AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **66(3)** (2016), Pages 425–435

On Kleinewillinghöfer types of finite Laguerre planes with respect to homotheties

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Abstract

Kleinewillinghöfer classified Laguerre planes with respect to central automorphisms and obtained a multitude of types. In this paper we investigate the Kleinewillinghöfer types of finite Laguerre planes with respect to homotheties and deal with some of the larger types.

1 Introduction

A finite *Laguerre plane* $\mathcal{L}$ of *order* n is the same as an orthogonal array of strength 3 on n symbols (levels), $n+1$ constraints and index 1; cf. [1], or equivalently, a transversal (or group divisible) design $\text{TD}_1(3, n+1, n)$. Since we have a more geometric point of view we rather use the term Laguerre plane instead of orthogonal array or transversal design. Explicitly, a finite Laguerre plane $\mathcal{L} = (P, \mathcal{C}, \mathcal{G})$ of order n is an incidence structure consisting of a set P of $n(n+1)$ points, a set $\mathcal{C}$ of n^3 circles and a set $\mathcal{G}$ of $n+1$ generators (where circles and generators are both subsets of P) such that $\mathcal{G}$ partitions P and each generator contains n points, such that each circle intersects each generator in precisely one point, and such that three points, no two of which are on the same generator, can be joined by a unique circle.

Models of finite Laguerre planes can be obtained as follows. Let $\mathcal{O}$ be an oval in the Desarguesian projective plane $\mathcal{P}_2 = \mathsf{PG}(2, q)$, q a prime power. Embed $\mathcal{P}_2$ into 3-dimensional projective space $\mathcal{P}_3 = \mathsf{PG}(3, q)$ and let v be a point of $\mathcal{P}_3$ not belonging to $\mathcal{P}_2$. Then P consists of all points of the cone with base $\mathcal{O}$ and vertex v except the point v. Generators are the traces of lines of $\mathcal{P}_3$ through v that are contained in the cone. Circles are obtained by intersecting P with planes of $\mathcal{P}_3$ not passing through v. In this way one obtains an *ovoidal Laguerre plane of order* q. If the oval $\mathcal{O}$ one starts off with is a conic, one obtains the *Miquelian Laguerre plane of order* q. All known finite Laguerre planes of odd order are Miquelian and all known finite Laguerre planes of even order are ovoidal. In fact, it is a long-standing problem whether or not these are the only finite Laguerre planes. (There are many non-ovoidal infinite Laguerre planes though.)

Analogous to the Lenz–Barlotti classification of projective planes Kleinewillinghöfer in [10] and [11] classified Laguerre planes with respect to *central automorphisms*, that is, automorphisms that fix at least one point and a central collineation is induced in the derived projective plane at this fixed point; see Section 2 for a definition of derived projective plane. More precisely, one considers subgroups of central automorphisms which are linearly transitive, that is, the induced groups of central collineations are transitive on each central line except for the obvious fixed points, the centre and the point of intersection with the axis. Central automorphisms of Laguerre planes come in a variety of types according to whether the axis of a central collineation in the derived projective plane at a fixed point is the line at infinity, a line that stems from a circle of the Laguerre plane or a line that comes from a generator of the Laguerre plane, and whether or not the centre is on the axis of the central collineation. Not surprisingly Kleinewillinghöfer obtained a multitude of types in Laguerre planes. She considered four kinds of central automorphisms, G-translations, $(G, B(q, C))$-translations, C-homologies and $\{p, q\}$-homotheties. In this paper we are dealing only with homotheties; see Section 3 for a definition.

In her classification with respect to homotheties, Kleinewillinghöfer labelled the possible configurations of centres of homotheties from 1 to 13; see the beginning of Section 3. We then say that the (Kleinewillinghöfer) type of a Laguerre plane $\mathcal{L}$ with respect to homotheties is n if the full automorphism group of $\mathcal{L}$ admits a configuration labelled by n. In this paper we investigate the Kleinewillinghöfer types of finite Laguerre planes with respect to homotheties, making use of the special combinatorial situation in such planes. As so often is the case in finite geometry, Laguerre planes of even order behave quite differently from those of odd order. This is reflected in the results we obtain for the Kleinewillinghöfer types of finite Laguerre planes. A similar programme has been carried out for topological, locally compact, 2-dimensional Laguerre planes, which led to the discovery of many new 2-dimensional Laguerre planes; see [20] and the references given there.

In Section 2 we give a brief review of basis properties of and results about (not necessarily finite) Laguerre planes. The following section deals with Laguerre homotheties, and the types of finite Laguerre planes with respect to these central automorphisms are investigated. In particular, we deal with the largest types (that is, types at least 11) and give a characterisation of elation Laguerre planes in terms of Kleinewillinghöfer types with respect to Laguerre homotheties.

2 Laguerre planes

In general, a Laguerre plane $\mathcal{L} = (P, \mathcal{C}, \mathcal{G})$ is an incidence structure consisting of a set P of points, a set $\mathcal{C}$ of circles and a set $\mathcal{G}$ of generators (where circles and generators are both subsets of P) such that $\mathcal{G}$ partitions P, such that each circle intersects each generator in precisely one point (parallel projection), such that three points no two of which are on the same generator can be joined by a unique circle (joining), such that given a point p on a circle C and a point q not on the generator through p, there is a unique circle that contains both points and *touches* C at p, that is, intersects C only in p or coincides with C (touching) and such that there is a circle with at least three points and there are four points not on a circle (richness); compare [22]. In case that P is finite (with n points on each generator) this set of axioms is equivalent to the definition of a Laguerre plane of order n given at the beginning of the introduction.

Being on the same generator defines an equivalence relation on P, and points on the same generator are often called *parallel*. We denote the generator that contains the point p by $[p]$.

It readily follows that for each point p of a Laguerre plane $\mathcal{L}$ the incidence structure $\mathcal{A}_p = (A_p, \mathcal{C}_p)$ whose point set A_p consists of all points of $\mathcal{L}$ that are not parallel to p and whose line set $\mathcal{C}_p$ consists of all restrictions to A_p of circles of $\mathcal{L}$ passing through p and of all generators not passing through p is an affine plane, called the *derived affine plane at p*. This affine plane extends to a projective plane $\mathcal{P}_p$, which we call the *derived projective plane at p*.

When forming the derived projective plane $\mathcal{P}_p$ of a Laguerre plane at a point p a circle C not passing through the distinguished point p induces an oval in $\mathcal{P}_p$ by removing from C the point on $[p]$ and adding in $\mathcal{P}_p$ the point ω at infinity of the lines that come from generators of $\mathcal{L}$. The line at infinity of $\mathcal{P}_p$ (relative to the derived affine plane $\mathcal{A}_p$) is a tangent to this oval. In $\mathcal{A}_p$ one has a *parabolic curve*.

Thus a Laguerre plane corresponds to a projective plane with enough of these ovals, that pairwise intersect in at most two affine points. This planar description of a Laguerre plane must then be extended by the points of one generator where one has to adjoin a new point to each line and to each oval as above of the affine plane.

In case of an ovoidal Laguerre plane each derived affine plane is Desarguesian. Then given a parabolic function f (that is, $\{(x, f(x)) \mid x \in \mathbb{F}\} \cup \{\omega\}$ is an oval in the Desarguesian projective plane over the field $\mathbb{F}$ where ω is the point at infinity of the y-axis in the projective plane), the circles of the corresponding ovoidal Laguerre plane $\mathcal{L}(f)$ can be represented as the sets

$$\{(x, y) \in \mathbb{F}^2 \mid y = af(x) + bx + c\} \cup \{(\infty, a)\},$$

where $a, b, c \in \mathbb{F}$. Moreover, every ovoidal Laguerre plane is isomorphic to a plane $\mathcal{L}(f)$ for a suitable parabolic function f.

Using the celebrated result of Segre [16] that every oval in a finite Desarguesian projective plane of odd order is a conic, the following characterization of finite Miquelian Laguerre planes was obtained in [3] or [13, VII.2].

THEOREM 2.1 *A finite Laguerre plane of odd order that admits a Desarguesian derivation is Miquelian.*

If $\mathcal{O}$ is an oval in a projective plane of even order, then the tangents to $\mathcal{O}$ all pass through a common point, the *nucleus* or *knot* of $\mathcal{O}$, see [15] or [4, 3.2.23]. This fact implies that in a Laguerre plane of even order the relation of 'touching' between circles is an equivalence relation, that is, if circles C_1 and C_2 touch each other at a point p and C_2 and C_3 touch each other at a point q (not necessarily the same as p), then C_1 and C_3 also touch each other at a point.

For small orders the above theorem and the results of [17] and [18] imply the following.

THEOREM 2.2 *A Laguerre plane of order at most ten is ovoidal and, in fact, Miquelian except in case of order eight.*

An automorphism of a Laguerre plane $\mathcal{L}$ is a permutation of the point set that maps circles onto circles and generators to generators. The collection of all automorphisms of $\mathcal{L}$ forms a group Γ with respect to composition, the automorphism group $\Gamma = \mathrm{Aut}(\mathcal{L})$ of $\mathcal{L}$. The collection of all automorphisms of $\mathcal{L}$ that fix each generator globally is a normal subgroup of Γ, called the *kernel* of $\mathcal{L}$. Finally, the collection of all automorphisms that fix each generator globally but fix no circle, together with the identity forms a normal subgroup Δ, see [19, Corollary 1.5], called the *elation group* of $\mathcal{L}$.

For example, each transformation

$$(x, y) \mapsto \begin{cases} (x, ry + af(x) + bx + c), & \text{if } x \in \mathbb{F} \\ (\infty, ry + a), & \text{if } x = \infty \end{cases}$$

where $a, b, c, r \in \mathbb{F}$, $r \neq 0$, is an automorphism of the ovoidal Laguerre plane $\mathcal{L}(f)$ in the planar description from above. These maps comprise the kernel of $\mathcal{L}(f)$. The elation group consists precisely of those maps where $r = 1$. Depending on the form of f there may be other automorphisms.

In [19] finite elation Laguerre planes were introduced and their basic structure investigated. They generalize finite ovoidal Laguerre planes and seem to give the best candidates so far to find finite non-ovoidal Laguerre planes. A finite *elation Laguerre plane* is a finite Laguerre plane that admits a group of automorphisms that acts trivially on the set of generators and regularly on the set of circles. The group in question is the elation group defined above. Each derived projective plane of a finite elation Laguerre plane $\mathcal{L}$ is a dual translation plane with translation centre the point ω. In particular, $\mathcal{L}$ has order a power of some prime. A finite elation Laguerre plane of order q is equivalent to a generalized oval (or pseudo-oval) with $q+1$ elements and thus to a translation generalized quadrangle of order q. In case of odd order one also obtains a translation generalized quadrangle with an antiregular point; compare [2], [9] and [21].

3 Laguerre Homotheties

A (Laguerre) homothety of a Laguerre plane $\mathcal{L}$ is determined by two non-parallel points p and q of $\mathcal{L}$; compare [10], [11]. An automorphism of $\mathcal{L}$ is a $\{p, q\}$-*homothety* if it fixes p and q and induces a homothety with centre q in the derived affine plane $\mathcal{A}_p$ at p, that is, in $\mathcal{A}_p$ every line through q is fixed. A group of $\{p, q\}$-homotheties is called $\{p, q\}$-*transitive* if it acts transitively on the points of each circle through p and q minus the two points p and q. In this case the group is also transitive on $[p] \setminus \{p\}$ and on $[q] \setminus \{q\}$. We say that the automorphism group Γ of $\mathcal{L}$ is $\{p, q\}$-transitive if Γ contains a $\{p, q\}$-transitive subgroup of $\{p, q\}$-homotheties. Note that a $\{p, q\}$-transitive group of $\{p, q\}$-homotheties of a finite Laguerre plane of order n has order $n - 1$ and, in particular, contains an involution in case of odd order.

With respect to homotheties Kleinewillinghöfer [10, Satz 3.2] or [11, Satz 1] obtained 13 types for Laguerre planes, labelled 1 to 13. If $\mathcal{H}$ denotes the set of all unordered pairs of non-parallel points $\{p, q\}$ for which the automorphism group of the Laguerre plane $\mathcal{L}$ is $\{p, q\}$-transitive, then exactly one of the following statements is valid.

1. $\mathcal{H} = \emptyset$.
2. $|\mathcal{H}| = 1$.
3. There are a point p and a generator G with $p \notin G$ such that
$$\mathcal{H} = \{\{p, q\} \mid q \in G\}.$$
4. There are a point p and a circle C through p such that
$$\mathcal{H} = \{\{p, q\} \mid q \in C \setminus \{p\}\}.$$
5. There are a circle C and a fixed-point-free involution $\phi : C \to C$ such that
$$\mathcal{H} = \{\{p, \phi(p)\} \mid p \in C\}.$$
6. There is a circle C such that $\mathcal{H} = \{\{p, q\} \mid p, q \in C, p \neq q\}$.
7. There are two distinct generators F, G and a bijection $\phi : F \to G$ such that
$$\mathcal{H} = \{\{p, \phi(p)\} \mid p \in F\}.$$
8. There are two distinct generators F, G such that $\mathcal{H} = \{\{p, q\} \mid p \in F, q \in G\}$.
9. Each point of $\mathcal{L}$ is in exactly one pair in $\mathcal{H}$.
10. There are a generator G and a bijection $\phi : G \to \mathcal{G} \setminus \{G\}$ such that
$$\mathcal{H} = \{\{p, q\} \mid p \in G, q \in \phi(p)\}.$$
11. There is a point p such that $\mathcal{H} = \{\{p, q\} \mid q \in P \setminus [p]\}$.
12. There is a generator G such that $\mathcal{H} = \{\{p, q\} \mid p \in G, q \in P \setminus G\}$.

13. $\mathcal{H}$ consists of all pairs of non-parallel points.

Clearly, a Miquelian Laguerre plane is of type 13. Conversely, the Miquelian Laguerre planes are characterized by this type; see [6, Satz 7].

THEOREM 3.1 *A Laguerre plane is of Kleinewillinghöfer type* 13 *if and only if it is Miquelian.*

The characterization of Miquelian Laguerre planes of characteristic $\neq 2$ in [6, Satz 8] does not require the $\{p, q\}$-transitivity for all pairs of non-parallel points. In case of finite Laguerre planes of odd order, this result shows that a configuration as in type 12 with respect to Laguerre homotheties already implies Miquelian; the additional condition made in [6, Satz 8], namely that each $\{p, q\}$-transitive group of $\{p, q\}$-homotheties contains an involution is automatically satisfied. In fact, one has the following.

LEMMA 3.2 *A finite Laguerre plane $\mathcal{L}$ of odd order that admits a point p such that $\mathcal{L}$ is $\{p, q\}$-transitive for all points q not parallel to p is Miquelian.*

Proof. Let p be a point of a finite Laguerre plane $\mathcal{L}$ such that $\mathcal{L}$ is $\{p, q\}$-transitive for all points q not parallel to p. Then the derived projective plane at p is, in the notation of [14], (q, W)-transitive for all points q not on the line W at infinity. Hence this derived plane is Desarguesian, see [14, 3.1.11 and 3.5.49]. But then $\mathcal{L}$ is Miquelian by Theorem 2.1 in case $\mathcal{L}$ has odd order. □

COROLLARY 3.3 *A finite Laguerre plane is of Kleinewillinghöfer type* 12 *if and only if it has even order and is ovoidal over a proper translation oval (not a conic).*

Proof. Since such a Laguerre plane $\mathcal{L}$ admits a point p as in Lemma 3.2 one sees that $\mathcal{L}$ must have even order because a Miquelian Laguerre plane is of type 13 by Theorem 3.1. The remaining statement on finite Laguerre planes of type 12 now follows from the characterization of type 12 among all finite Laguerre planes of even order in [6, Satz 9]. □

THEOREM 3.4 *A finite Laguerre plane that contains a group of automorphisms of Kleinewillinghöfer type* 11 *is Miquelian or ovoidal over a translation oval.*

Proof. Since such a Laguerre plane $\mathcal{L}$ admits a point p as in Lemma 3.2 one obtains that $\mathcal{L}$ is Miquelian and thus ovoidal unless $\mathcal{L}$ has even order. As seen in the proof of Lemma 3.2 the derived projective plane $\mathcal{P}_p$ at p is Desarguesian. Hence, assuming that $\mathcal{L}$ has even order, we can coordinatise the derived affine plane $\mathcal{A}_p$ at p over the field $\mathbb{F} = \mathsf{GF}(2^h)$ where 2^h is the order of $\mathcal{L}$

Let Σ be the group generated by all $\{p, q\}$-homotheties for all $q \notin [p]$. The restriction of Σ onto $\mathcal{A}_p$ consists of all transformations $(x, y) \mapsto (rx + b, ry + c)$ where $b, c, r \in \mathbb{F}$, $r \neq 0$, and where $p = (\infty, 0)$. Moreover, the full translation group $\Upsilon = \{(x, y) \mapsto (x + b, y + c) \mid b, c \in \mathbb{F}\}$ is induced by Σ and consists of Laguerre translations, that is, each of them acts trivially on $G = [p]$. To see the latter, note that Υ is an elementary abelian 2-group. It acts on G, which has 2^h points, fixes p and thus must fix at least another point $p' \neq p$ on G. Since Σ is transitive on $G \setminus \{p\}$ and Υ is normal in Σ, it follows that Υ fixes each point in the orbit of p'. This shows that Υ acts trivially on G.

Since Υ also fixes $p' \in G \setminus \{p\}$, this group induces a group of collineations of the derived plane $\mathcal{A}_{p'}$ at p'. Furthermore, because $\Upsilon \simeq \mathbb{F}^2$ is abelian, this group is either the translation group of $\mathcal{A}_{p'}$ or a shift group so that lines of $\mathcal{A}_{p'}$ are of the form $\{(x, f(x + b) + c) \mid x \in \mathbb{F}\}$ where $f : \mathbb{F} \to \mathbb{F}$ is planar. The latter case is not possible because the order of $\mathbb{F}$ has to be odd by [5, Lemma 9] in order for a planar function on $\mathbb{F}$ to exist. This shows that Υ consists of translations in $\mathcal{A}_{p'}$.

We consider the circle $C_b = \{(x, f_b(x)) \mid x \in \mathbb{F}\} \cup \{p'\}$ through p' that touches $y = bx$ at $(0, 0)$ where $b \in \mathbb{F}$. Without loss of generality we may assume that $p' = (\infty, 1)$. This circle induces a line L_b in $\mathcal{A}_{p'}$ and so the subgroup of Υ that stabilizes L_b is transitive on L_b. However, in $\mathcal{P}_p$ the circle C_b induces an oval $\mathcal{O}_b$ (and a parabolic curve in $\mathcal{A}_p$). Hence $\mathcal{O}_b$ is a translation oval. The classification of translation ovals in finite Desarguesian planes (compare [7] or [8, Theorem 8.41]) shows that $f_b(x) = x^{2^k} + bx$ where h and k are co-prime.

The transitivity of Σ on $G \setminus \{p\}$ implies that every circle not passing through p is of the form $\{(x, af_b(x/a) + c) \mid x \in \mathbb{F}\} \cup \{(\infty, a)\}$. Since $af_b(x/a) + c = a^{1-2^k}x^{2^k} + bx + c$ and because $a \mapsto a^{1-2^k}$ is a permutation of $\mathbb{F}$, circles of $\mathcal{L}$ are represented, after a suitable relabelling of the points on G, as $\{(x, af_0(x) + bx + c) \mid x \in \mathbb{F}\} \cup \{(\infty, a)\}$ where $a, b, c \in \mathbb{F}$. Hence $\mathcal{L}$ is the ovoidal Laguerre plane $\mathcal{L}(f_0)$. □

Note that in the situation of Theorem 3.4 the Laguerre plane then is of type 13 or 12. Also Corollary 3.3 follows from Theorem 3.4 because a finite Laguerre plane of type 12 admits a group of automorphisms as in Theorem 3.4.

Kleinewillinghöfer [10, proof of Satz 4.1] notes that in a finite Laguerre plane of even order there is no fixed-point-free involution on a circle, and similarly, the point set cannot be partitioned into pairs of non-parallel points as in type 9, because parallelity of points is preserved in this case and so one has an induced fixed-point-free involution on $\mathcal{G}$. A direct consequence of these observations is the following.

PROPOSITION 3.5 *A finite Laguerre plane of Kleinewillinghöfer type* 5 *or* 9 *has odd order.*

In summary, we now obtain the following possible Kleinewillinghöfer types for finite Laguerre planes with respect to Laguerre homotheties.

THEOREM 3.6 *A finite Laguerre plane of odd order is of Kleinewillinghöfer type* $1, 2, 3, 4, 5, 6, 7, 8, 9, 10,$ *or* 13.

A finite Laguerre plane of even order is of Kleinewillinghöfer type $1, 2, 3, 4, 6, 7, 8, 10, 12,$ *or* 13.

There are examples of finite ovoidal Laguerre planes of even order that are of Kleinewillinghöfer type 1, 8, 12 and 13. (All finite ovoidal Laguerre planes of odd order are of type 13.) The respective types are obtained, for example, from the oval $\mathcal{O}_f = \{(x, f_i(x)) \mid x \in \mathbb{F}_n\} \cup \{\omega\}$ in $\mathsf{PG}(2, 2^h)$ when f_i is one of the following functions

$$\begin{array}{rcll} f_1(x) & = & x^{1/6} + x^{3/6} + x^{5/6} & \text{where } h \geq 5 \text{ is odd;} \\ f_2(x) & = & x^6 & \text{where } h \geq 5 \text{ is odd;} \\ f_3(x) & = & x^{2^i} & \text{where } \gcd(i, h) = 1, i \geq 2, h \geq 4; \\ f_4(x) & = & x^2 & \text{any } h. \end{array}$$

For the fact that the above polynomials describe ovals in $\mathsf{PG}(2, 2^h)$, see [7] or [8, Section 8.4]. Since the Laguerre planes are ovoidal it suffices to determine those points p and q on the circle $y = 0$ for which the plane is $\{p, q\}$-transitive; the plane then will be $\{r, s\}$-transitive for all points r on $[p]$ and s on $[q]$; see the end of Section 2 for automorphisms of ovoidal Laguerre planes. In particular, an ovoidal Laguerre must be of type 1, 8, 12 or 13.

In the last case, when $f_4(x) = x^2$, the oval $\mathcal{O}_{f_4}$ is a conic and $\mathcal{L}(f_4)$ is Miquelian; thus $\mathcal{L}(f_4)$ has type 13 by Theorem 3.1. The second to last case where $f_3(x) = x^{2^i}$ results in a translation oval $\mathcal{O}_{f_3}$ that is not a conic (unless $h = 3$) so that $\mathcal{L}(f_3)$ has type 12 by Corollary 3.3. Since $f_2(x) = x^6$ does not describe a translation oval and because $H = \{(x, y) \mapsto (rx, ry) \mid r \in \mathbb{F}, r \neq 0\}$ is a linearly transitive group of $\{(\infty, 0), (0, 0)\}$-homotheties, one sees that $\mathcal{L}(f_2)$ is of type 8. Finally in the first case, when $f_1(x) = x^{1/6} + x^{3/6} + x^{5/6}$ (where exponents are modulo $2^h - 1$, see [12]), $\mathcal{O}_{f_1}$ is neither a translation oval nor is the above group H a group of automorphisms of $\mathcal{L}(f_1)$. In fact, no group of $\{(u, 0), (v, 0)\}$-homotheties where $u, v \in \mathbb{F} \cup \{\infty\}$, $u \neq v$, is linearly transitive in this case. Thus $\mathcal{L}(f_1)$ is of type 1.

In 2-dimensional Laguerre planes, type 8 leads to ovoidal Laguerre planes. In order to prove a weaker result for finite Laguerre planes we need the following lemma.

LEMMA 3.7 *Let p be a point of a finite Laguerre plane $\mathcal{L}$ of order n and let G be a generator not containing p. Assume that the automorphism group of $\mathcal{L}$ is $\{p, q\}$-transitive for all points q on G. Then the derived affine plane $\mathcal{A}_p$ at p is a dual near-field plane, n is a prime power, and, after suitable coordinatisation, the group Σ generated by all $\{p, q\}$-homotheties for all $q \in G$ consists of all transformations $(x, y) \mapsto (x * r, y * r + t)$ where $r, t \in \mathbb{F}_n$, $r \neq 0$, and where $+$ is addition in the Galois field $\mathbb{F}_n$ of order n and $*$ is multiplication in the near-field. (Here $p = (\infty, 0)$ and $G = \{0\} \times \mathbb{F}_n$.) Furthermore, the translations of $\mathcal{A}_p$ given by $(x, y) \mapsto (x, y + t)$ where $t \in \mathbb{F}_n$ extend to Laguerre translations, that is, they fix $[p]$ pointwise.*

Proof. It follows from [14, 3.5.46] that $\mathcal{A}_p$ is a dual near-field plane. Hence n is a prime power. Non-vertical lines of $\mathcal{A}_p$ are of the form $\{(x, m*x+t) \mid x \in \mathbb{F}_n\}$ where $*$

defines a group operation on $\mathbb{F}_n \setminus \{0\}$ and is left-distributive, that is, $(m_1+m_2)*x = m_1*x+m_2*x$ for all $m_1, m_2, x \in \mathbb{F}_n$. The form of elements in Σ now follows.

Let Δ be the kernel of the action of Σ on the set of generators. Obviously Δ consists of the translations of $\mathcal{A}_p$ in the vertical direction. Let $F = \{\infty\} \times \mathbb{F}_n$ be the generator containing p. Since Σ is transitive on $F \setminus \{p\}$, the normal subgroup Δ acts trivially on F or transitively on $F \setminus \{p\}$. Assume that Δ has a non-trivial orbit on F. Let p_0 and p_1 be two distinct points in this orbit and let $q = (0, 0) \in G$. By our transitivity assumptions there are a $\delta \in \Delta$ and a $\{p, q\}$-homothety α such that $\delta(p_0) = p_1$ and $\alpha(p_0) = p_1$, $\delta(x, y) = (x, y + t)$ and $\alpha(x, y) = (x * r, y * r)$ for some $r, t \in \mathbb{F}_n$, $r \neq 0, 1$, $t \neq 0$. Then $\beta = \delta^{-1}\alpha \neq id$ is given by $\beta(x, y) = (x * r, y * r - t)$ which is a (p, q')-homothety where $q' = (0, b) \in G$ and b is the unique element of $\mathbb{F}_n$ such that $b*r-b = t$. However, β also fixes p_0 which is impossible. This shows that Δ must act trivially on F. □

A configuration as described in Lemma 3.7 occurs in types 3, 8, 10. 11, 12 and 13. In case of type 8 we obtain the following.

THEOREM 3.8 *A finite Laguerre plane of type* 8 *is an elation Laguerre plane.*

Proof. Let $\mathcal{L}$ be a finite Laguerre plane of type 8 and order n. Lemma 3.7 shows that for each point p of the distinguished generators F and G the derived affine plane at p is dual near-field plane. Furthermore, given a point $p \in F$ the group $\Delta^{(p)}$ of translations of $\mathcal{A}_p$ in the vertical direction acts trivially on F and sharply transitively on G. Likewise, for $q \in G$ the group $\Delta^{(q)}$ of translations of $\mathcal{A}_q$ in the vertical direction acts trivially on G and sharply transitively on F. Both $\Delta^{(p)}$ and $\Delta^{(q)}$ are isomorphic to the additive group of $\mathbb{F}_n$. Furthermore, if $\delta^{(q)} \in \Delta^{(q)} \setminus \{id\}$, the conjugate $\delta^{(p)}\delta^{(q)}(\delta^{(p)})^{-1}$ acts trivially on G and fixed-point-free on $P \setminus G$ and fixes each generator. Thus this map induces a central collineation in $\mathcal{P}_q$ with centre ω and axis W, that is, $\delta^{(p)}\delta^{(q)}(\delta^{(p)})^{-1} \in \Delta^{(q)}$. One likewise obtains that $\delta^{(q)}(\delta^{(p)})^{-1}(\delta^{(q)})^{-1} \in \Delta^{(p)}$. Hence, the commutator of $\delta^{(p)}$ and $\delta^{(q)}$ belongs to $\Delta^{(p)} \cap \Delta^{(q)} = \{id\}$ for each $\delta^{(q)} \in \Delta^{(p)}$ and $\delta^{(q)} \in \Delta^{(q)}$. Therefore $\Delta^{(p)}$ and $\Delta^{(q)}$ commute, both are normal subgroups of Δ, the group generated by $\Delta^{(p)}$ and $\Delta^{(q)}$, and $\Delta \cong \mathbb{F}_n^2$ is commutative. Clearly, Δ is in the kernel of $\mathcal{L}$.

We fix $p \in F$ and $q \in G$ and coordinatise $\mathcal{L}$ in such a way that $p = (\infty, 0)$, $q = (0, 0)$ and that $\mathcal{A}_p$ is the affine plane whose non-vertical lines are given by $y = m*x+t$ where $m, t \in \mathbb{F}_n$. The automorphisms in $\Delta^{(p)}$ are then given by $\delta_t^{(p)}(x, y) = (x, y+t)$ for $t \in \mathbb{F}_n$. In the coordinates of $\mathcal{A}_p$ the automorphisms in $\Delta^{(q)}$ are of the form $\delta_s^{(q)}(x, y) = (x, \alpha_s(x, y))$ for $s \in \mathbb{F}_n$ and suitable functions α_s. However $\delta^{(p)}$ and $\delta^{(q)}$ commute. This implies that $\alpha_s(x, y) = \alpha_s(x, 0) + y$ for all $x, y \in \mathbb{F}_n$, $x \neq 0$. Furthermore, because $\Delta^{(q)} \cong \mathbb{F}_n$ one obtains that $\alpha_{s+s'}(x, 0) = \alpha_s(x, 0) + \alpha_{s'}(x, 0)$ so that $s \mapsto \alpha_s(x, 0)$ is linear over $\mathbb{F}_r$ where r is the characteristic of $\mathbb{F}_n$. If we consider $\mathbb{F}_n$ as a d-dimensional vector space over $\mathbb{F}_r$ (so that $n = r^d$), then $\alpha_s(x, 0) = s \cdot f(x)$ where $s \in \mathbb{F}_r^d$ and $f(x)$ is a $d \times d$ matrix over $\mathbb{F}_n$. Since the graphs of $x \mapsto s \cdot f(x)$ represent touching circles to $y = 0$ at q, one finds that each such function is parabolic.

Recall that $\Delta^{(q)}$ is transitive on $F = \{\infty\} \times \mathbb{F}_n$ and acts trivially on $G = \{0\} \times \mathbb{F}_n$. We therefore obtain that δ_s is given by

$$(x, y) \mapsto \begin{cases} (x, y + s \cdot f(x)), & \text{for } x \in \mathbb{F}_n, \\ (\infty, y + s), & \text{for } x = \infty. \end{cases}$$

We now obtain the circles of $\mathcal{L}$ through (∞, s) by applying δ_s to the circles through p. This shows that all circles of $\mathcal{L}$ are of the form $\{(x, a \cdot f(x) + b * x + c) \mid x \in \mathbb{F}_n\} \cup \{(\infty, a)\}$ where $a, b, c \in \mathbb{F}_n$. In particular, $(x, y) \mapsto (x, y + a \cdot f(x) + b * x + c)$ is an automorphism of $\mathcal{L}$. Since the collection of these automorphisms is in the kernel of $\mathcal{L}$ and is sharply transitive on $\mathcal{C}$, we see that $\mathcal{L}$ is an elation Laguerre plane. □

Conversely, the possible types of elation Laguerre planes are very restricted.

PROPOSITION 3.9 *A finite elation Laguerre plane is of type* 1, 8, 12 *or* 13 *with respect to homotheties.*

A finite non-ovoidal elation Laguerre plane must be of type 1 *or* 8.

Proof. Since the elation group of a finite elation Laguerre plane acts regularly on the circle set and transitively on each generator, one readily sees that whenever $\{p, q\}$ belongs to the configuration $\mathcal{H}$ in the classification of types then so does $\{p', q'\}$ for all $p' \in [p]$ and $q' \in [q]$. Hence the first statement follows.

Theorem 3.1 and Corollary 3.3 then imply the second statement. □

The examples following Theorem 3.6 show that each of the types 1, 8, 12 and 13 occurs as a type of a finite ovoidal Laguerre plane and thus as the type of a finite elation Laguerre plane.

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(Received 24 Mar 2016)