma L(3, q)$.) It is a nice exercise for the reader to identify the two standard maximal parabolic subgroups in this case by revisiting Example 1 with $n = 2$. These parabolics turn out to be the respective stabilizers of a canonically chosen point P and line l for which $\{P, l\}$ is a point-line incident pair in the projective plane.

6.2 Flag geometries

Let $\mathcal{G}$ be a rank-n geometry, that is, a geometry with objects of n distinct types. Let us designate the *type* of an object α by $t(\alpha)$. For convenience, we denote the set of all types by $I = \{1, 2, \ldots, n\}$.

To $\mathcal{G}$ we may associate a second geometry, denoted $\mathfrak{F}(\mathcal{G})$, called the *flag geometry* of $\mathcal{G}$. We call the objects of $\mathfrak{F}(\mathcal{G})$ *flags*, by which we mean sets of pairwise incident objects in the initial geometry $\mathcal{G}$. Let $\mathcal{F} = \mathcal{F}(\mathcal{G})$ denote the set of flags in $\mathfrak{F}(\mathcal{G})$. We define incidence in $\mathfrak{F}(\mathcal{G})$ as follows: For $F_1, F_2 \in \mathcal{F}$, we have F_1 incident to F_2 precisely when $F_1 \cup F_2 \in \mathcal{F}$. We define the *flag-type* of F to be the set $t(F) = \{t(\alpha) \mid \alpha \in F\}$, and we call a flag *maximal* if $t(F) = I$.

Let now $\mathcal{G}$ be a rank-n coset geometry of a group $G = G_n(q)$ of Lie type, in which the objects of $\mathcal{G}$ are cosets of the n maximal standard parabolic subgroups P_i of G. Here we may regard I as the set of nodes of the Dynkin diagram of G.

There is a very fruitful identification one can make in passing from $\mathcal{G}$ to $\mathfrak{F}(\mathcal{G})$. Namely, we identify a specified flag F of type J with the coset gP_J, where $P_J = \bigcap_{j\in J} P_j$ and $gP_j \in F$ for all $j \in J$. In this manner we see that flags themselves correspond to cosets of standard parabolic subgroups, just as their component objects do. This further allows us to identify flag-types with corresponding subsets of nodes of the Dynkin diagram.

From the above, it is immediate that G acts transitively on the set of flags of each fixed type. For this reason we refer to $\mathfrak{F}(\mathcal{G})$ as a *flag-transitive* geometry. Further note that this identification establishes that flag geometries themselves are coset geometries, although of a much higher rank.

6.3 Subgeometries

Let $\mathcal{G}$ be a rank-n geometry, and let $K \subseteq I$ be a specified set of object types of size $k = |K|$. Then one obtains a k-rank subgeometry $\mathcal{G}_K$ of $\mathcal{G}$ by restricting the set of objects of $\mathcal{G}$ to those of type i, for all $i \in K$. Incidence in $\mathcal{G}_K$ is simply the restriction of incidence in $\mathcal{G}$ to objects of said type.

It is straightforward to see that the natural embedding $\mathcal{G}_K \hookrightarrow \mathcal{G}$ yields a corresponding embedding of flag geometries $\mathfrak{F}(\mathcal{G}_K) \hookrightarrow \mathfrak{F}(\mathcal{G})$, where the flag-types in $\mathfrak{F}(\mathcal{G}_K)$ consist of all nonempty subsets of K. In this sense we may regard $\mathfrak{F}(\mathcal{G}_K)$ as a *sub-flag geometry* of $\mathfrak{F}(\mathcal{G})$. (In the literature, the term *residual geometry* is sometimes used.) Still, there are subgeometries of $\mathfrak{F}(\mathcal{G}_K)$ that are not sub-flag geometries yet are important in their own right.

For example, the rank-n geometry $\mathcal{G}$ from which $\mathfrak{F}(\mathcal{G})$ was initially constructed can be realized as such a subgeometry. Indeed, given any object α in $\mathcal{G}$, we may identify it with the minimal flag $\{\alpha\}$ in $\mathfrak{F}(\mathcal{G})$. Moreover, α and β are incident in $\mathcal{G}$ iff $\{\alpha, \beta\}$ is a flag in $\mathfrak{F}(\mathcal{G})$, that is, iff $\{\alpha\}$ and $\{\beta\}$ are incident in $\mathfrak{F}(\mathcal{G})$. Therefore, as required of any subgeometry, the incidence relation on $\mathcal{G}$ is the restriction of the incidence relation on $\mathfrak{F}(\mathcal{G})$.

6.4 Geometric actions

Let us now recount the four standard geometries that arise naturally from a group G of Lie type. Recall that these geometries are denoted $\mathcal{G}$, $\mathfrak{F}(\mathcal{G})$, $\mathcal{W}$, and $\mathfrak{F}(\mathcal{W})$. The first two will be referred to as *Lie geometries*, and the last two as *Weyl geometries*. Further, we call the action of G on any of these geometries a *geometric action*.

At this stage, we refine some of our existing notation to more vividly reflect the set of objects in a geometric action (G, Ω). Henceforth:

(a) $(G, \mathcal{G})$ will be denoted $(G, \gamma(G))$, where $\gamma(G) := \bigcup_{i=1}^{n}(G\!:\!P_i)$,

(b) $(G, \mathfrak{F}(\mathcal{G}))$ will be $(G, \mathcal{F}(G))$, where $\mathcal{F}(G) := \bigcup_{J\subseteq I}(G\!:\!P_J)$,

(c) $(W, \mathcal{W})$ will be denoted $(W, \gamma(W))$, where $\gamma(W) := \bigcup_{i=1}^{n}(W\!:\!W_i)$,

(d) $(W, \mathfrak{F}(\mathcal{W}))$ will be denoted $(W, \mathcal{F}(W))$, where $\mathcal{F}(W) := \bigcup_{J\subseteq I}(W\!:\!W_J)$.

In fact, all mentioned geometries live inside the flag geometry $\mathfrak{F}(\mathcal{G})$. Indeed, we have already observed that the Lie geometry $\mathcal{G}$ may be embedded in its flag geometry $\mathfrak{F}(\mathcal{G})$. Clearly, the same procedure may be applied to obtain an embedding of $\mathcal{W}$ in $\mathfrak{F}(\mathcal{W})$.

To complete the picture, we claim that $\mathfrak{F}(\mathcal{W})$ is a sub-flag geometry of $\mathfrak{F}(\mathcal{G})$. Observe that this is tantamount to showing that $\gamma(W)$ may be identified with a subset of $\gamma(G)$. Recall that the Lie geometry $\mathcal{G}$ is constructed subject to a fixed choice of Borel subgroup B of G. Let T be a maximal torus in B. Then the set $\gamma(W)$ may be identified with the set $\gamma(G)_T$ of T-invariant objects of $\gamma(G)$, see [4]. Thus the picture is complete.

Example 3. Let us return to the classical projective plane $PG(2, q)$ constructed from the group $G = A_2(q) \cong PGL(3, q)$ of Example 1 with $n = 2$. Here $\mathcal{G}$ is a rank-2 coset geometry, hence there are precisely three flag-types in $\mathfrak{F}(\mathcal{G})$, namely $\{1\}$, $\{2\}$ and $\{1, 2\}$. Flags of type $\{1\}$ and $\{2\}$ correspond, respectvely, to cosets of the maximal parabolics P_1 and P_2 of G. Those of type $\{1, 2\}$ correspond to cosets of $B = P_1 \cap P_2$ in G.

We may further use a familiar model of $PG(2, q)$ to identify flags. Namely, we associate type $\{1\}$ flags with one-dimensional subspaces V_1 of $GF(Q)^3$, type $\{2\}$ flags with two-dimensional subspaces V_2, and type $\{1, 2\}$ flags with chains of the form $V_1 < V_2$.

If we now pass to the Weyl geometry $\mathfrak{F}(\mathcal{W})$, the generalized 3-gon reduces to an ordinary 3-gon, that is, a triangle. This makes perfect sense if we recall that the Weyl group of $A_2(q)$ is $W \cong S_3$. Indeed S_3 is isomorphic to the dihedral group of order 6 which acts naturally on the triangle. □

More generally, one can consider the flag geometry $\mathfrak{F}(\mathcal{G})$ where $\mathcal{G}$ is the rank-n geometry of the group $G = A_n(q) \cong PGL(n + 1, q)$. In this case $\mathcal{G}$ is the classical projective geometry $PG(n, q)$, and just as in Example 3 we may identify flags as chains $V_1 < V_2 < \cdots < V_r$ $(r \leq n)$ of embedded subspaces of $GF(q)^{n+1}$ of increasing dimension. The flag-type here may be taken to be the set $\{n_1, n_2, \ldots, n_r\}$ of dimensions $n_i = \dim(V_i)$. Note that when $n = 2$, these flag-types coincide with the ones given above.

7 Orbitals and the double coset algebra

7.1 Double coset algebra

Let G be a group, and let H and K be subgroups of G. For each $g \in G$ we define $HgK := \{hgk \mid h \in H, k \in K\}$, and refer to it as an (H, K)-*double coset* of G. We denote the set of all (H, K)-double cosets of G by $H/G\backslash K$. It is easy to see that $H/G\backslash K$ forms a partition of G, and that each HgK is simultaneousy a union of left cosets of K in G and a union of right cosets of H in G.

Now consider all (H, K)-double cosets as H and K vary over a prescribed set of subgroups $\Sigma = \{L_1, L_2, \dots L_r\}$ of G. We denote the totality of double cosets varying over these subgroups by $\Sigma/G\backslash\Sigma$.

We may define a vector space structure on $\Sigma/G\backslash\Sigma$ by linear extension, that is, its elements are formal linear combinations of $L_i g L_j$, $1 \le i, j \le r$. In this manner, $\{L_i g L_j\}_{1\le i,j\le r}$ forms a natural basis for $\Sigma/G\backslash\Sigma$.

Now fix $H, K, L \in \Sigma$ and $g_1, g_2 \in G$. Decompose Hg_1K and Kg_2L as respective unions of right cosets of H and left cosets of L as mentioned above:

$$Hg_1K = \bigcup_i Ha_i, \qquad \text{and} \qquad Kg_2L = \bigcup_j b_jL.$$

For each $g \in G$, denote $c_g = \left|\{(i, j) \mid a_ib_j \in Hg\}\right|$. We may then define a multiplication on $\Sigma/G\backslash\Sigma$ according to the rule:

$$Hg_1K \cdot Kg_2L = \sum_{g\in\mathcal{T}} c_g(HgL)\,,$$

where the summation ranges over a transversal $\mathcal{T}$ of left cosets of H in G. This gives $\Sigma/G\backslash\Sigma$ the structure of an algebra. We call it a *double coset algebra*.

7.2 Relation to orbitals

Double coset algebras are especially important to us for two reasons. First, if the subgroups that comprise Σ are interpreted as stabilizers of orbit representatives of a group action (G, Ω), then each double coset H_igH_j corresponds to a unique orbital $\mathcal{O}$ of (G, Ω), via

$$\mathcal{O} = \{(xH_i, yH_j) \mid H_ix^{-1}yH_j = H_igH_j\}\,.$$

Second, if G ia a group of Lie type and W is its Weyl group, then the two double coset algebras that correspond to the geometric actions $(G, \mathcal{F}(G))$ and $(W, \mathcal{F}(W))$ are isomorphic, see [7].

This second property, which is a feature unique to groups of Lie type, establishes a natural bijection between the orbitals of $(G, \mathcal{F}(G))$ and those of $(W, \mathcal{F}(W))$. Indeed, due to the Bruhat decomposition $G = BWB$, coupled with the fact that B is contained in every standard parabolic subgroup P_J, we may express the double coset $P_{J_1}gP_{J_2}$ as $P_{J_1}wP_{J_2}$ for some $w \in W$. This establishes a canonical bijection of orbitals in the frames of double cosets, namely $P_{J_1}wP_{J_2} \leftrightarrow W_{J_1}wW_{J_2}$. A major consequence of this is that the orbital structure of $(G, \mathcal{F}(G))$ does not depend on the field over which the group G is defined.

7.3 Modes of representation

Communicating with machine requires that the chosen language be numerical. From our discussion above, we observe that one possible way to represent an orbital $\mathcal{O}$ consisting of ordered pairs of flags (F_1, F_2) is by a triple (J_1, J_2, M), where J_1 is the flag-type of F_1, J_2 is the flag-type of F_2, and M is the $|J_1| \times |J_2|$ matrix whose (j_1, j_2) entry is $P_{j_1} g P_{j_2}$ for suitable $g \in G$, where (j_1, j_2) ranges over $J_1 \times J_2$. But since $(G, \mathcal{F}(G))$ and $(W, \mathcal{F}(W))$ have isomorphic double coset algebras, we may also represent each $P_{j_1} g P_{j_2}$ by $W_{j_1} w W_{j_2}$ for suitable $w \in W$. As already mentioned, this latter representation is preferred since the action $(W, \mathcal{F}(W))$ is field-independent.

However, one is not confined to the use of double cosets for the entries of matrix M in our triple. Indeed, there are more convenient parameters that uniquely characterize the orbitals of $(G, \gamma(G))$ (see, for example, [13]). We illustrate this directly.

Example 4. Let us again consider the Chevalley group $G = A_n(q)$. Recall from Example 3, and the discussion immediately thereafter, that a flag F may be identified with a chain $V_1 < V_2 < \cdots < V_r$ of embedded subspaces of increasing dimension. Recall further, that the flag-type $t(F)$ may be identified with the sequence $\{\dim(V_1), \dim(V_2), \ldots, \dim(V_r)\}$.

Now, let $F_1, F_2 \in \mathcal{F}(G)$ have respective chains

$$X_1 < X_2 < \cdots < X_r \qquad \text{and} \qquad Y_1 < Y_2 < \cdots < Y_s\,.$$

For each $1 \le i \le r$ and $1 \le j \le s$, the orbital $\mathcal{O}$ of $(G, \gamma(G))$ containing (X_i, Y_j) can be characterized by the value $a_{ij} = \dim(X_i \cap Y_j)$. Consequently, we may now characterize the orbital $\mathcal{O}$ of $(G, \mathcal{F}(G))$ that contains (F_1, F_2) by the triple (J_1, J_2, A), where $J_1 = \{\dim(X_i) \mid 1 \le i \le r\}$, $J_2 = \{\dim(Y_j) \mid 1 \le j \le s\}$, and A is the $r \times s$ matrix whose (i, j) entry is a_{ij}.

In fact, alternate characterizations exist for the orbitals of $(G, \mathcal{F}(G))$ when $G = A_n(q)$. For example, in place of matrix $A = (a_{ij})$ one can instead consider matrix $B = (b_{ij})$, where b_{ij} is the dimension of the affine variety $(X_i - X_{i-1}) \cap (Y_j - Y_{j-1})$. The advantage here is that this description allows one to determine precisely which matrices B can actually arise. Indeed, a necessary and sufficient condition is that

$$\sum_{j=1}^{s} b_{ij} = \dim(X_i) \quad \text{and} \quad \sum_{i=1}^{r} b_{ij} = \dim(Y_j) \quad \forall 1 \le i \le r,\ \forall 1 \le j \le s\,.$$

Observe that in the case of maximal flags, both A and B correspond to permutation matrices.

As a final illustration, one can also characterize orbitals of $(G, \mathcal{F}(G))$ in terms of orbitals of $(W, \mathcal{F}(W))$. This amounts to replacing vector spaces with subsets of an $(n+1)$-element set. □

8 Characterization and recognition via incidence chains

We have just seen how the objects in $\gamma(G)$, where $G = A_n(q)$, may be represented as chains of embedded subspaces. This general mode of representation can be extended to other classical Chevalley groups by passing to, for example, totally isotropic subspaces. Still, it is difficult to apply this sort of model to the exceptional Chevalley groups. What we seek is a more universal approach, that is, one that will not vary with the choice of Chevalley group G. The following concept provides a promising pathway.

8.1 Incidence chains

Let $G = G_n(q)$ be a group of Lie type. We call an ordered sequence $\mathbf{x} = (x_0, x_1, \ldots, x_s)$ of objects $x_i \in \gamma(G)$ an *incidence chain* (or simply *chain*) from α to β provided: $x_0 = \alpha$, $x_s = \beta$, and $x_i \,\mathcal{I}\, x_{i+1}$ for all $0 \le i \le s-1$. We refer to $s = l(\mathbf{x})$ as the *length* of the chain, and to $t(\mathbf{x}) = (t(x_0), t(x_1), \ldots, t(x_s))$ as its *type*.

The notion of a chain applies equally well to objects from the Weyl geometry (just replace $\gamma(G)$ with $\gamma(W)$ in the above definition). However, this is already implied from the identification of $\gamma(W)$ with the set $\gamma(G)_T$ of T-invariant objects of $\gamma(G)$, see Subsec. 6.4.

A chain $\mathbf{x}$ from α to β will be called a *key chain* provided the following hold:

(i) $\mathbf{x}$ contains no proper subchain from α to β (i.e., $\mathbf{x}$ is irreducible),

(ii) $\mathbf{x}$ is the unique chain from α to β of type $t(\mathbf{x})$,

(iii) $\mathbf{x}$ is of minimum length among all chains satisfying (i) and (ii).

We denote by $\mathrm{KC}(\alpha, \beta)$ the set of key chains from α to β. Note that $\mathrm{KC}(\alpha, \beta)$ may be empty for certain $\alpha, \beta \in \gamma(G)$.

Lemma 1 *Let $\mathbf{x}$ be a chain that satisfies condition (ii) above. Then the double-point stabilizer $G_{\alpha,\beta}$ fixes $\mathbf{x}$.*

Proof. Let $\mathbf{y} = g{\cdot}\mathbf{x}$, where $g \in G_{\alpha,\beta}$. As g stabilizes α and β, it follows that $\mathbf{y}$ is also a chain from α to β. But $t(\mathbf{y}) = t(\mathbf{x})$, since object type is G-invariant. Thus $\mathbf{y} = \mathbf{x}$ follows from (ii). □

It turns out (see [4]) that each orbital of $(G, \gamma(G))$ contains exactly one representative coming from $\gamma(W) \times \gamma(W)$ of the form (P_i, wP_j), where w is a word of minimum length in $P_i w P_j$. (Here, words in W are expressed as products of fundamental reflections.) In this instance we call (P_i, wP_j) a *standard pair*.

Lemma 2 *Let (α, β) be a standard pair, and let $\mathbf{x} \in \mathrm{KC}(\alpha, \beta)$. Suppose $\mathbf{x} = (\alpha, x_1, \ldots, x_{s-1}, \beta)$. Then $x_i \in \gamma(W)$ for all $i = 1, 2, \ldots, s-1$.*

Proof. From Lemma 1, each x_i is invariant under the action of the double-point stabilizer $G_{\alpha,\beta}$. But here $G_{\alpha,\beta} = P_i \bigcap wP_jw^{-1}$ for some i, j, and some $w \in W$. As $T \subseteq wBw^{-1}$ for all $w \in W$, it immediately follows that T is a subgroup of $P_i \bigcap wP_jw^{-1}$. Thus each x_i is T-invariant, i.e., $x_i \in \gamma(W)$. □

8.2 Key data and the orbital key

Recall that $\mathrm{KC}(\alpha,\beta)$ may be empty for certain $\alpha,\beta\in\gamma(G)$. Clearly, this property is satisfied for all pairs in the orbital $\mathcal{O}$ to which (α,β) belongs. In such case, we refer to (α,β) as a *residual pair*, and to $\mathcal{O}$ as a *residual orbital*.

Conversely, all non-residual orbitals consist only of non-residual pairs. For (α,β) a non-residual standard pair, we denote by $\mathrm{Data}(\alpha,\beta)$ the set of all types $t(\mathbf{x})$ of chains $\mathbf{x}\in\mathrm{KC}(\alpha,\beta)$. Since $\mathrm{Data}(\alpha,\beta)$ is uniquely determined by the orbital $\mathcal{O}$ that contains (α,β), we may express it simply as $\mathrm{Data}(\mathcal{O})$. We call $\mathrm{Data}(\mathcal{O})$ the *key data* for $\mathcal{O}$.

The next lemma is critical to the fulfillment of our objective. Its proof is highly technical so will not be presented here. However in what follows we give a fairly rigorous account of the steps involved.

As a consequence of Lemma 2 and the existence of a standard pair, each key data set $\mathrm{Data}(\mathcal{O})$ may be completely characterized in terms of key chains in $\gamma(W)$. By Lemma 1 the objects comprised in these chains must be invariant under the appropriate double-point stabilizers. Objects in $\gamma(W)$ may be deduced from [19], at which point both the orbitals of $(W,\gamma(W))$ and the relevant double-point stabilizers may be computed (see [5] for the exceptional group case). These stabilizers suffice to determine a natural basis for the corresponding Boolean algebra, from which all invariant chains in $\gamma(W)$ may be determined. The key chains are easily identified among these invariant chains, leaving verification of the lemma to a straightforward comparison of the resulting key data sets.

In Section 10, we provide a proof of Lemma 3 for the special case of a Chevalley group of classical type A_n (see Lemma 7). As well, we construct a natural basis for the Boolean algebra corresponding to each Chevalley group of classical type B_n, C_n or D_n, leaving to the interested reader the task of determining (invariant) key chains in the corresponding Weyl geometry.

Lemma 3 (i) *Let $\gamma_i(G)$ denote the set of objects in $\gamma(G)$ of type i. Then for each i and j, there exists at most one residual orbital $\mathcal{O}$ of $(G,\gamma(G))$ that is contained in $\gamma_i(G)\times\gamma_j(G)$.*
(ii) $\mathrm{Data}(\mathcal{O}_1)\bigcap\mathrm{Data}(\mathcal{O}_2)=\emptyset$ *whenever $\mathcal{O}_1$ and $\mathcal{O}_2$ are distinct orbitals of $(G,\gamma(G))$.* □

By Lemma 3 (i), each set $\gamma_i(G)\times\gamma_j(G)$ contains a unique residual orbital or none at all. In the latter case, let us choose an orbital $\mathcal{O}\subset\gamma_i(G)\times\gamma_j(G)$ for which $\mathrm{Data}(\mathcal{O})$ contains a chain type $\mathbf{t}$ of maximum length $l(\mathbf{t})$. We call $\mathcal{O}$ a *pseudo-residual orbital*.

The role of the pseudo-residual orbital is to bring conformity to the procedure of orbital recognition. Our rationale in choosing an orbital with key chain of maximum length is simply because, computationally speaking, such a chain is the most logical one to avoid.

Unless otherwise indicated, we shall no longer make a distinction between residual and pseudo-residual orbitals, using the term *p-residual* for both. Thus every set $\gamma_i(G)\times\gamma_j(G)$ contains exactly one p-residual orbital.

We are now in a position to define the (i, j)-*orbital key* $\mathrm{Key}_{ij}(G)$. For each orbital $\mathcal{O} \subset \gamma_i(G) \times \gamma_j(G)$ that is not p-residual, we choose a single representative $\mathbf{t} \in \mathrm{Data}(\mathcal{O})$. We define $\mathrm{Key}_{ij}(G)$ to be the set of all such representatives.

8.3 An orbital recognition scheme

For each i and j, the orbital key $\mathrm{Key}_{ij}(G)$ completely characterizes the orbitals in $\gamma_i(G) \times \gamma_j(G)$. Note here that the p-residual orbital is characterized by default, as it is the only orbital in $\gamma_i(G) \times \gamma_j(G)$ not represented in the orbital key. Thus the union $\bigcup_{ij} \mathrm{Key}_{ij}(G)$ provides a complete characterization of all orbitals in $(G, \gamma(G))$.

The following theorem provides a fast orbital recognition scheme for the geometric action $(G, \gamma(G))$.

Theorem 4 *Given distinct objects η, $\xi \in \gamma(G)$, identify i and j for which $(\eta, \xi) \in \gamma_i(G) \times \gamma_j(G)$. If there exists a chain $\mathbf{x} \in \mathrm{KC}(\eta, \xi)$ whose type $t(\mathbf{x})$ coincides with the type $\mathbf{t} \in \mathrm{Data}(\mathcal{O}) \subset \mathrm{Key}_{ij}(G)$, then $(\eta, \xi) \in \mathcal{O}$. Otherwise, (η, ξ) is contained in the p-residual orbital in $\gamma_i(G) \times \gamma_j(G)$.*

Proof. This is an immediate consequence of Lemma 3. □

The characterization scheme in terms of orbital keys, and the recognition scheme in Theorem 4, are both universal, however our task is still not complete. Indeed, we have yet to provide a characterization of orbitals in the manner proposed, that is, one that enables fast orbital recognition in terms of systems of equations. To accomplish this, we will need to first transform our present characterization to the language of Coxeter traces. Following this, we will need to establish how $\gamma(G)$, subject to the benefits of this transformation, may be embedded in the upper Borel subalgebra $\mathfrak{L}^U$ of the Lie algebra for G. We are now prepared to address the first of these two tasks. The second task will be carried out in Sectiion 9.

8.4 The Coxeter trace

Consider the action of the Borel subgroup B on $\gamma(G)$. We call the orbits of $(B, \gamma(G))$ *Schubert cells*. Schubert cells provide a very useful partition of $\gamma(G)$. Indeed, it is well known that each Schubert cell C contains a unique T-invariant object $\alpha \in \gamma(W)$, see [4]. Thus we may write $C = C_\alpha$ for this particular cell. Further, for every $a \in C_\alpha$ we write $\alpha = \mathrm{Cox}(a)$ and refer to it as the *Coxeter trace* of a. Evidently, the Coxeter trace of each object in $\gamma(G)$ is an object in $\gamma(W)$.

Lemma 5 *Let a and b be incident objects in $\gamma(G)$. Then α and β are incident objects in $\gamma(W)$, where $\alpha = \mathrm{Cox}(a)$ and $\beta = \mathrm{Cox}(b)$.*

Proof. Let $a = gP_i$ and $b = g'P_j$. Since a and b are incident in $\gamma(G)$, we have $gP_i \cap g'P_j \neq \emptyset$, in which case we may assume $g = g'$. This implies that the restriction $\mathcal{I}'$ of the incidence relation on $\gamma(G)$ to the set $C_\alpha \cup C_\beta$ is nonempty. By transitivity of the unipotent subgroup U on the edges of $\mathcal{I}'$, we see that α and β are incident in $\gamma(G)$, whence $\alpha \cap \beta \neq \emptyset$. As α and β are objects in the Weyl geometry $\gamma(W)$, we conclude that they are incident in $\gamma(W)$ as well. □

As a consequence of Lemma 5, we see that given any incidence chain $\mathbf{x} = (x_0, x_1, \ldots, x_s)$ in the Lie geometry, we obtain a corresponding incidence chain $\text{Cox}(\mathbf{x}) = (\text{Cox}(x_0), \text{Cox}(x_1), \ldots, \text{Cox}(x_s))$ in the Weyl geometry. Let us refer to $\text{Cox}(\mathbf{x})$ as the *trace chain* of $\mathbf{x}$.

Note that it may well be the case that $\text{Cox}(x_i) = \text{Cox}(x_{i+2})$ even though $x_i \neq x_{i+2}$. Let us call a chain $\mathbf{y}$ in the Weyl geometry *thin* if $\mathbf{y}$ is the trace chain of a unique chain $\mathbf{x}$ in the Lie geometry. Otherwise, we call $\mathbf{y}$ *thick*. Observe that the trace chain of a key chain is always thin.

8.5 Orbital recognition scheme #2

Recall that for $1 \leq i, j \leq n$, the (i, j)-orbital key $\text{Key}_{ij}(G)$ is comprised of representative types $\mathbf{t} \in \text{Data}(\mathcal{O})$ taken over all orbitals $\mathcal{O}$ in $\gamma_i(G) \times \gamma_j(G)$ that are not p-residual. Further recall that all chains in $\text{Data}(\mathcal{O})$ may be completely characterized by trace chains in $\gamma(W)$. We are now prepared to exhibit an orbital recognition scheme based purely on these trace chains.

Starting with an ordered pair (a, b) of distinct objects $a, b \in \gamma(G)$, we follow the ordered sequence of steps indicated below:

1. Identify i and j such that $(a, b) \in \gamma_i(G) \times \gamma_j(G)$.

2. For each $\mathbf{t} \in \text{Key}_{ij}(G)$ create a list $\mathcal{L}(\mathbf{t})$ of all chains of type $\mathbf{t}$ that start at $\alpha = \text{Cox}(a)$ and end at $\beta = \text{Cox}(b)$.

3. If $\mathcal{L}(\mathbf{t})$ contains a thick trace chain, proceed to the next representative in $\text{Key}_{ij}(G)$. Otherwise, check $\mathcal{L}(\mathbf{t})$ for the existence of a thin trace chain.

4. If $\mathcal{L}(\mathbf{t})$ contains a thin trace chain then $(a, b) \in \mathcal{O}$, where $\mathbf{t}$ is the representative in $\text{Key}_{ij}(G)$ chosen from $\text{Data}(\mathcal{O})$. Otherwise, (a, b) is contained in the p-residual orbital in $\gamma_i(G) \times \gamma_j(G)$.

We shall soon see that steps 3 and 4 may be framed in the context of systems of equations.

9 Embedding geometric actions in the Lie algebra

Let $G = G_n(q)$ be a group of Lie type with corresponding Lie algebra $\mathfrak{L}$. Recall that our aim is to embed $\gamma(G)$ in the upper Borel subalgebra $\mathfrak{L}^U$ of $\mathfrak{L}$. From our previous discussions, this embedding will allow all geometric actions to be formulated inside of $\mathfrak{L}^U$.

We provide below a fairly complete picture of the embedding procedure, however the reader who wishes to acquire a deeper understanding should consult the articles [13, 20, 21, 22]. (See, also, [24] where this is accomplished using an alternate notation.)

9.1 The embedding procedure

Let G be a group of Lie type with fixed unipotent subgroup U, Borel subgroup $B = N_G(U)$, Weyl group W, and Lie algebra $\mathfrak{L}$. Recall that the Schubert cells C_α, where $\alpha \in \gamma(W)$, are the orbits of the permutation group $(B, \gamma(G))$.

It turns out that for each $\alpha \in \gamma(W)$, there is a unique subgroup U_α of U that acts regularly on C_α, see [7]. This allows us to identify $\gamma(G)$ with the set $\{(\alpha, \mathfrak{a}) \mid \alpha \in \gamma(W),\ \mathfrak{a} \in U_\alpha\}$. In light of this, our task of embedding $\gamma(G)$ in $\mathfrak{L}^U$ is reduced to two essential parts: embedding $\gamma(W)$ in $\mathfrak{L}^U$, and similarly embedding U_α in $\mathfrak{L}^U$ for each $\alpha \in \gamma(W)$.

Regarding the first part, it is in fact the case that $\gamma(W)$ actually embeds in the Cartan subalgebra $\mathfrak{H} \subset \mathfrak{L}^U$. We give here only a brief account of the details, referring the interested reader to [12].

Consider the root lattice L, that is, the discrete lattice generated by the roots in a fundamental basis Π. It is well known that L resides in the vector space $\mathfrak{H}^*$, that is, the dual space of the Cartan subalgebra $\mathfrak{H}$. Thus the dual root lattice L^* is contained in $\mathfrak{H}^{**} = \mathfrak{H}$. The Weyl group W acts on L^* via the contragredient action $w \cdot x^* = (w^{-1} \cdot x)^*$. With respect to this action, each orbit of (W, L^*) corresponds to the cosets of a unique parabolic subgroup of W. Conversely, since the dual basis Π^* resides in L^*, the coset spaces $(W : W_i)$ correspond to the orbits $W \cdot r_i^*$, where $r_i \in \Pi$. The embedding $\gamma(W) \hookrightarrow \mathfrak{H}$ is thereby demonstrated.

As for the second part, it turns out that each U_α actually embeds in the positive root space $\mathfrak{L}^+ \subset \mathfrak{L}^U$. To see this we first introduce a special family of subgroups of the unipotent group U.

For each root $r \in \Phi$, consider the one-parameter subgroup of U defined by $X_r = \{x_r(t) \mid t \in GF(q)\}$, where $x_r(t) = \exp(t\,\mathrm{ad}(e_r))$. Here e_r is the root vector corresponding to the root r, and ad: $\mathfrak{L} \to End(\mathfrak{L})$ is the adjoint map from $\mathfrak{L}$ into its endomorphism group, see [7].

We call X_r a *root subgroup* of G. Note that for every root $r \in \Phi$, we have $x_r(t_1)x_r(t_2) = x_r(t_1 + t_2)$. Thus each root subgroup is isomorphic to the additive group of the field $GF(q)$ over which G is defined.

The set of positive roots Φ^+ may be naturally ordered with respect to a Hasse diagram, e.g., see [15]. Let us assume this ordering to be $\Phi^+ = \{r_1, r_2, \ldots, r_N\}$. Then we may factorize U as a product of positive root subgroups:

$$U = \prod_{i=1}^{N} X_{r_i}(t) = X_{r_1}(t_1)X_{r_2}(t_2)\cdots X_{r_N}(t_N)\,.$$

This product is moreover a "standard factorization", meaning that each element $u \in U$ has a unique representation of the form

$$u = x_{r_1}(t_1)x_{r_2}(t_2)\cdots x_{r_N}(t_N), \quad t_i \in GF(q), \ \ 1 \le i \le N\,.$$

Thus we see that each $u \in U$ is uniquely determined by the ordered N-tuple $(t_1, t_2, \ldots, t_N) \in GF(q)^N$. But this N-tuple may be envisioned as a vector in the positive root space $\mathfrak{L}^+$, namely $t_u = \sum_{i=1}^N t_i e_{r_i}$. The identification of u with t_u embeds the entire unipotent subgroup U in $\mathfrak{L}^+$. In particular, the subgroups U_α are so embedded for every $\alpha \in \gamma(W)$.

In fact, there is a very nice and useful description of precisely how the subgroups U_α embed. For any vector sum $t = \sum_{i=1}^N t_i e_{r_i}$, we define the *support* of t to be $\mathrm{Supp}(t) = \{e_{r_i} \mid t_i \neq 0\}$. Now for each $\alpha \in \gamma(W)$, we define the set $\eta^-(\alpha) := \{e_{r_i} \mid \alpha(r_i) < 0\}$. (Here we are using the fact that since α embeds in $\mathfrak{H}$, it acts as a linear functional on Φ^+.) Then elements of the embedded subgroup U_α in $\mathfrak{L}^+$ are precisely those t_u for which $\mathrm{Supp}(t_u) \subseteq \eta^-(\alpha)$.

By uniting these two parts we arrive at the desired embedding of $\gamma(G)$ in $\mathfrak{L}^U = \mathfrak{H} \oplus \mathfrak{L}^+$. Note that since we may now interpret the ordered pair $(\alpha, \mathfrak{a})$ as an element of $\mathfrak{H} \oplus \mathfrak{L}^+$, we may alternatively express it in the form $\alpha + \mathfrak{a} \in \mathfrak{L}^U$. We shall adopt this convention moving forward.

9.2 Type in the embedded geometry

For each dual root r_i^* in Π^* we consider the Weyl orbit $W \cdot r_i^*$. To each object $\alpha \in W \cdot r_i^*$ we assign the type $t(\alpha) = i$. It is straightforward to see that the type function so defined is consistent with the one defined for the pre-embedded Weyl geometry.

Similarly, for each $\alpha \in W \cdot r_i^*$ we consider the Borel orbit $B \cdot \alpha$ (i.e., the Schubert cell C_α). To each $\alpha + \mathfrak{a} \in B \cdot \alpha$, we assign the type $t(\alpha + \mathfrak{a}) = i$. Note that objects in $B \cdot \alpha$ where $\alpha \in W \cdot r_i^*$ correspond to the type i objects of the pre-embedded Lie geometry. Lastly observe that $t(\alpha + \mathfrak{a}) = t(\alpha)$. This reflects the fact that α is the Coxeter trace of $\alpha + \mathfrak{a}$, see Subsec. 8.4.

Example 5. Let $G = A_2(q) \cong PGL(3, q)$. Here $\gamma(G) = (G\!:\!P_1) \cup (G\!:\!P_2)$. In terms of the embedded geometry $\gamma(G) \hookrightarrow \mathfrak{L}^U$, the coset space $(G\!:\!P_i)$ is represented by the union of Borel orbits $\bigcup_\alpha B \cdot \alpha$, where α ranges over the Weyl orbit $W \cdot r_i^*$. $i = 1, 2$. As we have already seen, we can express the elements in the Borel orbit $B \cdot \alpha$ in the form $\alpha + \mathfrak{a}$, where $\alpha \in \mathfrak{H}$ and $\mathfrak{a} \in \mathfrak{L}^+$.

In Table 1 we give an explicit description of all elements $\alpha + \mathfrak{a}$. Note that we indicate the coefficients in each expression by λ (suitably indexed). These coefficients assume values in the field $GF(q)$.

type	α	$\mathfrak{a}$	$\lvert B \cdot \alpha \rvert$
1	r_1^*	0	1
1	$-r_1^* + r_2^*$	$\lambda_{r_1} e_{r_1}$	q
1	$-r_2^*$	$\lambda_{r_2} e_{r_2} + \lambda_{r_1+r_2} e_{r_1+r_2}$	q^2
2	r_2^*	0	1
2	$r_1^* - r_2^*$	$\lambda_{r_2} e_{r_2}$	q
2	$-r_1^*$	$\lambda_{r_1} e_{r_1} + \lambda_{r_1+r_2} e_{2r_1+r_2}$	q^2

Table 1: Objects in the embedded geometry of type $A_2(q)$

Recall that the Lie geometries of type $A_2(q)$ are generalized 3-gons, i.e., classical projective planes of order q. Note that there are exactly q^2+q+1 objects of each type. One may associate type 1 objects with projective points, and type 2 objects with projective lines.

Example 6. Here we consider the group $G = B_2(q)$. As G has rank 2 we again have two types of objects in our geometry: $\gamma(G) = (G\!:\!P_1) \cup (G\!:\!P_2)$, where P_1, P_2 are the maximal parabolic subgroups of G. As in Example 5, we provide a detailed description of all elements $\alpha + \mathfrak{a}$ in the embedded Lie geometry, see Table 2.

type	α	$\mathfrak{a}$	$\lvert B\cdot\alpha\rvert$
1	r_1^*	0	1
1	$-r_1^*+2r_2^*$	$\lambda_{r_1}e_{r_1}$	q
1	$r_1^*-2r_2^*$	$\lambda_{r_2}e_{r_2}+\lambda_{r_1+r_2}e_{r_1+r_2}$	q^2
1	$-r_1^*$	$\lambda_{r_1}e_{r_1}+\lambda_{r_1+r_2}e_{r_1+r_2}+\lambda_{2r_1+r_2}e_{2r_1+r_2}$	q^3
2	r_2^*	0	1
2	$r_1^*-r_2^*$	$\lambda_{r_2}e_{r_2}$	q
2	$-r_1^*+r_2^*$	$\lambda_{r_1}e_{r_1}+\lambda_{2r_1+r_2}e_{2r_1+r_2}$	q^2
2	$-r_2^*$	$\lambda_{r_2}e_{r_2}+\lambda_{r_1+r_2}e_{r_1+r_2}+\lambda_{2r_1+r_2}e_{2r_1+r_2}$	q^3

Table 2: Objects in the embedded geometry of type $B_2(q)$

In this case $\gamma(G)$ is a generalized 4-gon. Note that there are q^3+q^2+q+1 objects of each type. We may associate the type 1 objects with points of the 4-gon, and type 2 objects with lines.

9.3 Incidence in the embedded geometries

Now that we understand that every object in $\gamma(G)$ may be identified with an element $\alpha+\mathfrak{a} \in \mathfrak{L}^U$, we need to address the issue of incidence. Clearly we require that $\alpha+\mathfrak{a}$ and $\beta+\mathfrak{b}$ be incident in $\mathfrak{L}^U$ if and only if they represent an incident pair of objects in $\gamma(G)$.

We first observe that since α and β are elements in $\mathfrak{H}$, they act on the root system Φ by linear extension of the action of the standard dual basis Π^* on Π: $r_i^*(r_j) = \delta_{ij}$, $\forall\, r_i, r_j \in \Pi$. This allows us to define incidence in the embedded Weyl geometry as follows.

Weyl incidence: $\alpha\,\mathcal{I}_{\mathcal{W}}\,\beta$ *provided* $\alpha(r)\beta(r) \geq 0$ *for all* $r \in \Phi^+$.

Let now $\alpha+\mathfrak{a}$ and $\beta+\mathfrak{b}$ be two objects of the embedded Lie geometry. Then, as is well known, the Lie product $\mathfrak{c} = [\alpha+\mathfrak{a}, \beta+\mathfrak{b}]$ resides in $\mathfrak{L}^+$. Let us define $\widehat{\mathfrak{c}}$ to be the projection of $\mathfrak{c}$ onto the subspace of $\mathfrak{L}^+$ spanned by $\mathrm{Supp}(\mathfrak{a}) \bigcap \mathrm{Supp}(\mathfrak{b})$. We may now formulate incidence in the Lie geometry as follows.

Lie incidence: $(\alpha+\mathfrak{a})\,\mathcal{I}_{\mathcal{G}}\,(\beta+\mathfrak{b})$ *provided* $\alpha\,\mathcal{I}_{\mathcal{W}}\beta$ *and* $\widehat{\mathfrak{c}} = 0$.

We colloquially refer to the requirement $\widehat{\mathfrak{c}} = 0$ as "vanishing of the Lie product". It is precisely here where the proposed systems of equations arise.

Theorem 6 *Incidence, as defined above for the embedded geometries, coincides with incidence in the (pre-embedded) geometries.*

Proof. A full verification may be found in [20]. □

Example 7. We revisit Example 5 for the purpose of examining incidence in the embedded Lie geometry of type A_2.

We first provide an incidence matrix for the embedded Weyl geometry, see Table 3. Note that the rows are indexed by objects of type 1 and the columns by objects of type 2.

	r_2^*	$r_1^* - r_2^*$	$-r_1^*$
r_1^*	1	1	0
$-r_1^* + r_2^*$	1	0	1
$-r_2^*$	0	1	1

Table 3: Incidence matrix for the embedded Weyl geometry of type A_2

As an example, let us demonstrate how one comes to the determination that $r_1^* \, \mathcal{I}_{\mathcal{W}} \, (r_1^* - r_2^*)$. Here the set of positive roots is given by $\Phi^+ = \{r_1, \, r_2, \, r_1 + r_2\}$. We easily compute

$$r_1^*(r_1) = 1, \;\; r_1^*(r_2) = 0, \;\; r_1^*(r_1 + r_2) = 1$$

$$(r_1^* - r_2^*)(r_1) = 1, \;\; (r_1^* - r_2^*)(r_2) = -1, \;\; (r_1^* - r_2^*)(r_1 + r_2) = 1$$

Thus by inspection, $r_1^*(r) \, (r_1^* - r_2^*)(r) \geq 0$ for all $r \in \Phi^+$ whence r_1^* and $r_1^* - r_2^*$ are incident in $\gamma(W)$. As an example of contrary nature, note that r_1^* is not incident to $-r_1^*$ since $r_1^*(r_1)(-r_1^*)(r_1) = -1$.

Regarding incidence in the embedded Lie geometry, we again provide the incidence matrix (Table 4), as well as one detailed example.

	r_2^*	$r_1^* - r_2^* + \gamma_{r_2} e_{r_2}$	$-r_1^* + \gamma_{r_1} e_{r_1} + \gamma_{r_1+r_2} e_{r_1+r_2}$
r_1^*	1	1	0
$-r_1^* + r_2^* + \lambda_{r_1} e_{r_1}$	1	0	δ_{ab}
$-r_2^* + \lambda_{r_2} e_{r_2} + \lambda_{r_1+r_2} e_{r_1+r_2}$	0	δ_{cd}	δ_{ef}

Table 4: Incidence matrix for the embedded Lie geometry of type A_2.

In Table 4, each of δ_{ab}, δ_{cd} and δ_{ef} is the kronecker delta function, where $a = 2\lambda_{r_1}$, $b = 3\gamma_{r_1}$; $c = 2\lambda_{r_2}$, $d = 3\gamma_{r_2}$; and $e = \lambda_{r_1+r_2}$, $f = \gamma_{r_1+r_2} + \lambda_{r_1}\gamma_{r_2}$.

It is fairly straightforward to verify that there are q^2+q+1 objects of each type, and that each object of type 1 is incident to exactly $q+1$ objects of type 2, and conversely. This is to be expected, since the Lie geometry is the projective plane $PG(2,q)$.

Finally, we illustrate how one determines the entry δ_{ef} in Table 4. So let us consider arbitrary elements $\alpha+\mathfrak{a} = -r_2^* + \lambda_{r_2}e_{r_2} + \lambda_{r_1+r_2}e_{r_1+r_2}$ and $\beta+\mathfrak{b} = -r_1^* + \gamma_{r_1}e_{r_1} + \gamma_{r_1+r_2}e_{r_1+r_2}$, where $\lambda_{r_2}, \lambda_{r_1+r_2}, \gamma_{r_1}, \gamma_{r_1+r_2} \in GF(q)$.

From Table 3 we see that $(-r_2^*)\,\mathcal{I}_{\mathcal{W}}\,(-r_1^*)$, so we proceed to the condition involving vanishing of the Lie product. Using bilinearity and skew-symmetry, together with the facts that $[\mathfrak{H},\mathfrak{H}] = [e_{r_1}, e_{r_1+r_2}] = [e_{r_2}, e_{r_1+r_2}] = [e_{r_1+r_2}, e_{r_1+r_2}] = 0$, we may reduce the computation of $[\alpha+\mathfrak{a}, \beta+\mathfrak{b}]$ to

$$-\gamma_{r_1}\,[r_2^*, e_{r_1}] - \gamma_{r_1+r_2}\,[r_2^*, e_{r_1+r_2}] + \lambda_{r_2}\,[r_1^*, e_{r_2}] - \lambda_{r_2}\,\gamma_{r_1}\,[e_{r_1}, e_{r_2}] + \gamma_{r_1+r_2}[r_1^*, e_{r_1+r_2}]\,.$$

The surviving Lie products may be calculated from the matrix representation provided in Example 2, namely

$$[r_2^*, e_{r_1}] = -e_{r_1},\ [r_i^*, e_{r_1+r_2}] = e_{r_1+r_2},\ [r_1^*, e_{r_2}] = -e_{r_2},\ [e_{r_1}, e_{r_2}] = e_{r_1+r_2}\,.$$

Thus $\mathfrak{c} = [\alpha+\mathfrak{a}, \beta+\mathfrak{b}] = \gamma_{r_1}e_{r_1} - \lambda_{r_2}e_{r_2} + (\lambda_{r_1+r_2} - \gamma_{r_1+r_2} - \lambda_{r_2}\gamma_{r_1})e_{r_1+r_2}$ whence $\widehat{\mathfrak{c}} = (\lambda_{r_1+r_2} - \gamma_{r_1+r_2} - \lambda_{r_2}\gamma_{r_1})e_{r_1+r_2}$. Setting $\widehat{\mathfrak{c}} = 0$ we obtain the incidence relation $\lambda_{r_1+r_2} = \gamma_{r_1+r_2} + \lambda_{r_2}\gamma_{r_1}$ which coincides with the information conveyed by entry δ_{ef} in Table 4. □

We customarily express this last incidence relation in Example 7 as

$$\lambda_{r_1+r_2} - \gamma_{r_1+r_2} = \lambda_{r_2}\gamma_{r_1}$$

to emphasize the fact that each element of the form $-r_1^* + \gamma_{r_1}e_{r_1} + \gamma_{r_1+r_2}e_{r_1+r_2}$ is incident to a *unique* element of the form $-r_2^* + \lambda_{r_2}e_{r_2} + \lambda_{r_1+r_2}e_{r_1+r_2}$ once a choice for the value of $\lambda_{r_2} \in GF(q)$ is confirmed. In fact, this property is satisfied for any incident pair of objects in any Lie gometry. When one extends this methodology from incident pairs to incidence chains, this is when systems of equations of this general type emerge.

10 Invariant chains in classical Chevalley groups

In Section 8 we saw that each orbital $\mathcal{O}$ of a coherent configuration of Chevalley type could be characterized by $\mathbf{t} = t(\mathbf{x}) \in \mathrm{Data}(\mathcal{O})$, where $\mathbf{x} \in \mathrm{KC}(\alpha,\beta)$ with (α,β) a standard pair in $\mathcal{O}$. Thus given any pair $(\eta,\xi) \in \gamma_i(G) \times \gamma_j(G)$, to determine the orbital to which (η,ξ) belongs it suffices to compare the types $t(\mathbf{y})$ of chains $\mathbf{y}$ connecting η to ξ with those from a specified list, i.e., the orbital key $\mathrm{Key}_{ij}(G)$. Moreover, we came to realize that these lists could be formulated purely in terms of the Weyl geometry, that is, T-invariant chains where T is a maximal torus in the fixed Borel subgroup. We referred to these as trace chains. Finally, as a consequence of Lemmas 1 and 2, we noted that the objects in each considered chain are additionally invariant under the action of the appropriate double-point stabilizer in the Weyl group W.

Thus, it behooves us to determine the relevant double-point stabilizers and their corresponding sets of invariant objects in each considered case. In this section we accomplish this for the Chevalley groups of classical type A_n, B_n, C_n, D_n. The exceptional cases G_2 and F_4 are fairly straightforward due to their small rank, while descriptions for the remaining exceptional cases, E_6, E_7, E_8, may be obtained via computer.

Such a description greatly simplifies the task of formulating orbital characterization and recognition schemes. In Section 11 we do exactly this for the classical Chevalley group of type A_n providing all relevant details.

10.1 The case A_n, $n \geq 1$

Recall that the Weyl group W of type A_n is isomorphic to the symmetric group S_{n+1}. Under this isomorphism we may identify the fundamental reflections $w_i \in W$ with the transpositions $(i, i+1)$ in S_{n+1}, $1 \leq i \leq n$. Recall that the maximal parabolic subgroups of W are given by $W_i = \langle w_j \mid j \in I,\, j \neq i \rangle$. Thus W_i is isomorphic to the subgroup $S_i \times S_{n+1-i}$ of S_{n+1}.

We may identify the geometry $\gamma(W)$ with the set $\mathcal{S} = \mathcal{S}(I)$ of proper nonempty subsets of $I = \{1, 2, \ldots, n+1\}$. Under this identification, $A \in \mathcal{S}$ has type $t(A) = |A|$, and distinct objects $A, B \in \mathcal{S}$ are incident if either $A \subseteq B$ or $B \subseteq A$.

The orbital of $(W, \gamma(W))$ that contains (A, B) is uniquely determined by the triple $(|A|, |B|, |A \cap B|)$. Thus, for each $1 \leq i, j \leq n$ the orbitals in $\gamma_i(G) \times \gamma_j(G)$ take the form $\mathcal{O}_k = \big\{(A, B) \mid |A| = i,\, |B| = j,\, |A \cap B| = k\big\}$.

It is easy to see that double-point stabilizer $W_{A,B}$ of $A, B \in \mathcal{S}$ is given by

$$W_{A,B} = \mathrm{Sym}(X) \times \mathrm{Sym}(A - X) \times \mathrm{Sym}(B - X) \times \mathrm{Sym}(I - Y),$$

where $X = A \cap B$ and $Y = A \cup B$. The invariant objects of $\gamma(W)$ under the action of $W_{A,B}$ are then the elements of the Boolean algebra with basis $\{X,\ A - X,\ B - X,\ I - Y\}$. From this we obtain the following description of $\mathrm{KC}(A, B)$.

Lemma 7 *Let A and B be distinct nonempty proper subsets of I. Then* $\mathrm{KC}(A, B)$ *coincides with exactly one of the following:*

(i) $\big\{(A, B)\big\}$ *if A, B are incident.*

(ii) $\big\{(A, X, B)\big\}$ *if A, B are nonincident, $X \neq \emptyset$, $Y = I$.*

(iii) $\big\{(A, Y, B)\big\}$ *if A, B are nonincident, $X = \emptyset$, $Y \neq I$.*

(iv) $\big\{(A, X, B), (A, Y, B)\big\}$ *if A, B are nonincident, $X \neq \emptyset$, $Y \neq I$.*

(v) $\emptyset$ *if A and B are nonincident, $X = \emptyset$, $Y = I$.*

Proof. If A and B are incident then clearly (i) obtains. So assume otherwise. By Lemma 1, all key chains in $\mathrm{KC}(A,B)$ must be fixed by the double-point stabilizer $W_{A,B}$. This implies that the only objects that can occur in such chains are precisely the invariant objects described in terms of the basis for the corresponding Boolean algebra. The result now follows from our previous description of all such objects. □

Example 8. We once again return to $PG(2,q)$, which is the geometry of the group $G = A_2(q) \cong PGL(3,q)$. We recall the two types of objects in $\gamma(G)$: points (type 1) and lines (type 2). The larger flag geometry additionally includes point-line incident pairs. We refer to point-line non-incident pairs as antiflags.

Let us frame our example in the context of the embedded Lie geometry. Thus, objects of $\gamma(G)$ assume the form $\alpha+\mathfrak{a} \in \mathfrak{L}^U$, where $\mathfrak{L}^U$ is the upper Borel subalgebra of $\mathfrak{L} = \mathfrak{L}(G)$. Recall that the Coxeter trace of $\alpha+\mathfrak{a}$ is given by $\mathrm{Cox}(\alpha+\mathfrak{a}) = \alpha$, so we may identify the sets A and B in Lemma 7 with α and β. Let's assume $t(\alpha) = 1$ and $t(\beta) = 2$.

Here $\gamma(G) = \gamma_1(G) \cup \gamma_2(G)$. The geometric action $(G, \gamma(G))$ provides a coherent configuration with eight orbitals. Two of these orbitals, Δ_1 and Δ_2, arise from fibers and correspond to type 1 and type 2 objects. Since G is doubly-transitive on points and lines of $PG(2,q)$, another two orbitals take the form $\Theta_1 = (\gamma_1(G) \times \gamma_1(G)) - \Delta_1$ and $\Theta_2 = (\gamma_2(G) \times \gamma_2(G)) - \Delta_2$. Since G is transitive on flags and antiflags, two of the four remaining orbitals partition $\gamma_1(G) \times \gamma_2(G)$, viz.

$$\mathcal{F} = \{(\alpha+\mathfrak{a}, \beta+\mathfrak{b}) \mid \{\alpha,\beta\} \text{ is a flag in } \gamma(W)\},$$

$$\mathcal{A} = \{(\alpha+\mathfrak{a}, \beta+\mathfrak{b}) \mid \{\alpha,\beta\} \text{ is an antiflag in } \gamma(W)\}.$$

That leaves as the two remaining orbitals the transposes $\mathcal{F}^{\,t}$ and $\mathcal{A}^{\,t}$ which partition $\gamma_2(G) \times \gamma_1(G)$.

Observe that the mergings $\Delta_1 \cup \Delta_2$, $\Theta_1 \cup \Theta_2$, $\mathcal{F} \cup \mathcal{F}^{\,t}$, $\mathcal{A} \cup \mathcal{A}^{\,t}$ yield a symmetric rank 4 fusion scheme. These mergings are induced by a polarity of the projective plane. We also note that the restriction of the geometric action $(G, \gamma(G))$ to either $\gamma_1(G)$ or $\gamma_2(G)$ results in a rank 2 association scheme (as a coherent sub-configuration).

Let us now better fit this example to the context of the present section. We start by setting $\alpha = \{1\}$, $\alpha' = \{2\}$, $\beta = \{1,2\}$, $\beta' = \{2,3\}$. This allows us to express the standard pair in each orbital as follows:

$$(\alpha,\alpha) \in \Delta_1, \quad (\beta,\beta) \in \Delta_2, \quad (\alpha,\alpha') \in \Theta_1, \quad (\beta,\beta') \in \Theta_2,$$
$$(\alpha,\beta) \in \mathcal{F}, \quad (\alpha,\beta') \in \mathcal{A}, \quad (\beta,\alpha) \in \mathcal{F}^{\,t}, \quad (\beta',\alpha) \in \mathcal{A}^{\,t}.$$

We next identify which specific case in Lemma 7 corresponds to each of these orbitals (excluding those arising from fibers):

$$\Theta_1 \leftrightarrow \text{(iii)}, \quad \Theta_2 \leftrightarrow \text{(ii)}, \quad \mathcal{F} \leftrightarrow \text{(i)}, \quad \mathcal{A} \leftrightarrow \text{(v)}, \quad \mathcal{F}^{\,t} \leftrightarrow \text{(i)}, \quad \mathcal{A}^{\,t} \leftrightarrow \text{(v)}.$$

We may now provide an explicit list of key chains in terms of $\alpha, \alpha', \beta, \beta'$:

$$\mathrm{KC}(\alpha,\alpha') = \{(\alpha,\beta,\alpha')\}, \ \mathrm{KC}(\beta,\beta') = \{(\beta,\alpha',\beta')\},$$
$$\mathrm{KC}(\alpha,\beta) = \{(\alpha,\beta)\}, \ \mathrm{KC}(\alpha,\beta') = \emptyset,$$
$$\mathrm{KC}(\beta,\alpha) = \{(\beta,\alpha)\}, \ \mathrm{KC}(\beta',\alpha) = \emptyset.$$

In terms of the embedded Weyl geometry (cf. Table 3) this list becomes:

$$\mathrm{KC}(\alpha,\alpha') = \big\{(r_1^*,\, r_2^*,\, -r_1^* + r_2^*)\big\},\ \mathrm{KC}(\beta,\beta') = \big\{(r_2^*,\, -r_1^* + r_2^*,\, -r_1^*)\big\},$$
$$\mathrm{KC}(\alpha,\beta) = \big\{(r_1^*, r_2^*)\big\},\ \mathrm{KC}(\alpha,\beta') = \emptyset,$$
$$\mathrm{KC}(\beta,\alpha) = \big\{(r_2^*, r_1^*)\big\},\ \mathrm{KC}(\beta',\alpha) = \emptyset.$$

The orbital keys are now formed as follows:

$$\mathrm{Key}_{11}(G) = \{(1,2,1)\},\quad \mathrm{Key}_{22}(G) = \{(2,1,2)\},$$
$$\mathrm{Key}_{12}(G) = \{(1,2)\},\quad \mathrm{Key}_{21}(G) = \{(2,1)\}.$$

Note that for every i, j there is exactly one p-residual orbital occurring in $\gamma_i(G) \times \gamma_j(G)$.

Given any pair of objects $\alpha + \mathfrak{a}, \beta + \mathfrak{b}$ in the embedded Lie geometry, one can use Table 4 to investigate $\mathrm{KC}(\alpha + \mathfrak{a}, \beta + \mathfrak{b})$. Incidences in chains correspond to solutions of linear equations of the kind witnessed in Example 7. If such a system of equations has a unique solution, then the trace of the investigated chain is thin. From here all that remains is a direct comparison of chain types using the orbital keys indicated above. □

10.2 The case B_n, $n \geq 2$

Here the Weyl group may be interpreted as a group of affine transformations of an n-dimensional vector space $V_n(2)$ over the field $GF(2)$. We explain this directly.

First observe that if we remove the rightmost node n from the Dynkin diagram of type B_n, we obtain the diagram of type A_{n-1}. This implies that the maximal parabolic subgroup W_n of W is isomorphic to the Weyl group of type A_{n-1}, that is, the symmetric group S_n. As such, we may identify the fundamental reflections $w_i \in W$, $i < n$, with the involutions $(i, i+1)$.

The parabolic W_n acts on $V_n(2)$ via permuation of coordinates of vectors. Now define the mapping $\tau : \mathbf{v} \mapsto \mathbf{v} + (0, \ldots, 0, 1)$. The W_n-conjugates of τ are the mappings $\tau_i : \mathbf{v} \mapsto \mathbf{v} + \mathbf{e}_i$, where $\mathbf{e}_i$ is the i-th standard unit vector in $V_n(2)$. Thus W_n normalizes $\langle\tau_1\rangle \times \langle\tau_2\rangle \times \cdots \times \langle\tau_n\rangle$ which gives W the structure of a wreath product, viz. $W = \langle w_n\rangle \wr W_n \cong S_2 \wr S_n$, where we have identified the fundamental reflection w_n with τ. Here S_n acts on $\langle\tau_1\rangle \times \langle\tau_2\rangle \times \cdots \times \langle\tau_n\rangle$ by permuting direct factors.

This identification enables us to determine the structure of all maximal parabolic subgroups W_i of W for $1 \leq i \leq n-1$. Namely, $W_i \cong S_i \times (S_2 \wr S_{n-i})$.

The Weyl geometry of type B_n may now be described as follows. We identify $\gamma(W)$ with the set $\mathcal{S} = \big\{[A,\varphi] \mid A \subseteq I,\ A \neq \emptyset,\ \varphi : A \to GF(2)\big\}$ where $I = \{1, 2, \ldots, n\}$. The type of $[A,\varphi]$ is defined to be $t\big([A,\varphi]\big) = |A|$. The objects $[A,\varphi], [B,\psi] \in \mathcal{S}$ are incident provided the following hold:

(i) $A \subseteq B$ or $B \subseteq A$,

(ii) $\varphi(x) = \psi(x)$ for all $x \in A \cap B$.

The action of the Weyl group W on $\gamma(W)$ is given as follows:

For $(\mathbf{v};\sigma) \in S_2 \wr S_n$ and $[A,\varphi] \in \mathcal{S}$, we have $(\mathbf{v};\sigma)\cdot[A,\varphi] = [B,\psi]$ where $\sigma\cdot A = B$ and $\psi(x) = \varphi(\sigma^{-1}\cdot x) + \mathbf{v}|_B \pmod 2$.

The orbital of $(W, \gamma(W))$ that contains $\big([A,\varphi],[B,\psi]\big)$ is uniquely determined by the 4-tuple $\big(|A|, |B|, r, s\big)$, where $r = |A \cap B|$ and s is the cardinality of the set $C_{(A,\varphi;B,\psi)} = \big\{x \in A \cap B \mid \varphi(x) = \psi(x)\big\}$. Thus, for each $1 \le i, j \le n$ the orbitals in $\gamma_i(G) \times \gamma_j(G)$ take the form

$$\mathcal{O}_{rs} = \big\{\big([A,\varphi],\, [B,\psi]\big)\big|\,|A| = i, |B| = j,\, |A\cap B| = r,\, |C_{(A,\varphi;B,\psi)}| = s\big\}.$$

Let us set $X = A \cap B$ and $C = C_{(A,\varphi;B,\psi)}$ in the above. It will be convenient to assume that φ is constant on each of the sets $C,\ X - C,\ A - X$, and similarly that ψ is constant on $C,\ X - C,\ B - X$. Indeed, such choices do not effect the orbital $\mathcal{O}_{rs}$, and they allow a very nice description of the double-point stabilizer $W_{[A,\varphi],[B,\psi]}$. Namely, letting $Y = A \cup B$, we have that $W_{[A,\varphi],[B,\psi]}$ is equal to

$$\mathrm{Sym}(C) \times \mathrm{Sym}(X - C) \times \mathrm{Sym}(A - X) \times \mathrm{Sym}(B - X) \times (S_2 \wr \mathrm{Sym}(I - Y))\,.$$

Thus, the invariant objects of $\gamma(W)$ under $W_{[A,\varphi],[B,\psi]}$ are precisely those elements $[D,\eta]$ for which D is an element of the Boolean algebra with basis $\{C,\ X - C,\ A - X,\ B - X\}$ and η is constant on the members of this basis.

10.3 The case C_n, $n \ge 2$

This will be a very brief discussion, since B_n and C_n have the same Coxeter diagram. This means that the procedure for identifying invariant chains in the present case is identical to the procedure just described for B_n.

This is not to suggest that the Lie geometries of $B_n(q)$ and $C_n(q)$ are isomorphic. For $n > 2$, they are not. While there is no distinction between the orbital keys of type B_n and C_n, the respective Lie algebras in which the geometries embed have different rules of Lie multiplication.

10.4 The case D_n, $n \ge 4$

As in the case of B_n, the Weyl group W of type D_n may also be considered as a group of affine transformations on $V_n(2)$, only here the transformations are of a more sophisticated nature.

We first observe that the Dynkin diagram for D_n has two rightmost nodes, $n-1$ and n. Removal of either node results in the diagram for A_{n-1}, hence W contains two copies of the symmetric group S_n. Let us denote these as $S_n^{(n-1)}$ and $S_n^{(n)}$ to reflect the node that has been removed. Without loss of generality, we may identify the fundamental reflections that generate $S_n^{(n)}$ as $w_i = (i, i+1)$ for $1 \le i \le n-1$.

Before describing the only remaining fundamental reflection w_n we observe that $S_n^{(n-1)}$ and $S_n^{(n)}$ correspond to the maximal parabolic subgroups W_n and W_{n-1} respectively. Furthermore, removal of node i from the diagram identifies the maximal parabolic W_i up to isomorphism as $S_i \times W(D_{n-i})$ for $1 \le i \le n-3$, and $S_{n-2} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ for $i = n-2$. Here, we denote by $W(D_{n-i})$ the Weyl group of type D_{n-i}, and emphasize that $W(D_3) \cong W(A_3) \cong S_4$.

We now describe the action of W on $V_n(2)$ via affine transformations, and identify the fundamental reflection $w_n \in W$ in the process.

For $\mathbf{v}, \mathbf{b} \in V_n(2)$ and $\sigma \in S_n^{(n)}$, consider all transformations of the form $\theta_{\sigma,\mathbf{b}}$: $\mathbf{v} \mapsto \sigma \cdot \mathbf{v} + \mathbf{b}$, subject to the requirement that $\Sigma_{x \in I} \mathbf{b}(x) = 0 \pmod 2$. These transformations all reside in W, in fact W may be generated by W_{n-1} and the single transformation $\theta_{\sigma,\mathbf{b}}$ with $\sigma = (n-1, n)$ and $\mathbf{b} = \mathbf{e}_{n-1} + \mathbf{e}_n$. Let us identify this latter transformation with w_n. We now see that W is isomorphic to a split extension of $\langle \mathbf{v} \rangle^\perp$ by S_n, where $\langle \mathbf{v} \rangle^\perp$ is the hyperplane in $V_n(2)$ orthogonal to $\mathbf{v} = (1, 1, \ldots, 1)$.

Regarding the objects of the Weyl geometry of type D_n, there is a proper embedding of $\gamma(W(D_n))$ in $\gamma(W(B_n))$, namely

$$\gamma(W(D_n)) = \big\{ [A, \varphi] \in \gamma(W(B_n)) \,\big|\, |A| \neq n-1 \big\}.$$

For objects $[A, \varphi] \in \gamma(W(D_n))$ with $|A| \leq n-2$, the notions of type and incidence agree exactly with those expressed in the case of B_n.

However, the situation is quite different for objects $[I, \varphi] \in \gamma(W(D_n))$, where $I = \{1, 2, \ldots, n\}$. Such objects are of two possible types, depending on the nature of φ. Specifically, we have

$$t([I, \varphi]) = \begin{cases} n-1 & \text{if } \sum_{x \in I} \varphi(x) = 1 \pmod 2 \\ n & \text{if } \sum_{x \in I} \varphi(x) = 0 \pmod 2 \end{cases}$$

Incidence is just as in the Weyl geometry of type B_n provided at most one of the objects under consideration has type $t \geq n-1$. Thus it remains to describe incidence between pairs of objects $[I, \varphi]$, $[I, \psi] \in \gamma(W(D_n))$ where $t([I, \varphi]) = n-1$ and $t([I, \psi]) = n$. Here $[I, \varphi]$ is incident to $[I, \psi]$ precisely when $|C| = n-1$, where $C = C_{I,\varphi;I,\psi}$ is as defined in the B_n case.

We now specify the orbitals in $\gamma_i(G) \times \gamma_j(G)$ for $1 \leq i, j \leq n$. These are most easily described in terms of the orbitals $\mathcal{O}_{rs}$ in the Weyl geometry of type B_n. There are seven cases that arise.

1. $(1 \leq i, j \leq n-2)$ $\quad \mathcal{O}_{rs}$ where $|A| = i$ and $|B| = j$.
2. $(1 \leq i \leq n-2,\ j = n-1)$
   $\mathcal{O}'_{rs} = \big\{ \big([A, \varphi], [B, \psi]\big) \in \mathcal{O}_{rs} \,\big|\, |A| = i,\ t([B, \psi]) = n-1 \big\}$
3. $(1 \leq i \leq n-2,\ j = n)$
   $\mathcal{O}''_{rs} = \big\{ \big([A, \varphi], [B, \psi]\big) \in \mathcal{O}_{rs} \,\big|\, |A| = i,\ t([B, \psi]) = n \big\}$
4. $(i = j = n-1)$
   $\mathcal{O}^{(1)}_{ns} = \big\{ \big([I, \varphi], [I, \psi]\big) \in \mathcal{O}_{ns} \,\big|\, t([I, \varphi]) = t([I, \psi]) = n-1 \big\}$
5. $(i = n-1,\ j = n)$
   $\mathcal{O}^{(2)}_{ns} = \big\{ \big([I, \varphi], [I, \psi]\big) \in \mathcal{O}_{ns} \,\big|\, t([I, \varphi]) = n-1,\ t([I, \psi]) = n \big\}$
6. $(i = n,\ j = n-1)$
   $\mathcal{O}^{(3)}_{ns} = \big\{ \big([I, \varphi], [I, \psi]\big) \in \mathcal{O}_{ns} \,\big|\, t([I, \varphi]) = n,\ t([I, \psi]) = n-1 \big\}$
7. $(i = j = n)$
   $\mathcal{O}^{(4)}_{ns} = \big\{ \big([I, \varphi], [I, \psi]\big) \in \mathcal{O}_{ns} \,\big|\, t([I, \varphi]) = t([I, \psi]) = n \big\}$

The reader will observe that $\mathcal{O}_{rs} = \mathcal{O}'_{rs} \bigcup \mathcal{O}''_{rs}$ and $\mathcal{O}_{ns} = \bigcup_{i=1}^4 \mathcal{O}^{(i)}_{ns}$.

Moreover $\mathcal{O}_{ns}^{(1)} = \mathcal{O}_{ns}^{(4)} = \emptyset$ for $n-s$ even, and $\mathcal{O}_{ns}^{(2)} = \mathcal{O}_{ns}^{(3)} = \emptyset$ for $n-s$ odd.

The double-point stabilizer $W_{[A,\varphi],[B,\psi]}$ of $[A,\varphi]$ and $[B,\psi]$ is given by

$$W_{[A,\varphi],[B,\psi]} = \mathrm{Sym}(C) \times \mathrm{Sym}(X-C) \times \mathrm{Sym}(A-X) \times \mathrm{Sym}(B-X) \times H(Y)$$

where, as usual, we denote $X = A \cap B$ and $Y = A \cup B$. In the above, $H(Y) \cong W(D_{n-|Y|})$ for $|Y| \leq n-3$, $H(Y) \cong Z_2 \times Z_2$ for $|Y| = n-2$, and $H(Y) = 1$ for $|Y| \geq n-1$. As in the B_n case, we assume φ and ψ are constant on the subsets C, $X-C$, $A-X$, $B-X$ on which they are defined.

A description of the invariant objects $[D,\eta] \in \gamma(W)$ under $W_{[A,\varphi],[B,\psi]}$ depends on the cardinality of the set Y, as we now explain.

For $|Y| \neq n-1$, $[D,\eta]$ is invariant precisely when D is an element of the Boolean algebra with basis $\{C,\ X-C,\ A-X,\ B-X\}$ and η is constant on all members of this basis. The same description obtains when $|Y| = n-1$, except that in this case we must include as invariant objects all $[I,\eta] \in \gamma(W)$ for which η is constant on the members of the basis.

11 Explicit characterization and recognition schemes for $A_n(q)$

Let $\alpha + \mathfrak{a}$ and $\beta + \mathfrak{b}$ be elements of the embedded Lie geometry of $A_n(q)$ in $\mathfrak{L}^U$. In this section we provide a recognition scheme for determining the orbital to which $(\alpha + \mathfrak{a}, \beta + \mathfrak{b})$ belongs. An accompanying flowchart is depicted in Fig. 2.

However, prior to orbital recognition we will need an efficient orbital characterization scheme in place. Observe that there are two practical choices for the representative of Data($\mathcal{O}$) corresponding to case (iv) of Lemma 7. At first glance it may seem that either choice violates the disjointness property of key data established in Lemma 3 (ii). But upon closer examination we see this is not the case.

For example, if we choose our orbital representative in case (iv) to be $\mathbf{t} = (i,r,j)$ where $r = |A \cup B|$, then necessarily $i+j < r$. Thus, there is no conflict with case (iii) which can only arise when $i+j = r$. Similarly, if we choose our orbital representative in (iv) to be $\mathbf{t} = (i,k,j)$ where $k = |A \cap B|$, then we must have $i+j < n+1$. This does not conflict with case (ii) which requires that $i+j = n+1$.

Also note that $\gamma_i(G) \times \gamma_j(G)$ contains a residual orbital only when $i+j = n+1$ (case (v) of Lemma 7). Thus for all i,j that satisfy $i+j < n+1$, we are free to choose a pseudo-residual orbital.

Let us base our orbital characterization scheme on the following choices:

(a) $\mathbf{t} = (i,k,j)$, as described above, will be the orbital representative corresponding to case (iv) of Lemma 7 for every i,j,

(b) the orbital corresponding to case (iii) of Lemma 7 will be the pseudo-residual orbital for every i,j for which $i+j < n+1$.

Such choices give us a clear computational advantage. Namely, all chain types of the form $(i,k,j) \in \mathrm{Key}_{ij}(G)$ must satisfy $k < \min(i,j)$.

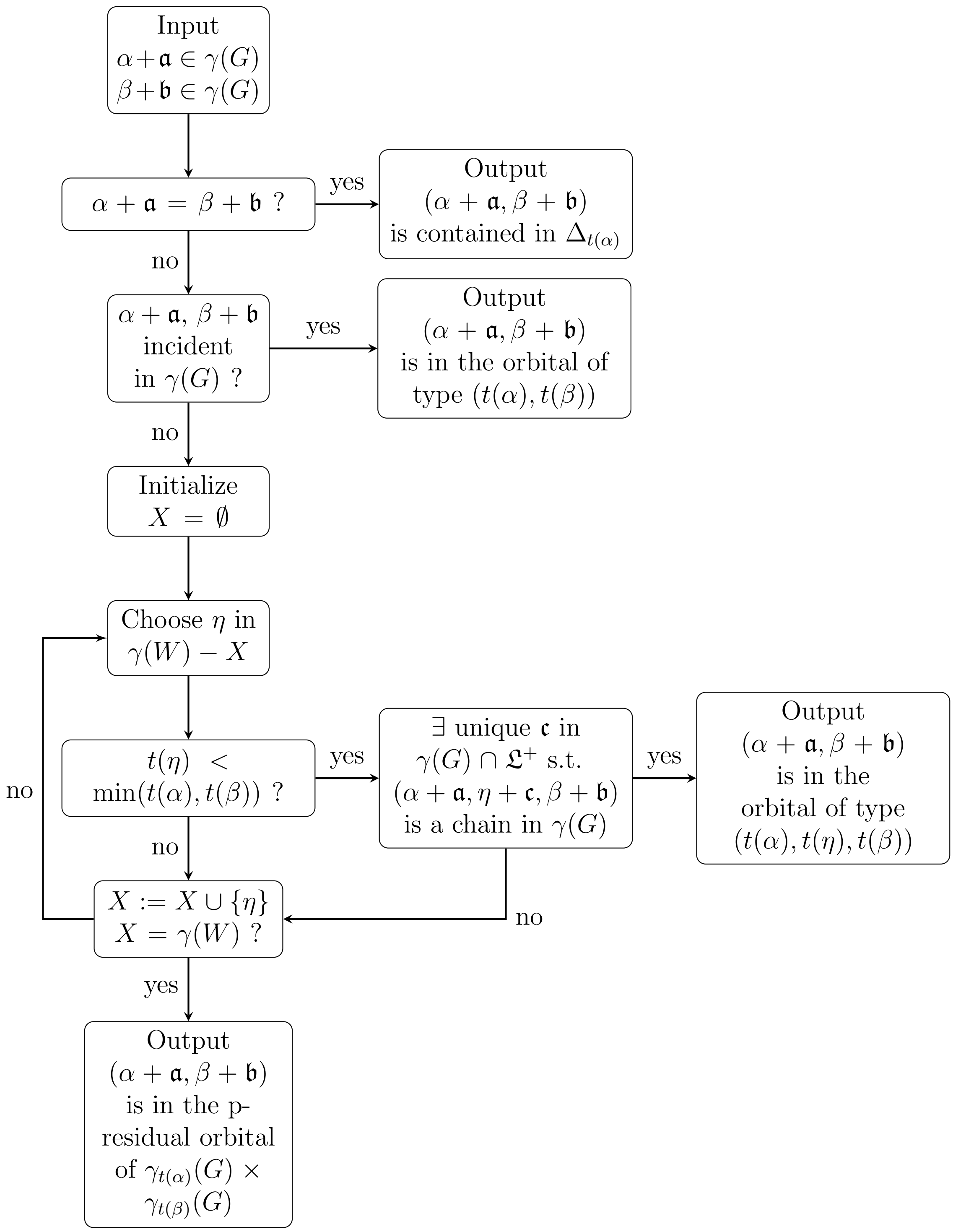

Figure 2: Orbital recognition scheme for the Chevalley groups $A_n(q)$

Theorem 8 *Let G be a classical Chevalley group of type A_n, and let W be its corresponding Weyl group. Let $\alpha + \mathfrak{a}$ and $\beta + \mathfrak{b}$ be elements of the embedded Lie geometry $\gamma(G) \hookrightarrow \mathfrak{L}^U$ with $t(\alpha) = i$ and $t(\beta) = j$. Then exactly one of the following occurs:*

(1) $\alpha + \mathfrak{a} = \beta + \mathfrak{b}$. *Here* $(\alpha + \mathfrak{a}, \beta + \mathfrak{b})$ *is contained in* Δ_i.

(2) $\alpha + \mathfrak{a}$ *and* $\beta + \mathfrak{b}$ *are incident in the Lie geometry. Here* $(\alpha + \mathfrak{a}, \beta + \mathfrak{b})$ *is in the unique orbital* $\mathcal{O}$ *for which* $\mathbf{t} = (i, j) \in \mathrm{Data}(\mathcal{O}) \subseteq \mathrm{Key}_{ij}(G)$.

(3) $\alpha + \mathfrak{a}$ *and* $\beta + \mathfrak{b}$ *are non-incident in the Lie geometry, and there exists an element* η *in the embedded Weyl geometry* $\gamma(W) \hookrightarrow \mathfrak{H}$ *such that:*

  (3.1) $t(\eta) < \min(i, j)$,

  (3.2) the system of linear equations in the variable $\mathfrak{c}$ *arising from the Lie geometric incidences* $(\alpha + \mathfrak{a})\, \mathcal{I}_G\, (\eta + \mathfrak{c})$ *and* $(\eta + \mathfrak{c})\, \mathcal{I}_G\, (\beta + \mathfrak{b})$ *has a unique solution.*

  Here $(\alpha + \mathfrak{a}, \beta + \mathfrak{b})$ *is in the unique orbital* $\mathcal{O} \in \gamma_i(G) \times \gamma_j(G)$ *for which* $\mathbf{t} = (i, t(\eta), j) \in \mathrm{Data}(\mathcal{O}) \subseteq \mathrm{Key}_{ij}(G)$.

(4) $\alpha + \mathfrak{a}$ *and* $\beta + \mathfrak{b}$ *are non-incident in the Lie geometry, and there does not exist* $\eta \in \gamma(W)$ *that fulfills 3. Here* $(\alpha + \mathfrak{a}, \beta + \mathfrak{b})$ *is in the p-residual orbital in* $\gamma_i(G) \times \gamma_j(G)$.

Proof. Clearly (1) and (2) are valid disjoint outcomes, so we may assume that $\alpha + \mathfrak{a}$ and $\beta + \mathfrak{b}$ are distinct non-incident objects in $\gamma(G)$. We claim that all cases in which α and β are distinct and non-incident objects in $\gamma(W)$ follow directly from Lemma 7. Indeed, if case (v) of Lemma 7 obtains then $(\alpha + \mathfrak{a}, \beta + \mathfrak{b})$ belongs to the residual orbital in $\gamma_i(G) \times \gamma_j(G)$, and if case (iii) obtains then $(\alpha + \mathfrak{a}, \beta + \mathfrak{b})$ belongs to the pseudo-residual orbital in $\gamma_i(G) \times \gamma_j(G)$. Note that these are disjoint cases, as they require different values for the sum $i + j$. Both cases result in outcome (4).

Likewise, cases (ii) and (iv) of Lemma 7 are disjoint in the manner just described, and in each case we must eventually encounter an object $\eta \in \gamma(W)$ for which outcome (3) obtains. This is unavoidable, due to the choices we made in formulating our orbital characterization scheme (fulfillment of (3.1)), and the fact that there must exist a chain in $\gamma(G)$ of suitable type whose trace chain is thin (fulfillment of (3.2)).

The remaining cases must be treated with care since Lemma 7 no longer applies to them. These are the cases in which the Coxeter traces α, β are either equal or incident, where we continue to assume that $\alpha + \mathfrak{a}$ and $\beta + \mathfrak{b}$ are distinct non-incident objects in $\gamma(G)$.

We claim that each case corresponds to one that we have previously treated. To prove this we consider the standard pair (η, ξ) in the orbital $\mathcal{O}$ to which $(\alpha + \mathfrak{a}, \beta + \mathfrak{b})$ belongs. Note that this standard pair will vary with each considered case, because when $\alpha = \beta$ we have $\mathcal{O} \subset \gamma_i(G) \times \gamma_i(G)$, whereas when α, β are incident we have $\mathcal{O} \subset \gamma_i(G) \times \gamma_j(G)$ where $i \neq j$.

We first observe that η must be non-incident to ξ since otherwise $\alpha + \mathfrak{a}$ would be incident to $\beta + \mathfrak{b}$, a contradiction. Also η and ξ must be distinct since otherwise $\mathcal{O}$ embeds in the diagonal Δ_Ω, again a contradiction. As η and ξ are distinct and non-incident, and as the types of chains in $\mathrm{KC}(\alpha + \mathfrak{a}, \beta + \mathfrak{b})$ coincide precisely with the types of chains in $\mathrm{KC}(\eta, \xi)$, we see that these two cases have already been treated. Indeed, each such case aligns with either outcome (3) or outcome (4). □

Remark. The authors have devised a similar orbital recognition scheme for the remaining classical Chevalley groups (curiously, the same recognition scheme works for B_n, C_n, D_n alike), but it is a bit too unwieldy to present here. They have also worked out explicit orbital characterizations for the twisted groups of Lie type, however the underlying theory involves an added layer of complication that would cause the present paper to exceed reasonable length. This is why we have chosen to restrict our attention to geometric actions of Chevalley groups. A treatment of orbital characterization and recognition for the twisted groups of Lie type will appear in a future article.

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