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A note on cycles of maximum length in bipartite digraphs

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Abstract

Adamus and Adamus in 2012 proposed the following conjecture: Let D be a bipartite digraph with colour classes X and Y such that $|X| = a \leq b = |Y|$. If $d^+(u)+d^-(v) > \frac{a+b+2}{2}$, whenever u and v lie in opposite colour classes and $uv \notin A(D)$, then D contains a cycle of length $2a$. Adamus and Adamus have proved that this conjecture is true when $a = b$. In this paper, we show that this conjecture is true when $b = a + 1$ or $b = a + 2$.

1 Introduction and terminology

We shall assume that the reader is familiar with the standard terminology on digraphs and refer the reader to [4] for terminology not defined here. Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. For a set $S \subset V(D)$, we denote by $N^+(S)$ the set of vertices dominated by the vertices of S, i.e., $N^+(S) = \{v \in V(D) : uv \in A(D)$ for some $u \in S\}$. Similarly, $N^-(S)$ denotes the set of vertices dominating the vertices of S, i.e., $N^-(S) = \{v \in V(D) : vu \in A(D)$ for some $u \in S\}$. If $S = \{u\}$, then the cardinality of $N^+(v)$ (resp. $N^-(v)$), denoted by $d^+(v)$ (resp. $d^-(v)$) is called the *out-degree* (resp. *in-degree*) of v in D. For $u \in V(D)$ and $W \subset V(D)$, we set $N^+_W(u)$ (resp. $N^-_W(u)$) to be the set of vertices of W dominated by (resp. dominating) u, and denote its cardinality by $d^+_W(u)$ (resp. $d^-_W(u)$).

Let $P = y_0y_1 \dots y_k$ and $Q = q_0q_1 \dots q_n$ be two vertex disjoint paths or cycles in D. For $i < j$, $y_i, y_j \in V(P)$ we denote by $P[y_i, y_j]$ the *subpath* of P from y_i to y_j. If there exist $y_i \in V(P)$ and $q_j \in V(Q)$ such that $y_iq_j \in A(D)$, then we will use $P[y_0, y_i]Q[q_j, q_n]$ to denote the path $y_0y_1 \dots y_iq_jq_{j+1} \dots q_n$.

Let D be a *bipartite digraph* with colour classes X and Y. We say that D is *balanced* if $|X| = |Y|$. A *matching* from X to Y is an independent set of arcs with

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origin in X and terminus in Y. When the cardinality of M is $|X|$, we say that M *saturates* X. A path or cycle is said to be *compatible with a matching* M from X to Y (or, M-*compatible*, for short) if its arcs are alternately in M and in $A(D) \setminus M$. For a matching M from X to Y and a vertex $x \in X$, we will denote by $M(x)$ the unique vertex $y \in Y$ such that $xy \in M$; similarly, for a vertex $y \in Y$, we will denote by $M^{-1}(y)$ the unique vertex $x \in X$ such that $xy \in M$.

There are numerous sufficient conditions for existence of cycles in bipartite digraphs (see [1, 2, 3, 6, 7]). In particular, Manoussakis and Milis [6] present a condition based on half-degrees, which suffices for the existence of a cycle of length twice the cardinality of the smaller colour class in a bipartite digraph.

Theorem 1.1. *[6] Let D be a bipartite digraph with colour classes X and Y such that $|X| = a \le b = |Y|$. If*

$$d^+(u) + d^-(v) \ge a + 2,$$

whenever $u, v \in V(D)$ and $uv \notin A(D)$, then D contains a cycle of length $2a$.

Motivated by this theorem, Adamus and Adamus proposed the following conjecture.

Conjecture 1.2. *[1] Let D be a bipartite digraph with colour classes X and Y such that $|X| = a \le b = |Y|$. If*

$$d^+(u) + d^-(v) > \frac{a+b+2}{2}, \tag{1}$$

whenever u and v lie in opposite colour classes and $uv \notin A(D)$, then D contains a cycle of length $2a$.

In Conjecture 1.2, if we write $b = a + k$, then the condition (1) can be rewritten into $d^+(u) + d^-(v) \ge a + 2 + \lfloor \frac{k}{2} \rfloor$. In [1], Adamus and Adamus have proved that when $k = 0$, the conjecture is true. In this paper, we shall show that the conjecture is true when $k = 1$ or $k = 2$.

2 The main result

Similarly to Definition 1.1 in [1], we introduce the following definition.

Definition 2.1. Let D be a bipartite digraph with colour classes X and Y such that $|X| = a \le b = |Y|$. For $n \ge 0$, we say that D satisfies condition A_n^* if and only if $d^+(u) + d^-(v) \ge a + n$, for all u and v from opposite colour classes and $uv \notin A(D)$.

The following easy facts will be very useful in our proofs of the main result. The proof of Lemmas 2.2, 2.4 and 2.5 is similar to the proof of Lemmas 2.1, 2.3, 2.4, respectively, in [1].

Lemma 2.2. *If D satisfies condition A_0^*, then D contains a matching from X to Y, which saturates X.*

Proof. By the well-known König-Hall theorem [5], it suffices to prove that, for every $S \subseteq X$, $|N^+(S)| \geq |S|$. If $|N^+(S)| \geq a$, then there is nothing to show; if not, for any $x \in S$ and $y \in Y \setminus N^+(S)$, then $xy \notin A(D)$. By assumption,

$$a \leq d^+(x) + d^-(y) \leq |N^+(S)| + a - |S|,$$

from which $|N^+(S)| \geq |S|$, as required. □

Let D be a bipartite digraph with colour classes X and Y, where $|X| = a$ and $|Y| = a + k$ for some $k \geq 1$. Throughout the rest of this paper, by a matching M, we shall always mean a matching which saturates X. For Lemmas 2.3-2.5, we choose a matching M and let $P = p_1p_2 \dots p_s$ be a path in D compatible with M, and of maximal length among paths compatible with M. Denote $Q = V(D) \setminus V(P)$, $Q_X = Q \cap X$, $Q_Y = Q \cap Y$, $Q' = \{u \in Q_Y \mid vu \notin M, \text{ for every } v \in Q_X\}$, $P_X = V(P) \cap X$ and $P_Y = V(P) \cap Y$.

Lemma 2.3. *If D satisfies condition A_0^*, then we can choose P such that each of the following holds:*

(a) *If s is even, then $p_1 \in X$ and $p_s \in Y$.*

(b) *If s is odd, then $p_1 \in Y$ and $p_s \in Y$.*

(c) *In the first case, $d^+_{Q_X}(p_s) = d^-_{Q_Y}(p_1) = 0$; in the second case, $d^+_{Q_X}(p_s) = d^-_{Q_Y \setminus Q'}(p_2) = 0$.*

Proof. If $p_s \in X$, then, by the maximality of P, it must be $M(p_s) \in V(P)$. But if s is odd, then it is impossible. Hence $p_s \in Y$. If s is even, then $M(p_s) = p_1$ and Pp_1 is, in fact, a cycle. We can renumber its vertices so that $p_1 \in X$ and hence, $p_s \in Y$.

Now clearly by the maximality of P, we have $d^+_{Q_X}(p_s) = 0$, and if s is even, then $d^-_{Q_Y}(p_1) = 0$. Suppose that s is odd. If $d^-_{Q_Y \setminus Q'}(p_2) > 0$, then there exists $y \in Q_Y \setminus Q'$ such that $yp_2 \in A(D)$. But the path $M^{-1}(y)yp_2 \dots p_s$ is a longer M-compatible path than P, a contradiction. Hence $d^-_{Q_Y \setminus Q'}(p_2) = 0$ □

Lemma 2.4. *Let D satisfy condition A_1^*. If $p_sp_1 \in A(D)$, then D has a cycle of length $2a$ compatible with M.*

Proof. Suppose, on the contrary, that the cycle $p_1p_2 \dots p_sp_1$ is not Hamiltonian. Hence $Q_Y \neq \emptyset$. Let $p_i \in P_X$ and $y \in Q_Y$. If $y \to p_i$, then the path $yP[p_i, p_s]P[p_1, p_{i-1}]$ is strictly longer than P and compatible with M, a contradiction. Thus $yp_i \notin A(D)$ for all $p_i \in P_X$ and all $y \in Q_Y$, and so

$$a + 1 \leq d^+(y) + d^-(p_i) \leq (a - \frac{1}{2}s) + \frac{1}{2}s = a,$$

a contradiction. □

Lemma 2.5. *Let D be a bipartite digraph with colour classes X and Y such that $|X| = a$ and $|Y| = a + k$, where $a \geq 2$ and $k = 1$ or $k = 2$. If $d^+(u) + d^-(v) \geq a + 2 + \lfloor \frac{k}{2} \rfloor$, where u and v lie in opposite colour classes and $uv \notin A(D)$, then D contains a cycle of length at least a, compatible with M.*

Proof. By Lemma 2.3, we may assume that $p_s \in Y$. Suppose that s is even. If $p_s p_1 \in A(D)$, then, by Lemma 2.4, D has a cycle of length $2a$, compatible with M. Next assume that $p_s p_1 \notin A(D)$. By Lemma 2.3(c), $d^-_Q(p_1) = d^+_Q(p_s) = 0$. Therefore, by assumption,

$$a + 2 \leq a + 2 + \lfloor \frac{k}{2} \rfloor \leq d^+(p_s) + d^-(p_1) = d^+_{V(P)}(p_s) + d^-_{V(P)}(p_1),$$

and hence $d^+_{V(P)}(p_s) \geq \frac{a+2}{2}$ or else $d^-_{V(P)}(p_1) \geq \frac{a+2}{2}$. In the first case, let $i_0 = \min\{i : p_s p_i \in A(D)\}$. Then $P[p_{i_0}, p_s]p_{i_0}$ is a cycle of length at least $2d^+_{V(P)}(p_s)$, which is greater than or equal to $a + 2$. In the latter case, the desired cycle is obtained similarly by considering the vertex p_j which dominates p_1 such that j is maximum.

Suppose that s is odd. First assume that $p_s \to p_2$. If $Q_X = \emptyset$, then $P[p_2, p_s]p_2$ is a cycle of length $2a$ and compatible with M. Thus assume that $Q_X \neq \emptyset$, which implies that $Q_Y \setminus Q' \neq \emptyset$. For any $y \in Q_Y \setminus Q'$ and $p_i \in P_X$, we have $y \nrightarrow p_i$, otherwise the path $M^{-1}(y)yP[p_i, p_s]P[p_2, p_{i-1}]$ is strictly longer than P and compatible with M, a contradiction. In particular, $y \nrightarrow p_2$. By assumption,

$$\begin{aligned} a + 2 + \lfloor \frac{k}{2} \rfloor \leq & d^+(y) + d^-(p_2) \leq |Q_X| + (|Q'| + |P_Y|) \\ = & (a - \frac{s-1}{2}) + (k - 1 + \frac{s-1}{2} + 1) = a + k, \end{aligned}$$

a contradiction. Next assume that $p_s \nrightarrow p_2$. By assumption,

$$\begin{aligned} a + 2 + \lfloor \frac{k}{2} \rfloor \leq & d^+(p_s) + d^-(p_2) = d^+_{V(P)}(p_s) + d^-_{V(P)}(p_2) + d^-_{Q'}(p_2) \\ \leq & d^+_{V(P)}(p_s) + d^-_{V(P)}(p_2) + k - 1 \end{aligned}$$

and hence $d^+_{V(P)}(p_s) + d^-_{V(P)}(p_2) \geq a + 3 + \lfloor \frac{k}{2} \rfloor - k \geq a + 2$. Thus, $d^+_{V(P)}(p_s) \geq \frac{a+2}{2}$ or else $d^-_{V(P)}(p_2) \geq \frac{a+2}{2}$. In the first case, we consider the vertex p_i dominated by p_s, such that i is minimum. The cycle $P[p_i, p_s]p_i$ has length at least $2d^+_{V(P)}(p_s)$, which is greater than or equal to $a + 2$. In the latter case, we consider the vertex p_j which dominates p_2 such that j is maximum. The cycle $P[p_2, p_j]p_2$ has length at least $2(d^-_{V(P)}(p_2) - 1)$, which is greater than or equal to a. □

The following theorem is our main result.

Theorem 2.6. *Let D be a bipartite digraph with colour classes X and Y such that $|X| = a$ and $|Y| = a + k$, where $a \geq 2$ and $k = 1$ or $k = 2$. If*

$$d^+(u) + d^-(v) \geq a + 2 + \lfloor \frac{k}{2} \rfloor,$$

whenever u and v lie in opposite colour classes and $uv \notin A(D)$, then D contains a cycle of length $2a$.

Proof. By Lemma 2.2, D contains a matching from X to Y, which saturates X and by Lemma 2.5, D contains a cycle of length at least a, compatible with some matching in D. Choose M a matching from X to Y, and a cycle C, of length $2m$, compatible with M in such a way that C is of maximal length among all the cycles in D compatible with some matching from X to Y. Write $C = x_1y_1 \dots x_my_m$, with $x_\nu \in X$ and $y_\nu \in Y$ for $1 \le \nu \le m$. By Lemma 2.5, we have $2m \ge a$.

We want to show that $m = a$. Suppose otherwise. Then we can choose a path P, of order $s \ge 2$, contained in $D \backslash V(C)$, compatible with M and of maximal length among such paths in $D \backslash V(C)$. Write $P = p_1p_2 \dots p_s$ and denote $\lfloor \frac{s}{2} \rfloor = p$. Let R denote the remaining vertices of D, i.e., $R = V(D) \backslash (V(C) \cup V(P))$. Also, we define that $R_X = X \cap R$, $R_Y = Y \cap R$, and $R' = \{u \in R_Y \mid vu \notin M, \text{for all } v \in X\}$. Denote $|R_X| = r$. If s is even, then $|V(P)| = s = 2p$, $|R'| = k$ and $|R_Y| = r + k$; If s is odd, then $|V(P)| = s = 2p+1$, $|R'| = k-1$ and $|R_Y| = r+k-1$. Therefore, $a = m+p+r$ and $2p + 2r = 2a - 2m \le a$.

Since P is a maximal path in $D \backslash V(C)$ compatible with M, similar to the proof of Lemma 2.3, then we can easily prove that the following hold for the path P:

(i) If s is even, then $d^+_R(p_s) = 0$ and $d^-_R(p_1) = 0$;

(ii) If s is odd, then $d^+_R(p_s) = 0$ and $d^-_{R/R'}(p_2) = 0$.

To complete the proof, we now consider the following two cases.

Case 1. s is even.

Subcase 1.1. $d^+_{V(C)}(p_s) = 0$ and $d^-_{V(C)}(p_1) > 0$.

Let then $y_i \in V(C)$ be such that $y_ip_1 \in A(D)$. It follows from the maximality of C that $d^-_{V(P)}(x_{i+1}) = 0$. In particular, $p_sx_{i+1} \notin A(D)$, and hence

$$\begin{aligned} a + 2 + \lfloor \frac{k}{2} \rfloor \le& d^+(p_s) + d^-(x_{i+1}) = d^+_{V(P)}(p_s) + (d^-_{V(C)}(x_{i+1}) + d^-_R(x_{i+1})) \\ \le& p + (m + r + k) = a + k, \end{aligned}$$

a contradiction.

Subcase 1.2. $d^+_{V(C)}(p_s) = 0$ and $d^-_{V(C)}(p_1) = 0$.

If $p_sp_1 \notin A(D)$, then, by assumption,

$$a + 2 + \lfloor \frac{k}{2} \rfloor \le d^+(p_s) + d^-(p_1) = d^+_{V(P)}(p_s) + d^-_{V(P)}(p_1) \le 2(p-1) < a,$$

a contradiction. Therefore $p_sp_1 \in A(D)$, and so P is a cycle. Hence $d^-_{V(R)}(p_i) = 0$ and $d^+_{V(R)}(p_j) = 0$, for all $p_i \in P_X$ and all $p_j \in P_Y$ by the maximality of P. Suppose now that $d^+_{V(C)}(p_j) = 0$ for a $p_j \in P_Y$, we have

$$\begin{aligned} a + 2 + \lfloor \frac{k}{2} \rfloor \le& d^+(p_j) + d^-(x_1) = d^+_{V(P)}(p_j) + (d^-_{V(C)}(x_1) + d^-_R(x_1)) \\ \le& p + (m + r + k) = a + k, \end{aligned}$$

a contradiction. Hence, there exist $x_i \in V(C) \cap X$ and $p_j \in P_Y$ such that $p_j x_i \in A(D)$. It follows from the maximality of C that $d^+_{V(P)}(y_{i-1}) = 0$. In particular, $y_{i-1}p_1 \notin A(D)$, and hence

$$\begin{aligned} a + 2 + \lfloor \frac{k}{2} \rfloor \le & d^+(y_{i-1}) + d^-(p_1) = (d^+_{V(C)}(y_{i-1}) + d^+_R(y_{i-1})) + d^-_{V(P)}(p_1) \\ \le & (m + r) + p = a, \end{aligned}$$

a contradiction.

Subcase 1.3. $d^+_{V(C)}(p_s) > 0$ and $d^-_{V(C)}(p_1) = 0$.

Let then $x_i \in V(C)$ be such that $p_s x_i \in A(D)$. It follows from the maximality of C that $d^+_{V(P)}(y_{i-1}) = 0$. In particular, $y_{i-1}p_1 \notin A(D)$, and hence

$$\begin{aligned} a + 2 + \lfloor \frac{k}{2} \rfloor \le & d^+(y_{i-1}) + d^-(p_1) = (d^+_{V(C)}(y_{i-1}) + d^+_R(y_{i-1})) + d^-_{V(P)}(p_1) \\ \le & (m + r) + p = a, \end{aligned}$$

a contradiction.

Subcase 1.4. $d^+_{V(C)}(p_s) > 0$ and $d^-_{V(C)}(p_1) > 0$.

There exist x_{j_0} and y_{i_0} on C such that $y_{i_0}p_1 \in A(D)$ and $p_s x_{j_0} \in A(D)$. Without loss of generality, assume that $j_0 = 1$ and $i_0 = m - l$. Denote $C' = C[x_{m-l+1}, y_m]$ and $C'' = C[y_1, x_{m-l}]$. Then the order of C' is $2l$ and the order of C'' is $2(m - l - 1)$. Note that $l \ge p$, because otherwise the cycle $p_s C[x_1, y_{m-l}]P[p_1, p_s]$ would be strictly longer than C and compatible with M, a contradiction. Assume, without loss of generality, that $y_\nu p_1 \notin A(D)$ for all $y_\nu \in V(C')$ and $p_s x_\nu \notin A(D)$ for all $x_\nu \in V(C')$.

Now we claim that

$$d^+_{V(C)}(p_s) + d^-_{V(C)}(p_1) \le m - l + 1 \text{ and } d^+_{V(C'')}(y_m) + d^-_{V(C'')}(x_{m-l+1}) \le m - l - 1.$$

Note that for every pair of vertices y_t, x_{t+1} from $V(C'')$ at most one of the arcs $p_s x_{t+1}$ and $y_t p_1$ belongs to $A(D)$. For else D would contain a cycle $p_s C[x_{t+1}, y_t]P[p_1, p_s]$, which is strictly longer than C and compatible with M, a contradiction. There is precisely $m-l-1$ of such pairs. Accounting for $y_{m-l}p_1$ and $p_s x_1$, we get the required estimate

$$d^+_{V(C)}(p_s) + d^-_{V(C)}(p_1) \le (m - l - 1) + 2 = m - l + 1.$$

Analogously, for every pair of vertices y_t, x_{t+1} from $V(C'')$ at most one of the arcs $y_m x_{t+1}$ and $y_t x_{m-l+1}$ belongs to $A(D)$. For otherwise D would contain a cycle $P[p_1, p_s]C[x_1, y_t]C[x_{m-l+1}, y_m]C[x_{t+1}, y_{m-l}]p_1$, which is strictly longer than C, a contradiction. There are precisely $m - l - 1$ of such pairs. Then, we get the required estimate $d^+_{V(C'')}(y_m) + d^-_{V(C'')}(x_{m-l+1}) \le m - l - 1$. Hence, the claim holds.

If $p_sp_1 \notin A(D)$, then

$$\begin{aligned}a + 2 + \lfloor \frac{k}{2} \rfloor \leq & d^+(p_s) + d^-(p_1)\\ =&(d^+_{V(C)}(p_s) + d^-_{V(C)}(p_1)) + (d^+_{V(P)}(p_s) + d^-_{V(P)}(p_1))\\ \leq&(m - p + 1) + 2(p - 1) = m + p - 1 < a,\end{aligned}$$

a contradiction. Therefore $p_sp_1 \in A(D)$, and so Pp_1 is, in fact, a cycle.

We shall show that $R_X = \emptyset$. Suppose otherwise. Let P' be a maximal path in R, compatible with M. Write $P' = p'_1p'_2 \dots p'_t$. By Lemma 2.3, we may, without loss of generality, assume that $p'_t \in R_Y$. Note that $p'_\nu \in R_X$ where $\nu = 1$ if t is even and $\nu = 2$ if t is odd. Since P is a maximal path in $D \backslash V(C)$ compatible with M, we have that $d^-_{V(P)}(p'_\nu) = d^+_{V(P)}(p'_t) = 0$. Moreover, $d^+_{V(C)}(p'_t) + d^-_{V(C)}(p'_\nu) \leq m$, because for every pair of vertices y_i, x_{i+1} on C at most one of the arcs $y_ip'_\nu$ and p'_tx_{i+1} exists (by the maximality of C). Hence

$$d^+(p'_t) + d^-(p'_\nu) = d^+_{V(C)}(p'_t) + d^-_{V(C)}(p'_\nu) + d^+_R(p'_t) + d^-_R(p'_\nu) \leq m + 2r + k,$$

and so

$$\begin{aligned}2a + 4 + 2\lfloor \frac{k}{2} \rfloor \leq& d^+(p'_t) + d^-(p_1) + d^+(p_s) + d^-(p'_\nu)\\ =&(d^+(p'_t) + d^-(p'_\nu)) + (d^+(p_s) + d^-(p_1))\\ \leq&(m + 2r + k) + d^+_{V(C)}(p_s) + d^-_{V(C)}(p_1) + d^+_{V(P)}(p_s) + d^-_{V(P)}(p_1)\\ \leq&(m + 2r + k) + (m - p + 1 + 2p)\\ =&2m + 2r + p + k + 1\\ <&2a + k + 1,\end{aligned}$$

a contradiction. Hence $r = 0$ and $a = m + p$.

By the choices of x_1 and y_{m-l}, we have $p_s \nrightarrow x_{m-l+1}$ and $y_m \nrightarrow p_1$. By the maximality of C, we have $d^+_{V(P)}(y_m) = d^-_{V(P)}(x_{m-l+1}) = 0$. Hence

$$\begin{aligned}2a + 4 + 2\lfloor \frac{k}{2} \rfloor \leq& d^+(p_s) + d^-(x_{m-l+1}) + d^+(y_m) + d^-(p_1)\\ =&(d^+(p_s) + d^-(p_1)) + (d^+(y_m) + d^-(x_{m-l+1}))\\ \leq&(m - l + 1 + 2p) + (m - l - 1 + 2 + 2l + k) = 2a + 2 + k,\end{aligned}$$

a contradiction.

Case 2. s is odd.

Subcase 2.1. $d^+_{V(C)}(p_s) = 0$ and $d^-_{V(C)}(p_2) > 0$.

Let then $y_i \in V(C)$ be such that $y_ip_2 \in A(D)$. It follows from the maximality of C that $d^-_{V(P)}(x_{i+1}) \leq 1$ (p_1x_{i+1} may belong to $A(D)$). Note that $p_sx_{i+1} \notin A(D)$, therefore

$$\begin{aligned}a+2+\lfloor\frac{k}{2}\rfloor \leq & d^+(p_s)+d^-(x_{i+1})\\ =& d^+_{V(P)}(p_s)+(d^-_{V(P)}(x_{i+1})+d^-_{V(C)}(x_{i+1})+d^-_{R}(x_{i+1}))\\ \leq & p+(1+m+r+k-1)=a+k,\end{aligned}$$

a contradiction.

Subcase 2.2. $d^+_{V(C)}(p_s)=0$ and $d^-_{V(C)}(p_2)=0$.

If $p_sp_2 \notin A(D)$, then, by assumption,

$$\begin{aligned}a+2+\lfloor\frac{k}{2}\rfloor \leq & d^+(p_s)+d^-(p_2)=d^+_{V(P)}(p_s)+(d^-_{V(P)}(p_2)+d^-_{R'}(p_2))\\ \leq & (p-1)+(p+k-1)=2p+k-2<a+k,\end{aligned}$$

a contradiction. Therefore, $p_sp_2 \in A(D)$ and denote $P^*=p_2p_3\dots p_sp_2$. For any $p_j \in V(P^*)\cap Y$, if there exists $u \in R_X$ such that $p_j \to u$, then the path $P[p_{j+1},p_s]P[p_2,p_j]uM(u)$ is strictly longer than P and compatible with M, a contradiction. Hence $d^+_R(p_j)=0$, for all $p_j \in V(P^*)\cap Y$. If $d^+_{V(C)}(p_j)=0$ for all $p_j \in V(P^*)\cap Y$, then, by assumption,

$$\begin{aligned}a+2+\lfloor\frac{k}{2}\rfloor \leq & d^+(p_j)+d^-(x_1)=d^+_{V(P)}(p_j)+(d^-_{V(C)}(x_1)+d^-_{R}(x_1)+d^-_{V(P)}(x_1))\\ \leq & p+(m+r+k-1+1)=a+k,\end{aligned}$$

a contradiction. Hence, there exist $x_i \in V(C)$ and $p_j \in V(P^*)\cap Y$ such that $p_jx_i \in A(D)$. It follows that $d^+_{V(P)}(y_{i-1})=0$. Hence, by assumption,

$$\begin{aligned}a+2+\lfloor\frac{k}{2}\rfloor \leq & d^+(y_{i-1})+d^-(p_2)\\ =& (d^+_{V(C)}(y_{i-1})+d^+_{R}(y_{i-1}))+(d^-_{V(P)}(p_2)+d^-_{R'}(p_2))\\ \leq & (m+r)+(p+1+k-1)=a+k,\end{aligned}$$

a contradiction.

Subcase 2.3. $d^+_{V(C)}(p_s)>0$ and $d^-_{V(C)}(p_2)=0$.

Let then $x_i \in V(C)$ be such that $p_sx_i \in A(D)$. It follows that $d^+_{V(P)}(y_{i-1})=0$. In particular, $y_{i-1}p_2 \notin A(D)$, and hence

$$\begin{aligned}a+2+\lfloor\frac{k}{2}\rfloor \leq & d^+(y_{i-1})+d^-(p_2)\\ =& (d^+_{V(C)}(y_{i-1})+d^+_{R}(y_{i-1}))+(d^-_{V(P)}(p_2)+d^-_{R'}(p_2))\\ \leq & (m+r)+(p+1+k-1)=a+k,\end{aligned}$$

a contradiction.

Subcase 2.4. $d^+_{V(C)}(p_s) > 0$ and $d^-_{V(C)}(p_2) > 0$.

There exist x_{j_0} and y_{i_0} on C such that $y_{i_0}p_2 \in A(D)$ and $p_s x_{j_0} \in A(D)$. Without loss of generality, assume $j_0 = 1$ and $i_0 = m - l$. Denote $C' = C[x_{m-l+1}, y_m]$ and $C'' = C[y_1, x_{m-l}]$. Then the order of C' is $2l$ and the order of C'' is $2(m - l - 1)$. Note that $l \geq p$, because otherwise the cycle $p_s C[x_1, y_{m-l}]P[p_2, p_s]$ would be strictly longer than C and compatible with M, a contradiction. Furthermore, we can choose the x_1 and y_{m-l} so that $y_\nu p_2 \notin A(D)$ for all $y_\nu \in V(C')$ and $p_s x_\nu \notin A(D)$ for all $x_\nu \in V(C')$.

Now we claim that

$$d^+_{V(C)}(p_s) + d^-_{V(C)}(p_2) \leq m - l + 1 \text{ and } d^+_{V(C'')}(y_m) + d^-_{V(C'')}(x_{m-l+1}) \leq m - l - 1.$$

Note that for every pair of vertices y_t, x_{t+1} from $V(C'')$ at most one of the arcs $p_s x_{t+1}$ and $y_t p_2$ belongs to $A(D)$. For else D would contain a cycle $p_s C[x_{t+1}, y_t]C[p_2, p_s]$, which is strictly longer than C. There are precisely $m-l-1$ of such pairs. Accounting for $y_{m-l}p_2$ and $p_s x_1$, we get the required estimate

$$d^+_{V(C)}(p_s) + d^-_{V(C)}(p_2) \leq (m - l - 1) + 2 = m - l + 1.$$

Analogously, for every pair of vertices y_t, x_{t+1} from $V(C'')$ at most one of the arcs $y_m x_{t+1}$ and $y_t x_{m-l+1}$ belongs to $A(D)$. For otherwise D would contain a cycle $P[p_2, p_s]C[x_1, y_t]C[x_{m-l+1}, y_m]C[x_{t+1}, y_{m-l}]p_2$, which is strictly longer than C, a contradiction. There are precisely $m - l - 1$ of such pairs. Then, we get the required estimate $d^+_{V(C'')}(y_m) + d^-_{V(C'')}(x_{m-l+1}) \leq m - l - 1$. Hence, the claim holds.

If $p_s p_2 \notin A(D)$, then

$$\begin{aligned} a + 2 + \lfloor \frac{k}{2} \rfloor \leq & d^+(p_s) + d^-(p_2) \\ = & (d^+_{V(C)}(p_s) + d^-_{V(C)}(p_2)) + (d^+_{V(P)}(p_s) + d^-_{V(P)}(p_2)) + d^-_{R'}(p_2) \\ \leq & (m - p + 1) + (2p - 1) + (k - 1) = m + p + k - 1 \leq a + k - 1, \end{aligned}$$

a contradiction. Therefore $p_s p_2 \in A(D)$ and let P^* denote the cycle $p_2 \dots p_s p_2$.

We shall show that $R_X = \emptyset$. Suppose otherwise. Let P' be a maximal path in R compatible with M. Write $P' = p'_1 p'_2 \dots p'_t$. By Lemma 2.3, we may, without loss of generality, assume that $p'_t \in R_Y$. Note that $p'_\nu \in R_X$ where $\nu = 1$ if t is even and $\nu = 2$ if t is odd. Since P is a maximal path in $D \backslash V(C)$ compatible with M, we have that $d^-_{V(P)}(p'_\nu) \leq 1$ and $d^+_{V(P)}(p'_t) = 0$. Moreover, $d^+_{V(C)}(p'_t) + d^-_{V(C)}(p'_\nu) \leq m$, because for every pair of vertices y_i, x_{i+1} on C at most one of the arcs $y_i p'_\nu$ and $p'_t x_{i+1}$ exists (by the maximality of C). Hence

$$\begin{aligned} d^+(p'_t) + d^-(p'_\nu) = & (d^+_{V(C)}(p'_t) + d^-_{V(C)}(p'_\nu)) + (d^+_R(p'_t) + d^-_R(p'_\nu)) \\ & + (d^+_{V(P)}(p'_t) + d^-_{V(P)}(p'_\nu)) \\ \leq & m + (2r + k - 1) + 1 = m + 2r + k \end{aligned}$$

and so

$$\begin{aligned}2a+4+2\lfloor\frac{k}{2}\rfloor \le & d^+(p'_t)+d^-(p_2)+d^+(p_s)+d^-(p'_\nu)\\ = & (d^+(p'_t)+d^-(p'_\nu))+(d^+(p_s)+d^-(p_2))\\ \le & (m+2r+k)+(m-p+1+2p+1+k-1)\\ = & 2m+2r+p+2k+1=2a-p+2k+1,\end{aligned}$$

a contradiction. Hence, we have shown that $r=0$ and $a=m+p$.

Note that $p_sx_{m-l+1}\notin A(D)$ and $y_mp_2\notin A(D)$. By the maximality of C, we have $d^+_{V(P)}(y_m)=0$ and $d^-_{V(P)}(x_{m-l+1})\le 1$. Hence

$$\begin{aligned}2a+4+2\lfloor\frac{k}{2}\rfloor \le & d^+(p_s)+d^-(x_{m-l+1})+d^+(y_m)+d^-(p_2)\\ = & (d^+(p_s)+d^-(p_2))+(d^+(y_m)+d^-(x_{m-l+1}))\\ \le & (m-l+1+2p+1)+(m-l-1+2+2l+k)\\ = & 2a+3+k,\end{aligned}$$

a contradiction. □

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