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Coloring of locally planar graphs with one color class small

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In memory of Professor Dan Archdeacon

Abstract

In this paper, we prove the following: for any orientable surface $\mathbb{S}_g$ of genus $g > 0$ and any $\varepsilon > 0$, there exists an integer $R = R(g, \varepsilon)$ such that:

(i) every graph G on $\mathbb{S}_g$ with representativity at least R has a 5-coloring such that one color class has cardinality at most $\varepsilon|V(G)|$;

(ii) every even-sided map G on $\mathbb{S}_g$ with representativity at least R has a 3-coloring such that one color class has cardinality at most $\varepsilon|V(G)|$; and

(iii) every even triangulation G on $\mathbb{S}_g$ with representativity at least R has a 4-coloring such that one color class has cardinality at most $\varepsilon|V(G)|$.

We also prove that $\varepsilon|V(G)|$ in (ii) and (iii) cannot be replaced with $o(|V(G)|)$.

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1 Introduction

A *surface* is a compact 2-dimensional manifold without boundary, and is known to be homeomorphic to either the orientable surface of genus $g \geq 0$, denoted by $\mathbb{S}_g$, or the nonorientable surface of genus k, demoted by $\mathbb{N}_k$. A simple closed curve γ on a surface $\mathbb{F}$ is *contractible* (respectively, *essential*) if γ does (does not) bound a closed 2-cell on $\mathbb{F}$. A *map* on a surface $\mathbb{F}$ means a fixed embedding of a graph on $\mathbb{F}$, and *essential* and *contractible* cycles of G are defined similarly to those closed curves on $\mathbb{F}$. For a graph G, let $|G|$ denote the number of vertices. A *k-cycle* is a cycle of length k, and it is *even* (respectively, *odd*) if the length is even (respectively, odd). A map G is a *triangulation* (respectively, *quadrangulation*) if each face is bounded by a 3-cycle (respectively, 4-cycle). A triangulation is *even* if each vertex has even degree. For a map G and its vertex v, the *link* of v is the boundary walk of the 2-cell region formed by all faces incident to v in G.

A *k-coloring* of a graph G is a map $c : V(G) \to \{1, 2, \ldots, k\}$ such that for any edge xy of G, $c(x) \neq c(y)$. A graph G is *k-colorable* if G admits a k-coloring. The *chromatic number* of G, denoted $\chi(G)$, is the smallest integer k such that G is k-colorable. A graph G is *k-chromatic* if $\chi(G) = k$.

One of the most famous theorems in topological graph theory is the Four Color Theorem [5], which states that *every planar graph is 4-colorable.* The statement is so simple, but only computer-assisted proofs are known; see also [21]. The work around this problem influenced many results in graph theory.

Heawood [10] pointed out that every map on a surface $\mathbb{F}$ is $H(\mathbb{F})$-colorable, where $H(\mathbb{F})$ is the *Heawood number*

$$H(\mathbb{F}) = \left\lceil \frac{7 + \sqrt{24g(\mathbb{F}) + 1}}{2} \right\rceil,$$

and $g(\mathbb{F})$ is the *Euler genus* of $\mathbb{F}$, which equals $2g$ and k for $\mathbb{S}_g$ and $\mathbb{N}_k$, respectively. In the 1970s, Ringel and Youngs [20] proved that the complete graph with exactly $H(\mathbb{F})$ vertices is embeddable in $\mathbb{F}$, except when $\mathbb{F}$ is the Klein bottle. This result solves the so-called "Map Color Theorem" completely. That is, the estimate of chromatic numbers by the Heawood number is best possible except for the Klein bottle.

Though the map color theorem was solved, Albertson [1] wondered if the Four Color Theorem should be essential for coloring maps on surfaces. That is, he asked whether or not every map on a surface is 4-colorable after deleting a constant number of vertices, as in the following (see also [15, Page 62]).

Conjecture 1 (Albertson's Four color problem) *For any surface $\mathbb{F}$, there exists an integer $N = N(\mathbb{F})$ such that every map on $\mathbb{F}$ is 4-colorable after deleting at most N vertices.*

The *representativity* of a map G on a non-spherical surface $\mathbb{F}$ is the minimum number of crossing points of G and γ, where γ ranges over all essential simple closed curves on $\mathbb{F}$ [23]. (For a map on the sphere, we define its representativity to be the infinity.) Here, we may suppose that G and γ intersect only at vertices, and the vertices in $G \cap \gamma$ attaining the representativity are *representative* of G. A map G is *k-representative* if G has representativity at least k. We say that a *locally planar map on a surface $\mathbb{F}$ satisfies property $\mathcal{P}$* if there exists an integer $N(\mathbb{F})$ such that every $N(\mathbb{F})$-representative map on $\mathbb{F}$ satisfies $\mathcal{P}$.

Conjecture 1 is still open even for the torus. Now we give the following conjecture, which is a restatement of Albertson's problem:

Conjecture 2 *For any surface $\mathbb{F}$, there exists a pair of integers $N = N(\mathbb{F})$ and $R = R(\mathbb{F})$ such that every R-representative map on $\mathbb{F}$ is 4-colorable after deleting at most N vertices.*

Here we explain that Conjecture 2 is indeed a restatement of Conjecture 1. We use induction on the genus of surfaces. For the sphere, the two statements are equivalent since the Four Color Theorem holds and spherical maps have the representativity infinity. Assume that the assertion of Conjecture 1 is true for surfaces with lower genus. Consider a map G on a surface $\mathbb{F}$ with representativity r. If $r \geq R(\mathbb{F})$, then directly applying the assertion of Conjecture 2, we can find a vertex set $S \subseteq V(G)$ with $|S| \leq N(\mathbb{F})$ and $\chi(G - S) \leq 4$. On the other hand, if $r < R(\mathbb{F})$, then removing the set T of the r representative vertices from G, we get a map G' on a surface $\mathbb{F}'$ of genus lower than $\mathbb{F}$. By induction hypothesis, G' has a vertex set $S' \subseteq V(G')$ with $|S'| \leq N(\mathbb{F}')$ and $\chi(G' - S') \leq 4$, and hence we have a vertex set $S = S' \cup T$ with $|S| \leq R(\mathbb{F}) + N(\mathbb{F}')$ and $\chi(G - S) = \chi(G' - S') \leq 4$.

Conjecture 2 is still open, but in this paper we prove the following result, focusing on a 5-coloring of maps with one color class small.

Theorem 3 *For any orientable surface $\mathbb{S}_g$ of genus $g > 0$ and any positive number ε, there exists an integer $R = R(g, \varepsilon)$ such that if G is an R-representative map on $\mathbb{S}_g$, then G admits a 5-coloring such that one color class has at most $\varepsilon|G|$ vertices.*

Thomassen [24] proved that every locally planar map on any surface is 5-colorable, where "5" is best possible. That is, any non-spherical surface admits non-4-colorable maps with arbitrarily large representativity. Hence Theorem 3 improves Thomassen's 5-color theorem with respect to the size of one color class. We discuss optimality of the condition $\varepsilon|G|$ in Section 3. Moreover, Theorem 3 also gives a result on a large independent set in a locally planar map as follows, which was shown in [3]. To obtain the corollary, take a largest class of four color classes of $G - S$ as an independent set, where S is a color class of a 5-coloring of G with $|S| \leq \varepsilon|G|$ in Theorem 3.

Corollary 4 *For any orientable surface $\mathbb{S}_g$ of genus $g > 0$ and any positive number ε, there exists an integer $R = R(g, \varepsilon)$ such that every R-representative map on $\mathbb{S}_g$ has an independent set S with $|S| \geq \frac{1-\varepsilon}{4}|G|$.*

Let us consider an analogy of Theorem 3 for quadrangulations and even triangulations on surfaces. For those two classes of maps on surfaces, the following is folklore:

Proposition 5 *(i) Every quadrangulation on the plane is 2-colorable.*

(ii) Every even triangulation on the plane is 3-colorable.

Hutchinson [12] proved that every locally planar quadrangulation on any orientable surface is 3-colorable, and Hutchinson, Richter and Seymour [14] proved that every locally planar even triangulation on any orientable surface is 4-colorable, where "3" and "4" are known to be best possible in both classes of maps on orientable surfaces.

We also prove an extension of those results for locally planar quadrangulations and even triangulations with respect to the size of smallest color class:

Theorem 6 *For any orientable surface* $\mathbb{S}_g$ *of genus* $g > 0$ *and any positive number* ε*, there exist integers* $R_2 = R_2(g, \varepsilon)$ *and* $R_3 = R_3(g, \varepsilon)$ *satisfying the following, respectively.*

(i) If G *is an* R_2*-representative quadrangulation on* $\mathbb{S}_g$*, then* G *admits a 3-coloring such that one color class has at most* $\varepsilon|G|$ *vertices.*

(ii) If G *is an* R_3*-representative even triangulation on* $\mathbb{S}_g$*, then* G *admits a 4-coloring such that one color class has at most* $\varepsilon|G|$ *vertices.*

We also prove in Section 3 that the bounds "$\varepsilon|G|$" cannot be replaced with $o(|G|)$ in Theorem 6 (i) and (ii). Similarly to Corollary 4, we have the following for large independent sets.

Corollary 7 *For any orientable surface* $\mathbb{S}_g$ *of genus* $g > 0$ *and any positive number* ε*, there exist integers* $R_2 = R_2(g, \varepsilon)$ *and* $R_3 = R_3(g, \varepsilon)$ *satisfying the following, respectively.*

(i) If G *is an* R_2*-representative quadrangulation on* $\mathbb{S}_g$*, then* G *has an independent set with size at least* $\frac{1-\varepsilon}{2}|G|$ *vertices.*

(ii) If G *is an* R_3*-representative even triangulation on* $\mathbb{S}_g$*, then* G *has an independent set with size at least* $\frac{1-\varepsilon}{3}|G|$ *vertices.*

2 Proof of theorems

The proofs of our theorems follow the combination of the standard methods, which were used in several papers, for example, [2, 4, 12, 14, 16, 18].

2.1 Preliminary

We first introduce an important tool for dealing with locally planar maps on surfaces. Let K and G be two maps on the same surface $\mathbb{F}$. We say that K is a *surface minor* of G if K is a map on $\mathbb{F}$ obtained from G by a sequence of contractions and deletions of edges on $\mathbb{F}$.

Lemma 8 *(Robertson and Seymour [22]) For any map K on a non-spherical surface $\mathbb{F}$, there exists an integer $R = R(\mathbb{F}, K)$ such that every R-representative map on $\mathbb{F}$ has a surface minor of K, up to homeomorphism.*

Let $[C, C']$ denote an *annulus triangulation*, that is, a triangulation on the annulus with disjoint boundary cycles C and C'. If $[C, C']$ is an annulus triangulation, then C and C' are homotopic. Similarly, we can define an *annulus quadrangulation*. Let $(C, C']$ be the map obtained from $[C, C']$ by removing all vertices of C, and let (C, C') be the map obtained from $[C, C']$ by removing all vertices of C and C'.

For the orientable surface $\mathbb{S}_g$ of genus $g > 0$, it is known that there are $2g$ simple closed curves $a_1, b_1, a_2, \ldots, a_g, b_g$ on $\mathbb{S}_g$ such that for $i = 1, \ldots, g$, a_i and b_i cross exactly once transversely, and each of a_i and b_i crosses no other a_j and b_j with $i \neq j$. See Figure 1 for an example. We call the set $\{a_1, b_1, \ldots, a_g, b_g\}$ *canonical generators of the fundamental group of* $\mathbb{S}_g$. This will play an essential role for the proofs of our main theorems.

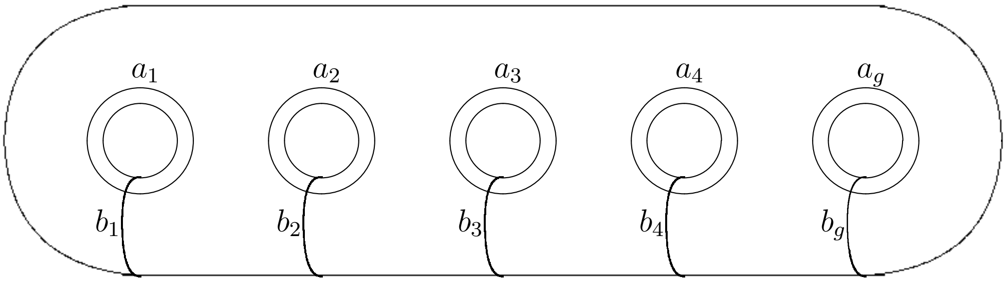

Figure 1: Canonical generators of the fundamental group of $\mathbb{S}_g$.

For a cycle C in a graph G, a *chord* of C is an edge in G that is not an edge of C but the two ends are contained in C. If C does not have a chord, then C is *chordless*. Also a cycle C in a graph G is *nice* if either C has even length or C contains a vertex of degree exactly 4 in G.

For a triangulation G on a surface $\mathbb{F}$, the induced subgraph H of G is *orderly* if it satisfies the following two conditions:

(i) every contractible 3-cycle in H bounds a face of G, and

(ii) every contractible 4-cycle in H is either the boundary of two triangular faces of G sharing an edge, or the link of a vertex of degree exactly 4 in G.

For a graph G and $U \subseteq G$, we denote by $N(U)$ the set of vertices that are not in U but are adjacent to at least one vertex in U. Inductively, we define $N^{i+1}(U)$ for $i \geq 1$ as the set of vertices not in $N^i(U) \cup N^{i-1}(U)$ but adjacent to at least one vertex in $N^i(U)$, where $N^0(U) = U$ and $N^1(U) = N(U)$. For a 2-sided cycle C in a map G on a surface $\mathbb{F}$, we define $R(C)$ and $L(C)$ as the *right* and *left* neighbors of C, respectively.

So, $N(U) = R(U) \cup L(U)$. Furthermore, we define $R^i(C)$ and $L^i(C)$, similarly to $N^i(C)$. Let G_C be the subgraph of G induced by $V(C) \cup N(C) \cup \cdots \cup N^4(C)$.

The following is the Nice Cycle Lemma proved by Albertson and Hutchinson [2]; see also [4].

Lemma 9 (Nice Cycle Lemma [2]) *Let G be a 15-representative triangulation on a non-spherical surface $\mathbb{F}$ and let C be a chordless essential nonseparating cycle in G. If G_C is orderly, then G_C admits a chordless nice cycle homotopic to C.*

2.2 Proof of Theorem 3

In this subsection, we prove Theorem 3, using a similar idea to the one in [2].

Proof of Theorem 3. We first prepare g pairwise disjoint simple *non-homologous* closed curves $a_1, \ldots, a_g$ on $\mathbb{S}_g$, i.e. no subset of them is surface-separating. See the simple closed curves $a_1, \ldots, a_g$ as in Figure 1 for example. Let $\ell = \lceil \frac{1}{\varepsilon} \rceil$, and let K be a map on $\mathbb{S}_g$ such that for each simple closed curve a_i, there are $11\ell + 2$ pairwise disjoint homotopic cycles, and that all the $(11\ell+2)g$ cycles are pairwise disjoint in K. By Lemma 8, there exists an integer $R' = R'(\mathbb{S}_g, K)$ such that any R'-representative map G on $\mathbb{S}_g$ has K as a surface minor. Let $R = \max\{R', 15\}$, where we note that R depends only on g and ε.

Let G be an R-representative map on $\mathbb{S}_g$, and we prove that G admits a 5-coloring c with the condition desired in Theorem 3. We may assume that G is a triangulation on $\mathbb{S}_g$.

Since G has K as a surface minor, G admits $(11\ell + 2)g$ pairwise disjoint cycles $C_1^0, \ldots, C_1^{11\ell+1}, C_2^0, \ldots, C_2^{11\ell+1}, \ldots, C_g^0, \ldots, C_g^{11\ell+1}$ such that $C_i^0, \ldots, C_i^{11\ell+1}$ are all homotopic to a_i on $\mathbb{S}_g$ for $i = 1, \ldots, g$. We may assume that those cycles $C_i^0, \ldots, C_i^{11\ell+1}$ appear on the annulus in this order. Consider the $11\ell g$ cycles of them, avoiding $C_1^0, C_1^{11\ell+1}$, $C_2^0, \ldots, C_g^{11\ell+1}$. If some cycle, say C_i^j with $1 \leq i \leq g$ and $1 \leq j \leq 11\ell$, of the $11\ell g$ cycles has a chord $e = xy$, then C_i^j can be bypassed by using e and we obtain shorter cycles homotopic to C_i^j, since $e \cup P$ or $e \cup P'$ bounds a disk on $\mathbb{S}_g$ (because of the cycle C_i^0 or $C_i^{11\ell+1}$), where $P \cup P' = C_i^j$ and $P \cap P' = \{x, y\}$. Therefore, all of the $11\ell g$ cycles are chordless. Since they are homotopic to a_i for some i, all of them are supposed to be essential and nonseparating. Recall that for $0 \leq j \leq \ell - 1$, $\left[C_i^{11j+1}, C_i^{11j+11}\right]$ denotes the annulus triangulation between the cycles C_i^{11j+1} and C_i^{11j+11}. (When $\mathbb{F}$ is the torus, we have two choices for such annulus triangulations, but we choose the one containing all cycles $C_i^{11j+2}, \ldots, C_i^{11j+10}$.) It is easy to see that for some integer k with $0 \leq k \leq \ell - 1$, we have

$$\sum_{i=1}^{g} \left| \left[C_i^{11k+1}, C_i^{11k+11}\right] \right| \leq \frac{|G|}{\ell} \leq \varepsilon |G|. \tag{1}$$

Thus, it suffices to prove that G has a 5-coloring $c : V(G) \to \{1, 2, 3, 4, 5\}$ such that

$$c^{-1}(5) \subseteq \bigcup_{i=1}^{g} \left[C_i^{11k+1}, C_i^{11k+11}\right].$$

Let i be an integer with $1 \leq i \leq g$. Recall that $G_{C_i^{11k+5}}$ is the subgraph of the annulus triangulation $\left[C_i^{11k+1}, C_i^{11k+9}\right]$ induced by $V(C_i^{11k+5}) \cup \cdots \cup N^4(C_i^{11k+5})$. Then we are going to use Lemma 9 for C_i^{11k+5} after the following modification to satisfy the orderly condition. In fact, we perform the following two operations in $\left[C_i^{11k+1}, C_i^{11k+9}\right]$.

- If $G_{C_i^{11k+5}}$ contains a contractible 3-cycle with interior having vertices of G, then delete all of the vertices in the interior.
- If $G_{C_i^{11k+5}}$ contains a contractible 4-cycle with interior having at least two vertices of G, then replace the interior with one vertex and connect it to all of the four vertices in the contractible 4-cycle.

Let $\widetilde{G}$ be the resulting triangulation by all these possible operations, and let $\widetilde{C}_i^{11k+5}$ be the cycle after the above modification from C_i^{11k+5}. To be exact, if C_i^{11k+5} passes through the interior of a contractible 4-cycle, then we reroute C_i^{11k+5} to pass through the added vertex in the contractible 4-cycle. Note that $\widetilde{G}_{\widetilde{C}_i^{11k+5}}$ is orderly, where $\widetilde{G}_{\widetilde{C}_i^{11k+5}}$ is the subgraph of $\widetilde{G}$ induced by $V(\widetilde{C}_i^{11k+5}) \cup \cdots \cup N^4(\widetilde{C}_i^{11k+5})$. Therefore, by Lemma 9, there exists a chordless cycle, say D_i, in $\widetilde{G}_{\widetilde{C}_i^{11k+5}}$ such that D_i is nice and homotopic to $\widetilde{C}_i^{11k+5}$. Note that $D_i \subseteq \left[C_i^{11k+1}, C_i^{11k+9}\right]$. By the symmetry of the left and the right sides of D_i, we may assume that the cycles $C_i^{11k+1}, \ldots, C_i^{11k+9}$ appear on the annulus triangulation $\left[C_i^{11k+1}, C_i^{11k+9}\right]$ from left to right, and hence $R(D_i) \subseteq \left[C_i^{11k+1}, C_i^{11k+10}\right]$.

We now remove the cycles $D_1, \ldots, D_g$ from $\widetilde{G}$, and then we naturally obtain a map on the sphere, say G_0, with exactly $2g$ boundaries, which correspond to $L(D_1), R(D_1), \ldots, L(D_g)$ and $R(D_g)$, respectively.

Let G_1 be the triangulation on the sphere obtained from G_0 by adding $2g$ new vertices $v_1^L, v_1^R, \ldots, v_g^L, v_g^R$ so that for $i = 1, \ldots, g$ and $X = L, R$, the vertex v_i^X is put on the disk bounded by $X(D_i)$ and joined to all vertices in $X(D_i)$. Then, by the Four Color Theorem, G_1 has a 4-coloring $c_1 : V(G_1) \to \{1, 2, 3, 4\}$.

Now we will bring the cycles $D_1, \ldots, D_g$ back to G_0, and construct a 5-coloring $\widetilde{c}$ of $\widetilde{G}$. Let i be an integer with $1 \leq i \leq g$. We first suppose that $c_1(v_i^L) = c_1(v_i^R)$. By symmetry, say $c_1(v_i^L) = c_1(v_i^R) = 1$. In this case, there are no vertices u in $L(D_i) \cup R(D_i)$ such that $c_1(u) = 1$. If D_i has even length, then we can color the cycle D_i by the colors 1 and 5 alternately; Otherwise, D_i contains a vertex of degree exactly 4 in $\widetilde{G}$, and hence we can color $D_i - u$ by the colors 1 and 5 alternately and then we color the vertex u by a color that does not appear in the neighbors of u.

Suppose next that $c_1(v_i^L) \neq c_1(v_i^R)$. By symmetry, say $c_1(v_i^L) = 1$ and $c_1(v_i^R) = 2$. Then there are no vertices u in $L(D_i)$ with $c_1(u) = 1$ and no vertices u' in $R(D_i)$ with $c_1(u') = 2$. In this case, we recolor those vertices in $(D_i \cup D_i^{R^2}]$ as follows, where $D_i^{R^2}$ is an essential cycle in $R^2(D_i)$ that is homotopic to D_i. Since $R(D_i) \subseteq \left[C_i^{11k+1}, C_i^{11k+10}\right]$, we have $D_i^{R^2} \subseteq \left[C_i^{11k+1}, C_i^{11k+11}\right]$.

First we recolor all vertices in $D_i^{R^2}$ colored by 2 to the color 5. Then we exchange the colors 1 and 2 for all vertices in $(D_i \cup D_i^{R^2}]$. Since, after the first step, there are no vertices in $D_i^{R^2}$ colored by 2, these two steps construct a proper 5-coloring of G_0 such that the colors 1 and 5 do not appear in $L(D_i) \cup R(D_i)$. (Recall that $R(D_i)$ has no vertices u' with $c_1(u') = 2$.) Hence by the same way as in the previous paragraph, we can color the cycle D_i.

Let $\widetilde{c}$ be the 5-coloring of $\widetilde{G}$ obtained by the above procedures for all i. Now we construct a 5-coloring of G by adding all deleted vertices to $\widetilde{G}$. Suppose that $G_{C_i^{11k+5}}$ contains a contractible 3-cycle, say C, with interior having vertices of G for some i. Note that the three vertices in C have all distinct colors by $\widetilde{c}$. Since the interior of C, together with C, forms a plane triangulation, it has a 4-coloring. By changing the colors to meet the colors of C by $\widetilde{c}$, we can extend the coloring $\widetilde{c}$ to the interior of C.

On the other hand, suppose next that $G_{C_i^{11k+5}}$ contains a contractible 4-cycle, say $C = xyzw$, with interior having at least two vertices of G for some i. Note that the quadrilateral region bounded by C has no diagonal xz nor yw, since, for otherwise, we can go to the case for non-facial triangular regions. Depending on the colors of x, y, z and w by $\widetilde{c}$, we have the following three cases.

(1) The vertices x, y, z and w have all distinct colors by $\widetilde{c}$.

(2) The vertices x, y, z and w have three colors by $\widetilde{c}$ in total.

(3) The vertices x, y, z and w have two colors by $\widetilde{c}$ in total.

Let H be the subgraph induced by all vertices in the interior of C, together with x, y, z and w. Note that H is a plane map with all faces triangular, except for the outer quadrilateral face bounded by C. In either case, we show that H has a 5-coloring such that the colors of x, y, z and w coincide with these by $\widetilde{c}$. Note that we only consider the interior of C, which is contained in $\left[C_i^{11k+1}, C_i^{11k+11}\right]$. We use the symmetry between the colors $1, 2, 3, 4$ and 5 in the following arguments.

Case (1) We may assume that the colors of x, y, z, w by $\widetilde{c}$ are $1, 2, 3$ and 4, respectively. Let $H_{(1)}$ be the map obtained from H by adding the edge connecting x and z through the outside of C. Note that $H_{(1)}$ is a plane triangulation. By the Four Color Theorem, $H_{(1)}$ has a 4-coloring $c_{(1)}$, using the colors $1, 2, 3$ and 4. Since x, y and z form a triangle in $H_{(1)}$, we may assume that the colors of them are $1, 2$ and 3, respectively. Note that $c_{(1)}(w) = 2$ or 4, but if $c_{(1)}(w) = 4$, then we are done. So, we may assume that $c_{(1)}(w) = 2$. In this case, we first change the color 4 with 5, and then change the color 2 with 4 except for y. So, y will be the only vertex of color 2. This gives a 5-coloring of H as desired.

Case (2) We may assume that the colors of x, y, z, w by $\widetilde{c}$ are $1, 2, 1$ and 3, respectively. Let $H_{(2)}$ be the map obtained from H by identifying the vertices x and z. Note that $H_{(2)}$ is a plane triangulation, which has no loop since the interior of C has no diagonal. By the Four Color Theorem, $H_{(2)}$ has a 4-coloring $c_{(2)}$ using the colors $1, 2, 3$ and 4. Note that $c_{(2)}$ directly gives a 4-coloring of H with $c_{(2)}(x) = c_{(2)}(z)$. Then by the same way as in Case (1), we obtain a 5-coloring of H as desired.

Case (3) We may assume that the colors of x, y, z, w by $\widetilde{c}$ are $1, 2, 1$ and 2, respectively. Let $H_{(3)}$ be the map obtained from H by identifying the vertices x and z. Note that $H_{(3)}$ is a plane triangulation with no loop. By the Four Color Theorem, $H_{(3)}$ has a 4-coloring $c_{(3)}$ using the colors $1, 2, 3$ and 4. By the symmetry of the colors, we may assume that $c_{(3)}(x) = c_{(3)}(z) = 1$ and $c_{(3)}(y) = 2$. If $c_{(3)}(w) = 2$, then we are done. So, we may also assume that $c_{(3)}(w) = 3$. In this case, change the color 2 with 5, and then put the color 2 to both y and w. So, y and w will be the only vertices of color 2. This gives a 5-coloring of H as desired.

In all cases, H has a 5-coloring such that the colors of the vertices in C coincide with those by $\widetilde{c}$. We can extend the 5-coloring $\widetilde{c}$ of $\widetilde{G}$ to the 5-coloring c of G by repeating the above procedures. Note that

$$c^{-1}(5) \subseteq \bigcup_{i=1}^{g} \left[C_i^{11k+1}, C_i^{11k+11}\right].$$

Hence it follows from the inequality (1) that

$$|c^{-1}(5)| \leq \sum_{i=1}^{g} \left| \left[C_i^{11k+1}, C_i^{11k+11}\right] \right| \leq \varepsilon |G|.$$

This completes the proof. □

2.3 Proof of Theorem 6 (i)

For an even-sided map G on a surface $\mathbb{F}$, the following holds:

Lemma 10 *(Lemma 9 in [19]) Let G be an even-sided map on a surface $\mathbb{F}$. Then two closed walks have the same parity of length if they are homotopic on $\mathbb{F}$.*

Lemma 11 *Let $A = [D_1, D_4]$ be an annulus quadrangulation with a 2-coloring $c_0 : V(A) \to \{1, 2\}$. Suppose that A has four pairwise disjoint homotopic essential cycles D_1, D_2, D_3, D_4 appearing on the annulus in this order. Then A has a 3-coloring $c : V(A) \to \{1, 2, 3\}$ such that*

- *(i) for any $v \in V(D_1)$, $c(v) = c_0(v)$,*
- *(ii) for any $v \in V(D_4)$, $c(v) = 3 - c_0(v)$, and*
- *(iii) if $c(v) = 3$, then v is contained in $(D_1, D_3]$.*

Proof. Let $c_1 : V(A) \to \{1, 2, 3\}$ be the 3-coloring of A such that for any $v \in V(D_1)$, $c_1(v) = c_0(v)$, and for any $v \notin V(D_1)$,

$$c_1(v) = \begin{cases} 1 & \text{if } c_0(v) = 1, \\ 3 & \text{if } c_0(v) = 2. \end{cases}$$

Let $c_2 : V(A) \to \{1,2,3\}$ be the 3-coloring of A such that for any $v \in V([D_1, D_2])$, $c_2(v) = c_1(v)$, and for any $v \notin V([D_1, D_2])$,

$$c_2(v) = \begin{cases} 3 & \text{if } c_1(v) = 3, \\ 2 & \text{if } c_1(v) = 1. \end{cases}$$

Let $c_3 : V(A) \to \{1,2,3\}$ be the 3-coloring of A such that for any $v \in V([D_1, D_3])$, $c_2(v) = c_1(v)$, and for any $v \notin V([D_1, D_3])$,

$$c_3(v) = \begin{cases} 2 & \text{if } c_2(v) = 2, \\ 1 & \text{if } c_2(v) = 3. \end{cases}$$

Then the 3-coloring c_3 of A is a 3-coloring as required. □

We prove Theorem 6 (i).

Proof of Theorem 6 (i). Take canonical generators $\{a_1, b_1, \dots, a_g, b_g\}$ of the fundamental group on $\mathbb{S}_g$. (See Figure 1 for example.) Moreover, for $i = 1, \dots, g$, let c_i be a simple closed curve on $\mathbb{S}_g$ which is homotopic to the concatenation of a_i and b_i.

Let $\ell = \lceil \frac{1}{\varepsilon} \rceil$, and let K be a map on $\mathbb{S}_g$ with a set of $4\ell \times 3g$ essential cycles

$$\bigcup_{i=1}^{g} \mathcal{A}_i \cup \bigcup_{i=1}^{g} \mathcal{B}_i \cup \bigcup_{i=1}^{g} \mathcal{C}_i,$$

where $\mathcal{A}_i = \{A_i^1, \dots, A_i^{4\ell}\}$, $\mathcal{B}_i = \{B_i^1, \dots, B_i^{4\ell}\}$ and $\mathcal{C}_i = \{C_i^1, \dots, C_i^{4\ell}\}$ such that

(i) $A_i^1, \dots, A_i^{4\ell}$ are 4ℓ pairwise disjoint cycles homotopic to a_i,

(ii) $B_i^1, \dots, B_i^{4\ell}$ are 4ℓ pairwise disjoint cycles homotopic to b_i,

(iii) $C_i^1, \dots, C_i^{4\ell}$ are 4ℓ pairwise disjoint cycles homotopic to c_i, and

(iv) for any $D_i \in \mathcal{A}_i \cup \mathcal{B}_i \cup \mathcal{C}_i$ and $D_j \in \mathcal{A}_j \cup \mathcal{B}_j \cup \mathcal{C}_j$ with $i \neq j$, then D_i and D_j are disjoint.

By Lemma 8, there exists an integer $R = R(\mathbb{S}_g, K)$ such that every R-representative map has K as a surface minor, where R depends only on g and ε.

Let G be an R-representative quadrangulation on $\mathbb{S}_g$. Then G has K as a surface minor. Thus, G has a set of $4\ell \times 3g$ pairwise disjoint cycles corresponding to the above $4\ell \times 3g$ cycles of K, for which we denote the cycles in G using the same symbols as those in K.

Here we claim that for each i, at least one of A_i^1, B_i^1 and C_i^1 has even length. If at least one of A_i^1 and B_i^1 contains a cycle of even length, then we are done. Otherwise, i.e., if both A_i^1 and B_i^1 have odd length, then C_i^1 must have even length, since it is homotopic to the concatenation of two odd cycles A_i^1 and B_i^1. By Lemma 10, for each $\mathcal{A}_i$, $\mathcal{B}_i$ and $\mathcal{C}_i$, all members in the set have the same parity of length. Put $\mathcal{D}_i = \{D_i^1, \dots, D_i^{4\ell}\}$ be a set of 4ℓ pairwise disjoint cycles of even length, for

$i = 1, \ldots, g$, where $\mathcal{D}_i$ is one of $\mathcal{A}_i$, $\mathcal{B}_i$ and $\mathcal{C}_i$. We may suppose that $D_i^1, \ldots, D_i^{4\ell}$ lie on the surface in this order.

Recall that $\left[D_i^{4j+1}, D_i^{4j+3}\right]$ denotes the annulus quadrangulation between the cycles D_i^{4j+1} and D_i^{4j+3}. (When $\mathbb{F}$ is the torus, we have two choices for such annulus quadrangulations, but we choose the one containing the cycle D_i^{4j+2}.) Note that the annulus map $\left[D_i^{4j+1}, D_i^{4j+3}\right]$ is bipartite, since it can be regarded as an even-sided map on the sphere. It is easy to see that for some k with $0 \leq k \leq \ell - 1$,

$$\sum_{i=1}^{g} \left| \left[D_i^{4k+1}, D_i^{4k+3}\right] \right| \leq \frac{|G|}{\ell} \leq \varepsilon |G|.$$

Let G_0 be the map on the sphere obtained from G by cutting along D_i^{4k+4} and pasting a disk to the two boundary components corresponding to D_i^{4k+4}, for $i = 1, \ldots, g$. (See [24] for the detail of cutting.) Let ${D'}_i^{4k+4}$ and ${D''}_i^{4k+4}$ denote the facial cycles of G_0 corresponding to D_i^{4k+4} in G, and for each $v \in V\left(D_i^{4k+4}\right)$, let $v' \in V\left({D'}_i^{4k+4}\right)$ and $v'' \in V({D''}_i^{4k+4})$ be the vertices corresponding to v, for $i = 1, \ldots, g$.

Since G_0 can be regarded as an even-sided map on the sphere, G_0 is bipartite and hence has a unique 2-coloring c_0. We construct a desired 3-coloring c of G, modifying the 2-coloring c_0 of G_0. For all vertices $v \in V(G) - \bigcup_{i=1}^{g} \left(D_i^{4k+1}, D_i^{4k+4}\right)$, we let $c(v) = c_0(v)$. Observe that for $i = 1, \ldots, g$, exactly one of the two cases happens:

- $c_0(v') = c_0(v'')$ for each $v \in V\left(D_i^{4k+4}\right)$, or
- $c_0(v') \neq c_0(v'')$ for each $v \in V\left(D_i^{4k+4}\right)$.

In the former, we also let $c(v) = c_0(v)$ for any vertex v of $\left(D_i^{4k+1}, D_i^{4k+4}\right)$. In this case, the third color is not used in $\left(D_i^{4k+1}, D_i^{4k+4}\right)$. On the other hand, in the latter case, introducing the third color, we exchange the two colors in the annulus map $\left[D_i^{4k+1}, D_i^{4k+4}\right]$, by Lemma 11.

In this case, $c^{-1}(3) \subseteq \bigcup_{i=1}^{g} V\left(\left[D_i^{4k+1}, D_i^{4k+3}\right]\right)$. Hence we have

$$|c^{-1}(3)| \leq \sum_{i=1}^{g} \left| \left[D_i^{4k+1}, D_i^{4k+3}\right] \right| \leq \frac{|G|}{\ell} \leq \varepsilon |G|. \qquad \square$$

2.4 Proof of Theorem 6 (ii)

We proceed to even triangulations on surfaces. For dealing with them, we use the following lemma, which allows us to reduce even triangulations G on $\mathbb{S}_g$ to a 3-colorable plane map by cutting G along a set of essential cycles.

Lemma 12 *For any orientable surface $\mathbb{S}_g$ of genus $g > 0$, there exists an integer $R' = R'(g)$ satisfying the following; Let G be an R'-representative even triangulation on $\mathbb{S}_g$, and let $\{a_1, b_1, \ldots, a_g, b_g\}$ be canonical generators of the fundamental group of $\mathbb{S}_g$. (See Figure 1.) Then there exist g pairwise non-homotopic cycles $D_1, \ldots, D_g$ in G satisfying the following three conditions.*

(i) *Each D_i is homotopic to a simple closed curve obtained by the concatenation of at most four simples closed curves in $\{a_1, b_1, \dots, a_g, b_g\}$.*

(ii) *The cycles $D_1, \dots, D_g$ are pairwise disjoint.*

(iii) *Let G_0 be the map obtained from G by cutting along $D_1, \dots, D_g$, and pasting a disk to the $2g$ boundary components corresponding to them. Then G_0 is a 3-colorable plane map.*

This lemma was essentially proved by Hutchinson, Richter and Seymour [14], considering an algebraic invariant for even triangulations, called the "monodromy". (See [11] for more detailed definition.) Therefore, we briefly explain how to modify the proof in [14] for proving Lemma 12.

Let G be an even triangulation on a surface $\mathbb{F}$. Let $W = f_0 f_1 \cdots f_k$ with $f_0 = f_k$ be a sequence of faces of G, called a *closed face walk*, such that f_i and f_{i+1} share an edge, for $i = 0, 1, \dots, k-1$. Let $W^i = f_0 \cdots f_i$ for $i = 0, 1, \dots, k$. (So, $W^k = W$.) Define the bijection $\sigma_{G,W^i,f_0} : V(f_0) \to V(f_i)$ recursively until $i = k$, as follows. For $i = 0$, $\sigma_{G,W^0,f_0} = \text{id}$, where "id" represents the identity map. For $i > 0$, define σ_{G,W^i,f_0} so that σ_{G,W^i,f_0} and σ_{G,W^{i-1},f_0} coincide on $V(f_{i-1}) \cap V(f_i)$. Then σ_{G,W,f_0} determines a unique element in the symmetric group $\mathbb{S}_3$ of degree 3.

It is easy to see that

- if two closed face walks W_1 and W_2 of G containing f are *homotopic* (i.e., the two closed walks W_1^* and W_2^* of the surface dual G^* of G corresponding to W_1 and W_2 respectively are homotopic on $\mathbb{F}$ as simple closed curves), then we have $\sigma_{G,W_1,f} = \sigma_{G,W_2,f}$, and
- if W is *contractible* on $\mathbb{F}$ (i.e., W^* bounds a closed 2-cell on $\mathbb{F}$), then $\sigma_{G,W,f} = \text{id}$.

So, by $\sigma_{G,W,f}$ for each closed face walk W containing f, we can define a homomorphism $\sigma_{G,f} : \pi_1(\mathbb{F}, x) \to \mathbb{S}_3$, called the *monodromy* of G, regarding W^* as an element of the fundamental group $\pi_1(\mathbb{F}, x)$ of $\mathbb{F}$ with base point x, where x is a point on $\mathbb{F}$ corresponding to f^* of G^*.

The proof of the lemma in [14] was done by induction on g. For each step, they cut the map on $\mathbb{S}_g$ along "k-wide handle T with $T \cap X = \emptyset$ and with balanced end-circuits", which means k pairwise disjoint essential homotopic identity-assigned closed face walks. (See [14, Page 235].) This handle T corresponds to a cycle D_i in Lemma 12. Since the set X corresponds to the "holes" obtained by the previous cutting, the condition "$T \cap X = \emptyset$" guarantees condition (ii). In order to find such a "handle with balanced end-circuit", they used the statement (3.4) in [14, Page 232], but the main point on the homotopy type was shown in the statement (2.5) in Page [14, Page 229]. In fact, they prepare three homotopy types α_1, α_2 and α_3, and proved that at least one concatenation of at most four simple closed curves in such three homotopy types is "balanced" (i.e., identity-assigned). This implies condition (i). After cutting the graph along all D_i's, we finally obtain the plane map, which is indeed the map G_0 with condition (iii). Then by the statement (4.2) in [14, p. 233] and the condition of "balanced end-circuits", the map G_0 is 3-colorable, and hence condition (iii) is also satisfied. This proves Lemma 12.

We also use the following lemma, which can be proved similarly to Lemma 11. (We can find a similar idea in the proof of Theorem (4.1) in [14, pp. 236–237].) Therefore, we omit the proof of it.

Lemma 13 *Let $A = [D_1, D_5]$ be an annulus triangulation which has five pairwise disjoint homotopic essential cycles D_1, D_2, D_3, D_4, D_5 lying on the annulus in this order. Suppose that A has a 3-coloring $c_0 : V(A) \to \{1,2,3\}$. Then for any element $s \in \mathbb{S}_3$, A has a 4-coloring $c_s : V(A) \to \{1,2,3,4\}$ such that*

(i) for any $v \in V(D_1)$, $c_s(v) = c_0(v)$,

(ii) for any $v \in V(D_5)$, $c_s(v) = s\big(c_0(v)\big)$, and

(iii) if $c_s(v) = 4$, then v is contained in $(D_1, D_4]$.

Now we are ready to prove Theorem 6(ii).

Proof of Theorem 6(ii). Take canonical generators $\{a_1, b_1, \ldots, a_g, b_g\}$ of the fundamental group on $\mathbb{S}_g$. (See Figure 1 for example.) Let $\ell = \lceil \frac{1}{\varepsilon} \rceil$, and let K be a map on $\mathbb{S}_g$ such that

- for any simple closed curve γ obtained by the concatenation of at most four simple closed curves in $\{a_1, b_1, \ldots, a_g, b_g\}$, K contains 5ℓ pairwise disjoint cycles homotopic to γ, and
- for any two such cycles D_γ and $D_{\gamma'}$ homotopic to γ and γ', respectively, if γ and γ' do not intersect, then D_γ and $D_{\gamma'}$ are disjoint.

So, the first condition requires K to have $5\ell\big\{(2g)^4 + (2g)^3 + (2g)^2 + 2g\big\}$ distinct cycles on $\mathbb{S}_g$. By Lemma 8, there exists an integer $R'' = R''(\mathbb{S}_g, K)$ such that every R''-representative map has K as a surface minor. Let $R = \max\{R', R''\}$, where $R' = R'(g)$ is the integer as in Lemma 12. Note that R depends only on g and ε.

Let G be an R-representative even triangulation on $\mathbb{S}_g$. Then G has K as a surface minor. In particular, G contains g pairwise non-homotopic essential cycles $D_1, \ldots, D_g$ satisfying conditions (i), (ii) and (iii) of Lemma 12. By conditions (i), (ii) and the conditions on K, K contains $5\ell g$ pairwise disjoint cycles $D_1^1, \ldots, D_1^{5\ell}, \ldots, D_g^1, \ldots, D_g^{5\ell}$ such that all of $D_i^1, \ldots, D_i^{5\ell}$ are homotopic to D_i for $i = 1, \ldots, g$.

It is easy to see that for some k with $0 \le k \le \ell - 1$,

$$\sum_{i=1}^{g} \left| \big[D_i^{5k+1}, D_i^{5k+4}\big] \right| \le \frac{|G|}{\ell} \le \varepsilon |G|,$$

where $\big[D_i^{5k+1}, D_i^{5k+4}\big]$ is the annulus triangulation between the cycles D_i^{5k+1} and D_i^{5k+4}. (When $\mathbb{F}$ is the torus, we have two choices for such annulus triangulations, but we choose the one containing the cycles D_i^{5k+2} and D_i^{5k+3}.) Let G_0 be the map on the sphere obtained from G by cutting along D_i^{5k+5} and pasting a disk to the two boundary components corresponding to D_i^{5k+5}, for $i = 1, \dots, g$. Let ${D'}_i^{5k+5}$ and ${D''}_i^{5k+5}$ denote the facial cycles of G_0 corresponding to D_i^{5k+5} in G, and for each $v \in V\big(D_i^{5k+5}\big)$, let $v' \in V\big({D'}_i^{5k+5}\big)$ and $v'' \in V({D''}_i^{5k+5})$ be the two vertices corresponding to v, for $i = 1, \dots, g$. By condition (iii) in Lemma 12, G_0 is 3-colorable, and let c_0 be a 3-coloring of G_0.

We construct a desired 4-coloring c of G, modifying the 3-coloring c_0 of G_0. For all vertices $v \in V(G) - \bigcup_{i=1}^{g} \big(D_i^{5k+1}, D_i^{5k+5}\big)$, we let $c(v) = c_0(v)$. Observe that for $i = 1, \dots, g$, all vertices $v \in V(D_i^{5k+5})$ satisfies

- $c_0(v') = s\big(c_0(v'')\big)$ for some $s \in \mathcal{S}_3$,

since any two homotopic closed face walks of G are assigned the same element in $\mathcal{S}_3$.

So, by using Lemma 13, we can exchange the three colors in the annulus map $\big[D_i^{5k+1}, D_i^{5k+5}\big]$ by introducing the fourth color. Let c be the resulting 4-coloring of G. In this case,

$$|c^{-1}(4)| \le \sum_{i=1}^{g} \Big|\big(D_i^{5k+1}, D_i^{5k+4}\big]\Big| \le \frac{|G|}{\ell} \le \varepsilon|G|,$$

since we use the forth color of c only in $\big(D_i^{5k+1}, D_i^{5k+4}\big]$. □

3 Optimality of the bounds in Theorems 3 and 6

In this paper, we dealt with locally planar maps on orientable surfaces, related to Albertson's 4-color problem (Conjecture 1) and Conjecture 2. We also considered an analogy for locally planar quadrangulations and even triangulations on orientable surfaces.

In order to strengthen Theorem 3, we wonder if the following theorem can be used (see also [3, 7]):

Theorem 14 (Hutchinson and Miller [13]) *Every map G with n vertices on an orientable surface $\mathbb{S}_g$ admits a vertex set $S \subseteq V(G)$ with $|S| = O(\sqrt{gn})$ such that $G - S$ is planar.*

Theorem 14 does not assume locally planarity of maps on surfaces, and hence Theorem 14 gives a better estimate, without using ε, for a vertex set S with $G - S$ 4-colorable. (Indeed, this directly shows the existence of an independent set T in a map G with n vertices on an orientable surface $\mathbb{S}_g$ with $|T| = \frac{n}{4} - O(\sqrt{gn})$, which is an improvement of Corollary 4.) Hence we ask the following as a common extension of Theorems 3 and 14:

Question 15 *Does every locally planar map G with n vertices on an orientable surface $\mathbb{S}_g$ admit a 5-coloring $c : V(G) \to \{1, 2, 3, 4, 5\}$ such that $|c^{-1}(5)| = O(\sqrt{gn})$?*

Furthermore, considering Albertson's 4-color problem (Conjecture 1), we may be able to improve Question 15 so that $|c^{-1}(5)|$ does not depend on n (while it must depend on g).

On the other hand, the following examples show that the bounds "$\varepsilon|G|$" in Theorem 6 (i) and (ii) are best possible in the sense that they cannot be replaced with $o(|G|)$.

First we consider the case for quadrangulations in Theorem 6 (i). Let G' be an r-representative non-bipartite quadrangulation on $\mathbb{S}_g$. Since G' is a non-bipartite quadrangulation, G' has an essential odd cycle C. Let G be a non-bipartite quadrangulation on $\mathbb{S}_g$ obtained from G' by cutting along C, and inserting an annulus quadrangulation $C \square P_m$ between the two boundary components, where $C \square P_m$ denotes the Cartesian product of C and the path P_m with m vertices. Then G is an r-representative quadrangulation on $\mathbb{S}_g$ with m pairwise disjoint odd cycles. Hence, for any 3-coloring of G, each of the m odd cycles contains at least one vertex from each of the three color classes. If we take such an integer $m = \omega(|G'|)$ and regard $|C|$ as a constant, we obtain $m = \frac{|G|-|G'|+|C|}{|C|} = \Theta(|G|)$, and hence the bound "$|c^{-1}(3)| \leq \varepsilon|G|$" in Theorem 6 (i) cannot be replaced with $o(|G|)$.

In a similar way, for any positive integer r, we can construct an r-representative even triangulation G on $\mathbb{S}_g$ with m pairwise disjoint non-3-colorable closed face walks, where $m = \Theta(|G|)$. Observe that each of such closed face walks requires at least one vertex from each of the four color classes, for any 4-coloring of G. This implies that the bound "$|c^{-1}(4)| \leq \varepsilon|G|$" in Theorem 6 (ii) cannot be replaced with $o(|G|)$.

4 Remarks for nonorientable surfaces

Our theorems are only for locally planar maps on orientable surfaces. Let us consider what we can say about those on nonorientable surfaces. For nonorientable surfaces $\mathbb{N}_k$ of genus k, the following are known:

(i) every locally planar map on $\mathbb{N}_k$ is 5-colorable [24],

(ii) every locally planar quadrangulation on $\mathbb{N}_k$ is 4-colorable [6, 9], and

(iii) every locally planar even triangulation on $\mathbb{N}_k$ is 5-colorable [6, 9],

where each of the estimate is best possible. Furthermore, for (ii) and (iii), 4-chromatic quadrangulations and 5-chromatic even triangulations on $\mathbb{N}_k$ were characterized in [6] and [17], respectively. Hence we ask the following:

Question 16 *Does every 3-colorable locally planar quadrangulation on $\mathbb{N}_k$ admit a 3-coloring with one color class small? Does every 4-colorable locally planar even triangulation on $\mathbb{N}_k$ admit a 4-coloring with one color class small?*

Mohar and Seymour [16] proved that a locally planar 4-chromatic quadrangulation G on $\mathbb{N}_k$ is 4-*critical* (i.e., $G - v$ is 3-colorable for any vertex v) if and only if every contractible 4-cycle of G bounds a face. This implies that every locally planar 4-chromatic quadrangulation G on $\mathbb{N}_k$ has a vertex v such that $G - v$ is 3-colorable. Hence we ask the following question:

Question 17 *Does every locally planar quadrangulation* G *on* $\mathbb{N}_k$ *admit a 4-coloring* $c : V(G) \to \{1, 2, 3, 4\}$ *such that* $|c^{-1}(4)| = 1$ *and* $|c^{-1}(3)| = \varepsilon|G|$*?*

As a partial solution, Esperet and Stehlık [8] proved that every quadrangulation G on the projective plane admits a 4-coloring $c : V(G) \to \{1, 2, 3, 4\}$ such that $|c^{-1}(4)| = 1$ and $|c^{-1}(3)| = O(\sqrt{\Delta|G|})$, where Δ is the maximum degree of G.

The first author of the present paper proved that a locally planar even triangulation G on $\mathbb{N}_k$ is 5-chromatic if and only if G is the *face subdivision* of some even-sided map H including a 4-chromatic quadrangulation H' as a subgraph, i.e. G is obtained from H by adding a single vertex to each face of H and joining it to all vertices on the corresponding boundary [17]. Hence G has a vertex v such that $G - v$ is 4-colorable, by choosing v from $V(H)$. Therefore, we finally ask:

Question 18 *Does every locally planar 5-chromatic even triangulation* G *on* $\mathbb{N}_k$ *admit a 5-coloring* $c : V(G) \to \{1, 2, 3, 4, 5\}$ *such that* $|c^{-1}(5)| = 1$ *and* $|c^{-1}(4)| = \varepsilon|G|$*?*

Final Notes

The authors would like to dedicate this paper to Professor Dan Archdeacon to mourn his untimely passing. The first author of this paper had one joint paper with him [6] on chromatic number of quadrangulations. The present paper deals with an extension of the results of the joint paper, and the authors hope that Dan has an interest in it.

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References

[1] M.O. Albertson, Open problem 2, *The Theory and Applications of Graphs*, ed. G. Chartrand et al., Wiley (1981) 609.

[2] M.O. Albertson and J.P. Hutchinson, Extending precolorings of subgraphs of locally planar graphs, *European J. Combin.* **25** (2004), 863–871.

[3] M.O. Albertson and J.P. Hutchinson, On the independence ratio of a graph, *J. Graph Theory* **2** (1978), 1–8.

[4] M.O. Albertson, J.P. Hutchinson and R.B. Richter, Revisiting a Nice Cycle Lemma and its Consequences, preprint in arXiv: https://arxiv.org/abs/1602.06985

[5] K. Appel and W. Haken, Every planar map is four colorable, *Bull. Amer. Math. Soc.* **82** (1976), 449–456.

[6] D. Archdeacon, J.P. Hutchinson, A. Nakamoto, S. Negami and K. Ota, Chromatic numbers of quadrangulations on closed surfaces, *J. Graph Theory* **37** (2001), 100–114.

[7] H.N. Djidjev and S.M. Venkatesan, Planarization of graphs embedded on surfaces, Graph-Theoretic Concepts in Computer Science (LNCS), **1017** (1995), 62–72.

[8] L. Esperet and M. Stehlík, The width of quadrangulations of the projective plane, preprint.

[9] S. Fisk and B. Mohar, Coloring graphs without short non-bounding cycles, *J. Combin. Theory Ser. B* **60** (1994), 268–276.

[10] P.J. Heawood, Map colour theorem, *Quart. J. Math.* **24** (1890), 332–338.

[11] Y. Higuchi, A. Nakamoto, K. Ota and T. Sakuma, N-Flips in even triangulations and Dehn twists on the torus, *Discrete Math.* **311** (2011), 1128–1135.

[12] J.P. Hutchinson, Three-coloring graphs embedded on surfaces with all faces even-sided, *J. Combin. Theory Ser. B* **65** (1995), 139–155.

[13] J.P. Hutchinson and G.L. Miller, On deleting vertices to make a graph of positive genus planar, In: *Discrete Algorithms and Complexity Theory*, Academic Press (1987), 81–98.

[14] J.P. Hutchinson, R.B. Richter and P. Seymour, Coloring Eulerian triangulations, *J. Combin. Theory Ser. B* **84** (2002), 225–239.

[15] T.R. Jensen and B. Toft, *Graph Coloring Problem*, Wiley (1995).

[16] B. Mohar and P.D. Seymour, Coloring locally bipartite graphs on surfaces, *J. Combin. Theory Ser. B* **84** (2002), 301–310.

[17] A. Nakamoto, 5-Chromatic even triangulations on surfaces, *Discrete Math.* **308** (2008), 2571–2580.

[18] A. Nakamoto, S. Negami, and K. Ota, Chromatic numbers and cycle partitions of quadrangulations on nonorientable closed surfaces, *Discrete Math.* **285** (2004), 211–218.

[19] S. Negami and A. Nakamoto, Diagonal transformations in graphs on closed surfaces, *Sci. Rep. Yokohama Nat. Univ., Sec. I* **40** (1993), 71–97.

[20] G. Ringel, *Map Color Theorem*, Springer, 1974.

[21] N. Robertson, D. Sanders, P.D. Seymour and R. Thomas, The four-color theorem, *J. Combin. Theory Ser. B* **70** (1997), 2–44.

[22] N. Robertson and P.D. Seymour, Graph minors. VII. Disjoint paths on a surface, *J. Combin. Theory Ser. B* **45** (1988), 212–254.

[23] N. Robertson and R. Vitray, Representativity of surface embeddings, "Paths, flows, and VLSI-layout" (Bonn, 1988), 293–328, *Algorithms Combin.* **9**, Springer, Berlin, (1990).

[24] C. Thomassen, 5-coloring maps on surfaces, *J. Combin. Theory Ser. B* **59** (1993), 89–105.

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