AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **66(2)** (2016), Pages 276–287

Generating functions for purely crossing partitions

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Abstract

The generating function for the number of purely crossing partitions of $\{1,\dots,n\}$ is found in terms of the generating function for Bell numbers. Further results about generating functions for asymptotic moments of certain random Vandermonde matrices are derived.

1 Introduction

Let $\mathcal{P}(n)$ denote the lattice of all set partitions of $[n] := \{1,\dots,n\}$. The elements of $\pi \in \mathcal{P}(n)$ are called blocks of the partition π; we write $i \overset{\pi}{\sim} j$ if and only if i and j belong to the same block of π and we write $i \overset{\pi}{\nsim} j$ otherwise.

Recall that a partition $\pi \in \mathcal{P}(n)$ is said to be noncrossing whenever there do not exist $i,j,k,l \in [n]$ with $i < j < k < l$, $i \overset{\pi}{\sim} k$ and $j \overset{\pi}{\sim} l$, but $i \overset{\pi}{\nsim} j$. We let $\mathrm{NC}(n)$ denote the set of all noncrossing partitions of $[n]$.

Definition 1.1 We say that a subset $S \subseteq [n]$ *splits* a partition $\pi \in \mathcal{P}(n)$ if S is the union of some of the blocks of π. In other words, S splits π if and only if $B \in \pi$ and $B \cap S \neq \emptyset$ implies $B \subseteq S$.

Definition 1.2 For $n \geq 1$, a partition $\pi \in \mathcal{P}(n)$ is said to be *purely crossing* if:

(a) no proper subinterval $\{p+1, p+2, \dots, p+q\}$ of $[n]$ splits π (by proper subinterval we mean with $0 \leq p < p+q \leq n$ and $q < n$);

(b) no block of π contains neighbors, namely, $k \overset{\pi}{\nsim} k+1$ for all $k \in [n-1]$;

(c) $1 \overset{\pi}{\nsim} n$.

We let $\mathrm{PC}(n)$ denote the set of all purely crossing partitions of $[n]$ and we let $\mathrm{PC} = \bigcup_{n=1}^{\infty} \mathrm{PC}(n)$.

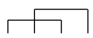
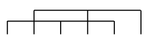
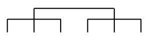
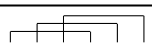
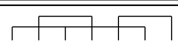
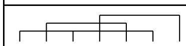
* Research supported in part by NSF grant DMS-1202660.

Note that condition (a) implies that π has no singleton blocks. Moreover, if $\pi \in \mathrm{PC}(n)$, then every partition obtained from π by cyclic permutation of $[n]$ is also purely crossing.

After an early version of this paper, we learned (from an anonymous referee) that Definition 1.1 and Definition 1.2(a) appear in the work of Beissinger [1] as instances of a treatment of a more general framework. However, parts (b) and (c) of 1.2 seem not to fit into her framework. See also [7].

The purely crossing partitions were introduced in [3] in connection with certain random Vandermonde matrices. Is it easy to see that $\mathrm{PC}(n)$ is empty for $n \in \{1, 2, 3, 5\}$ and that the only purely crossing partitions of sizes $n \in \{4, 6, 7\}$ are those found in Table 1 and their orbits under cyclic permutations of $[n]$.

Table 1: Purely crossing partitions of sizes 4, 6 and 7

n	picture	blocks	# in orbit
4		$\{1,3\},\{2,4\}$	1
6		$\{1,3,5\},\{2,4,6\}$	1
		$\{1,3\},\{2,5\},\{4,6\}$	3
		$\{1,4\},\{2,5\},\{3,6\}$	1
7		$\{1,3,6\},\{2,4\},\{5,7\}$	7
		$\{1,3,6\},\{2,5\},\{4,7\}$	7

In this note, we study $\mathrm{PC}(n)$ and describe how arbitrary partitions can be realized in terms of purely crossing ones and noncrossing partitions. Thereby, we find an algebraic description of the generating function for $|\mathrm{PC}(n)|$ in terms of the generating function for Bell numbers.

More generally, in Section 2 we study generating functions based on certain weighted sums over partitions. An intermediate stage is to consider connected partitions, which have been enumerated by F. Lehner [5], who proved that their cardinalities form the free cumulants of a Poisson distribution and found a generating series for them. We will make an equivalent derivation of the generating series, thereby reproving Lehner's result, but with an eye toward the generalization that follows. A motivation for and a possible application of this study in random matrices are described in Section 3. This includes more complicated results about algebra-valued generating functions that are similar to those in Section 2.

Notation. We will refer to the elements of $[n]$ as the atoms of the partition $\pi \in \mathcal{P}(n)$. We let $1_n = \{[n]\}$ be the partition of $[n]$ into one block and $0_n = \{\{1\}, \{2\}, \ldots, \{n\}\}$ be the partition of $[n]$ into n blocks (all singletons).

If $\phi : A \to B$ is a bijective mapping and if π is a set partition of A, then (by abuse of notation) we let $\phi(\pi)$ denote the set partition of B that results from applying ϕ

to every block of π.

The usual order $\leq$ on $\mathcal{P}(n)$ is $\sigma \leq \pi$ if and only if every block of π splits σ.

If π is a partition of a set X, then the restriction of π to a subset $Y \subseteq X$ is $\{Y \cap B \mid B \in \pi\} \backslash \{\emptyset\}$.

2 Generating functions and connected partitions

For each $n \geq 1$ and $\pi \in \mathrm{PC}(n)$, let $a(\pi) \in \mathbb{C}$. Consider the formal power series

$$A(x) = \sum_{n=1}^{\infty} a_n x^n = a_4 x^4 + a_6 x^6 + a_7 x^7 + \cdots, \quad \text{where} \quad a_n = \sum_{\pi \in \mathrm{PC}(n)} a(\pi).$$

Note that, if $a(\pi) = 1$ for all $\pi \in \mathrm{PC}(n)$, then

$$A(x) = \sum_{n=1}^{\infty} |\mathrm{PC}(n)| x^n$$

is the generating series for $|\mathrm{PC}(n)|$. This will be the principal case of interest in this paper; however, we prove results in the greater generality of arbitrary coefficients $a(\pi)$.

Definition 2.1 Let $\mathrm{PC}^+(n)$ be the set of all $\pi \in \mathcal{P}(n)$ such that (a) and (b) of Definition 1.2 hold.

The following lemma is straightforward and we omit a proof.

Lemma 2.2 $\mathrm{PC}^+(1) = \{0_1\}$ *and, for* $n \geq 2$, *we have the disjoint union*

$$\mathrm{PC}^+(n) = \mathrm{PC}(n) \cup \{\tilde{\sigma} \mid \sigma \in \mathrm{PC}(n-1)\},$$

where $\tilde{\sigma}$ *is obtained from* σ *by adjoining* n *to the block of* σ *that contains* 1.

Let $b(0_1) = 1$ and for $\pi \in \mathrm{PC}^+(n)$, $n \geq 2$, let

$$b(\pi) = \begin{cases} a(\pi) & \pi \in \mathrm{PC}(n), \\ a(\sigma) & \pi = \tilde{\sigma},\ \sigma \in \mathrm{PC}(n-1). \end{cases}$$

Consider the formal power series

$$B(x) = \sum_{n=1}^{\infty} b_n x^n = x + b_4 x^4 + b_5 x^5 + \cdots, \quad \text{where} \quad b_n = \sum_{\pi \in \mathrm{PC}^+(n)} b(\pi).$$

Thus, if $a(\pi) = 1$ for all $\pi \in \mathrm{PC}(n)$ and all n, then

$$B(x) = \sum_{n=1}^{\infty} |\mathrm{PC}^+(n)| x^n.$$

Lemma 2.3 *We have* $b_1 = 1$ *and for every* $n \geq 2$, $b_n = a_n + a_{n-1}$. *Thus, we have*

$$B(x) = x + (1 + x)A(x).$$

Proof: By definition of $\mathrm{PC}^+(1)$, $b_1 = 1$. For $n \geq 2$, by Lemma 2.2, we have

$$b_n = \sum_{\pi \in \mathrm{PC}(n)} a(\pi) + \sum_{\sigma \in \mathrm{PC}(n-1)} a(\sigma) = a_n + a_{n-1}.$$

The desired equality follows immediately. □

Definition 2.4 For $n \geq 1$, let $\mathrm{CO}(n)$ be the set of all connected partitions $\pi \in \mathcal{P}(n)$, namely, such that (a) of Definition 1.2 holds.

Definition 2.5 Given $\pi \in \mathcal{P}(n)$, let $\hat{\pi}$ denote the smallest noncrossing partition so that $\pi \leq \hat{\pi}$, in the usual order on $\mathcal{P}(n)$. The is called the *noncrossing closure* of π.

Lemma 2.6 *Let* $\pi \in \mathcal{P}(n)$. *Then* $\pi \in \mathrm{CO}(n)$ *if and only if* $\hat{\pi} = 1_n$.

Proof: If $\pi \notin \mathrm{CO}(n)$, then there is a proper subinterval I of $[n]$ that splits π. The partition $\{I, [n]\backslash I\}$ of $[n]$ is noncrossing and lies above π in the usual order on $\mathcal{P}(n)$. Thus, $\hat{\pi} \leq \{I, [n]\backslash I\}$ and $\hat{\pi} \neq 1_n$.

Now suppose $\hat{\pi} \neq 1_n$. Then at least one block of $\hat{\pi}$ is a proper subinterval of $[n]$, and this block necessarily splits π. So $\pi \notin \mathrm{CO}(n)$. □

Lemma 2.7 *Let* $n \geq 1$. *Then there is a bijection*

$$\Phi_n : \bigcup_{1 \leq \ell \leq n} \big\{(\sigma, k_1, \ldots, k_\ell \big| \sigma \in \mathrm{PC}^+(\ell),\, k_1, \ldots, k_\ell \geq 1,\, k_1 + \cdots + k_\ell = n\big\} \to \mathrm{CO}(n),$$

defined by $\pi = \Phi_n(\sigma, k_1, \ldots, k_\ell)$ *is the partition obtained from* σ *by replacing the* j*-th atom of* σ *with an interval of length* k_j. *More precisely, for the intervals*

$$I_j = \{k_1 + \cdots + k_{j-1} + 1, k_1 + \cdots + k_{j-1} + 2, \ldots, k_1 + \cdots + k_{j-1} + k_j\}$$

we have

$$\pi = \{\cup_{j \in B} I_j \mid B \in \sigma\}.$$

Proof: Let $\pi = \Phi_n(\sigma, k_1, \ldots, k_\ell)$. Suppose $K \subseteq [n]$ is a nonempty interval of $[n]$ that splits π. Since each block of π is a union of some of the intervals $(I_j)_{j \in [\ell]}$, we have $K = \bigcup_{j \in X} I_j$ for some subset X of $[\ell]$. Since K is a nonempty interval, X must be a nonempty interval of $[\ell]$. Since K splits π, we see that X splits σ. Since $\sigma \in \mathrm{PC}^+(\ell) \subseteq \mathrm{CO}(\ell)$, we have $X = [\ell]$, so $K = [n]$. This shows that Φ_n maps into $\mathrm{CO}(n)$.

We now describe the inverse map of Φ_n. Given $\pi \in \mathrm{CO}(n)$, consider the collection of interval subsets of $[n]$ that are maximal with respect to the property of being contained in some block of π. The collection of these is an interval partition of $[n]$, and we may arrange them in increasing order $(I_j)_{1\le j\le \ell}$. Let σ be the partition of $[\ell]$ given by $j_1 \overset{\sigma}{\sim} j_2$ if and only if I_{j_1} and I_{j_2} belong to the same block of π. We have $\sigma \in \mathrm{PC}^+(\ell)$ because if $j \overset{\sigma}{\sim} j+1$, then $I_j \cup I_{j+1}$ would be the subset of a single block of π, contradicting maximality of I_j. Letting $k_j = |I_j|$, we have $\sum_{j=1}^{\ell} = n$. Let $\widetilde{\Phi}_n(\pi) = (\sigma, k_1, \ldots, k_\ell)$. Then

$$\widetilde{\Phi}_n : \mathrm{CO}(n) \to \bigcup_{1\le \ell\le n} \big\{(\sigma, k_1, \ldots, k_\ell)\big|\sigma \in \mathrm{PC}^+(\ell),\, k_1, \ldots, k_\ell \ge 1,\, k_1 + \cdots + k_\ell = n\big\}$$

and $\Phi_n \circ \widetilde{\Phi}_n$ is the identity map on $\mathrm{CO}(n)$. We easily verify that $\widetilde{\Phi}_n \circ \Phi_n$ is the identity map on the domain of Φ_n. □

For $\pi \in \mathrm{CO}(n)$, let $c(\pi) = b(\sigma)$, where $\pi = \Phi_n(\sigma, k_1, \ldots, k_\ell)$, with $\sigma \in \mathrm{PC}^+(\ell)$. Let

$$C(x) = \sum_{n=1}^{\infty} c_n x^n, \quad \text{where} \quad c_n = \sum_{\pi\in\mathrm{CO}(n)} c(\pi).$$

Note that, if $a(\pi) = 1$ for all $\pi \in \bigcup_{n=1}^{\infty} \mathrm{PC}(n)$, then

$$C(x) = \sum_{n=1}^{\infty} |\mathrm{CO}(n)| x^n$$

is the generating series for $|\mathrm{CO}(n)|$.

Lemma 2.8 *As formal power series, we have*

$$C(x) = B\left(\frac{x}{1-x}\right).$$

Proof: From the bijection described in Lemma 2.7 and the definition of $c(\pi)$, we have

$$c_n x^n = \sum_{\pi\in\mathrm{CO}(n)} c(\pi) x^n = \sum_{\ell=1}^{n} \sum_{\sigma\in\mathrm{PC}^+(\ell)} b(\sigma) \sum_{\substack{k_1,\ldots,k_\ell\ge 1\\ k_1+\cdots+k_\ell=n}} x^n = \sum_{\ell=1}^{n} b_\ell \sum_{\substack{k_1,\ldots,k_\ell\ge 1\\ k_1+\cdots+k_\ell=n}} \prod_{j=1}^{\ell} x^{k_j},$$

so

$$C(x) = \sum_{n=1}^{\infty} c_n x^n = \sum_{\ell=1}^{\infty} b_\ell \Big(\sum_{k=1}^{\infty} x^k\Big)^{\ell} = B\left(\frac{x}{1-x}\right).$$

□

Lemma 2.9 *For each $n \ge 1$, there is a bijection*

$$\Psi_n : \big\{(\tau, \sigma_1, \sigma_2, \ldots, \sigma_{|\tau|}) \mid \tau = \{B_1, \ldots, B_{|\tau|}\} \in \mathrm{NC}(n),\, \sigma_j \in \mathrm{CO}(|B_j|)\big\} \to \mathcal{P}(n),$$

defined as follows. The idea is to replace each block B_j of τ with the corresponding partition of B_j determined by σ_j. For specificity, the blocks $B_1, \ldots, B_{|\tau|}$ above are written in order of increasing minimal elements. Given $B \subseteq [n]$, let $\psi_B : [|B|] \to B$ be the unique order-preserving bijection. Given $(\tau, \sigma_1, \ldots, \sigma_{|\tau|})$ in the domain of Ψ_n and writing $\tau = \{B_1, \ldots, B_{|\tau|}\}$ as above, we consider the partition $\psi_{B_j}(\sigma_j)$ of B_j for each j. Then

$$\Psi(\tau, \sigma_1, \ldots, \sigma_{|\tau|}) = \bigcup_{j=1}^{|\tau|} \psi_{B_j}(\sigma_j).$$

Proof: Given $\pi \in \mathcal{P}(n)$, let $\hat{\pi}$ be as in Definition 2.5 be the noncrossing cover of π. For each block B of $\hat{\pi}$, the restriction of π to B yields a partition that, after applying the order-preserving bijection of B onto $\{1, \ldots, |B|\}$, yields a partition $\sigma_B \in \mathcal{P}(|B|)$ with $\hat{\sigma}_B = 1_{|B|}$, since otherwise B would not be a block of $\hat{\pi}$. Thus, by Lemma 2.6, $\sigma_B \in \mathrm{CO}(|B|)$. Consequently, if we number the blocks $\hat{\pi}$ in conventional order, $\hat{\pi} = \{B_1, B_2, \ldots, B_{|\hat{\pi}|}\}$, then we immediately see $\pi = \Psi_n(\hat{\pi}, \sigma_{B_1}, \ldots, \sigma_{B_{|\hat{\pi}|}})$. We define

$$\widetilde{\Psi}_n : \mathcal{P}(n) \to \big\{(\tau, \sigma_1, \ldots, \sigma_{|\tau|}) \mid \tau = \{B_1, \ldots, B_{|\tau|}\} \in \mathrm{NC}(n),\ \sigma_j \in \mathrm{CO}(|B_j|)\big\}$$

by $\widetilde{\Psi}_n(\pi) = (\hat{\pi}, \sigma_{B_1}, \ldots, \sigma_{B_{|\hat{\pi}}})$ as given above. Thus, the composition $\Psi_n \circ \widetilde{\Psi}_n$ is the identity on $\mathcal{P}(n)$. Similarly, it is not difficult to see that $\widetilde{\Psi}_n \circ \Psi_n$ is the identity on the domain of Ψ_n. □

For $\pi = \Psi_n(\tau, \sigma_1, \ldots, \sigma_{|\tau|}) \in \mathcal{P}(n)$, let

$$d(\pi) = \prod_{j=1}^{|\tau|} c(\sigma_j). \tag{1}$$

Let

$$D(x) = 1 + \sum_{n=1}^{\infty} d_n x^n, \quad \text{where} \quad d_n = \sum_{\pi \in \mathcal{P}(n)} d(\pi).$$

For convenience and consistency, we are using the convention $\mathcal{P}(0) = \{\emptyset\}$ and we set $d(\emptyset) = 1$. Note that if $a(\pi) = 1$ for all $\pi \in \mathrm{PC}$, then $d(\pi) = 1$ for all $\pi \in \bigcup_{n \geq 1} \mathcal{P}(n)$. In this case, $d_n = |\mathcal{P}(n)|$ is the n-th Bell number and $D(x)$ is the generating function for the Bell numbers.

Lemma 2.10 *For each $n \geq 1$, there is a bijection*

$$\Theta_n : \mathcal{P}(n) \to \bigcup_{\substack{1 \leq \ell \leq n \\ k(1), \ldots, k(\ell) \geq 0 \\ k(1) + \cdots + k(\ell) = n - \ell}} \{(\sigma, \pi_1, \pi_2, \ldots, \pi_\ell) \mid \sigma \in \mathrm{CO}(\ell),\ \pi_j \in \mathcal{P}(k(j))\},$$

defined as follows. Given $\pi \in \mathcal{P}(n)$, let $\hat{\pi}$ be as in Definition 2.5 and let B_1 be the block of $\hat{\pi}$ containing 1. Let $\ell = |B_1|$ and write $B_1 = \{s(1), s(2), \ldots, s(\ell)\}$ with

$s(j-1) < s(j)$. Let ϕ be the order-preserving bijection from B_1 onto $[\ell]$ and let σ be the result of ϕ applied to the restriction of π to B_1. For $1 \le j \le \ell - 1$, let π_j be the result of applying the mapping $i \mapsto i - s(j)$ to the restriction of π to the interval $\{s(j)+1, \ldots, s(j+1)-1\}$. Let π_ℓ be the result of applying the mapping $i \mapsto i - s(\ell)$ to the interval $\{s(\ell)+1, \ldots, n\}$. Then $\Theta_n(\pi) = (\sigma, \pi_1, \ldots, \pi_\ell)$.

Proof: Take $k(j) = s(j+1) - s(j) - 1$ if $1 \le j \le \ell - 1$ and let $k(l) = n - s(\ell)$. We have $\hat{\sigma} = 1_{|B_1|}$, so by Lemma 2.6, $\sigma \in \mathrm{CO}(|B_1|)$ and Θ_n takes values in the indicated range.

We define

$$\widetilde{\Theta}_n : \bigcup_{\substack{1 \le \ell \le n \\ k(1),\ldots,k(\ell) \ge 0 \\ k(1)+\cdots+k(\ell) = n-\ell}} \{(\sigma, \pi_1, \pi_2, \ldots, \pi_\ell) \mid \sigma \in \mathrm{CO}(\ell),\ \pi_j \in \mathcal{P}(k(j))\} \to \mathcal{P}(n)$$

as follows. Given $(\sigma, \pi_1, \ldots, \pi_\ell)$ in the indicated domain of $\widetilde{\Theta}_n$, let $s(j) = k(1) + k(2) + \cdots + k(j-1) + j$, with $s(1) = 1$, and let

$$L = \{s(1), s(2), \ldots, s(\ell)\};$$

let $\tilde{\phi} : [\ell] \to L$ be the order-preserving bijection and consider the partition $\tilde{\phi}(\sigma)$ of L. For each j, let $\tilde{\pi}_j$ be the partition of the interval $\{s(j)+1, \ldots, s(j)+k(j)\}$ obtained by applying the mapping $i \mapsto i + s(j)$ to π_j. Let

$$\Theta_n(\sigma, \pi_1, \ldots, \pi_\ell) = \tilde{\phi}(\sigma) \cup \bigcup_{j=1}^{\ell} \tilde{\pi}_j.$$

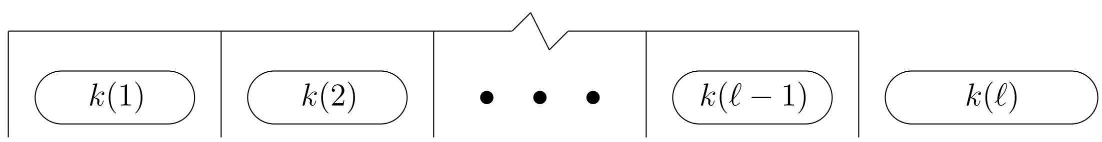
It it not difficult to show that Θ_n and $\widetilde{\Theta}_n$ are inverses of each other. □

Lemma 2.11 *Suppose $\pi \in \mathcal{P}(n)$ and $\Theta_n(\pi) = (\sigma, \pi_1, \ldots, \pi_\ell)$. Then, with d as defined in* (1)*, we have*

$$d(\pi) = c(\sigma) \prod_{j=1}^{\ell} d(\pi_j).$$

Proof: Let $\pi = \Psi_n(\tau, \sigma_1, \ldots, \sigma_{|\tau|})$. Then $\sigma = \sigma_1$ and we draw the first block B_1 of τ in Figure 1, with the gaps between elements of B_1 of length $k(1), \ldots, k(\ell-1)$

Figure 1: The first block of a noncrossing partition.



and with $k(\ell)$ elements of $[n]$ to the right of B_1. The other blocks of τ all lie in the gaps depicted by the ovals, either between the adjacent elements of B_1 or to the right of B_1. In particular, each π_j is formed from those in the list $\sigma_2, \ldots, \sigma_{|\tau|}$ whose corresponding blocks of τ lie in the oval labeled $k(j)$. Thus, we get

$$\prod_{j=1}^{\ell} d(\pi_j) = \prod_{i=2}^{|\tau|} c(\sigma_j).$$

By the definition (1) of $d(\pi)$, this proves the lemma. □

Lemma 2.12 *As formal power series, we have*

$$D(x) = 1 + C(xD(x)). \tag{2}$$

Letting $F(x) = xD(x)$ and letting $F^{\langle -1 \rangle}$ denote the inverse with respect to composition of F, we have

$$F^{\langle -1 \rangle}(w) = \frac{w}{1 + C(w).} \tag{3}$$

Proof: Using the bijection from Lemma 2.10 and using Lemma 2.11, we have

$$\begin{aligned} D(x) &= 1 + \sum_{\ell=1}^{\infty} \left(\sum_{\sigma \in \mathrm{CO}(\ell)} c(\sigma) x^{\ell} \right) \sum_{k(1),\ldots,k(\ell) \geq 0} \sum_{\substack{\pi_1 \in \mathcal{P}(k(1)),\ldots \\ \pi_\ell \in \mathcal{P}(k(\ell))}} \prod_{q=1}^{\ell} d(\pi_q) x^{k(q)} \\ &= 1 + \sum_{\ell=1}^{\infty} c_\ell x^\ell \left(\sum_{k=0}^{\infty} d_k x^k \right)^{\ell} = 1 + C(xD(x)). \end{aligned}$$

Multiplying both sides of (2) by x, we get $F(x) = x(1 + C(F(x)))$, and letting $w = F(x)$ we get $w = F^{\langle -1 \rangle}(w)(1 + C(w))$, from which (3) follows. □

Taking now $a(\pi) = 1$ for all π, as we observed:

- $D(x)$ is the generating function for the Bell numbers, $|\mathcal{P}(n)|$.
- $C(x)$ is the generating function for $|\mathrm{CO}(n)|$, and can be found from D using Lemma 2.12.
- $B(x)$ is the generating function for $|\mathrm{PC}^{+}(n)|$, and can be found from C using Lemma 2.8.
- $A(x)$ is the generating function for $|\mathrm{PC}(n)|$, and can be found from B using Lemma 2.3.

The Bell numbers $|\mathcal{P}(n)|$ are well known. Using Mathematica [8], we calculated the first several terms of each generating series and we obtained the values displayed in Table 2.

Table 2: Cardinalities of sets of partitions.

n	$\|\mathrm{PC}(n)\|$	$\|\mathrm{PC}^+(n)\|$	$\|\mathrm{CO}(n)\|$	$\|\mathcal{P}(n)\|$
1	0	1	1	1
2	0	0	1	2
3	0	0	1	5
4	1	1	2	15
5	0	1	6	52
6	5	5	21	203
7	14	19	85	877
8	62	76	385	4 140
9	298	360	1 907	21 147
10	1 494	1 792	10 205	115 975
11	8 140	9 634	58 455	678 570
12	47 146	55 286	355 884	4 213 597
13	289 250	336 396	2 290 536	27 644 437
14	1 873 304	2 162 554	15 518 391	190 899 322
15	12 756 416	14 629 720	110 283 179	1 382 958 545

3 Connection with random Vandermonde matrices

Purely crossing partitions arose in [3], appearing in the study of asymptotic moments of certain random Vandermonde matrices X_N. In particular, by Theorem 3.28 of [3], for the n-th asymptotic $*$-moment, we have

$$m_n := \lim_{N\to\infty} \mathbb{E}\circ\mathrm{tr}\left((X_N X_N^*)^n\right) = \sum_{\pi\in\mathcal{P}(n)} w_\pi \tag{4}$$

with weight

$$w_\pi = \tau\big(\Lambda_\pi(\underbrace{1,1,\ldots,1}_{n-1\text{ times}})\big),$$

where Λ_π is the multilinear function from $n-1$ copies of $C[0,1]$ into $C[0,1]$ described in Section 2 of [3] and where τ is the trace on $C[0,1]$ obtained by integrating with respect to Lebesgue measure. These w_π are precisely the volumes of certain polytopes first described by Ryan and Debbah in [6], who had also obtained the formula (4). In Section 4 of [3], we show how, for arbitrary $\pi\in\mathcal{P}(n)$, Λ_π and, thus w_π, can be computed via a reduction procedure in terms of the Λ_σ for $\sigma\in\bigcup_{k=1}^n \mathrm{PC}(k)$. This procedure is akin to that used in Section 2, but more complicated, involving nested evaluations of various Λ_ρ. In this section, we carry out this analysis. Let us also mention that noncrossing $C[0,1]$-valued cumulants for the asymptotic $*$-moments of X_N are expressed in terms of purely crossing partitions, at least for shorter lengths. See Proposition 4.14 of [3].

Let τ be the trace on $C[0,1]$ given by integration with respect to Lebesgue measure. Consider the following formal power series in variable $g \in C[0,1]$, with each n-th term being an n-fold $\mathbb{C}$-multilinear map of $C[0,1] \times \cdots \times C[0,1]$ into $C[0,1]$ evaluated in the variable repeated n times:

$$A(g) = \sum_{n=1}^{\infty} a_n(g), \qquad a_n(g) = \sum_{\pi \in \mathrm{PC}(n)} \Lambda_\pi(g, \ldots, g)g$$

$$B(g) = \sum_{n=1}^{\infty} b_n(g), \qquad b_n(g) = \sum_{\pi \in \mathrm{PC}^+(n)} \Lambda_\pi(g, \ldots, g)g$$

$$C(g) = \sum_{n=1}^{\infty} c_n(g), \qquad c_n(g) = \sum_{\pi \in \mathrm{CO}(n)} \Lambda_\pi(g, \ldots, g)g$$

$$D(g) = 1 + \sum_{n=1}^{\infty} d_n(g), \qquad d_n(g) = \sum_{\pi \in \mathcal{P}(n)} \Lambda_\pi(g, \ldots, g)g.$$

(A more general and formal treatment of such formal power series, in terms of the multilinear function series of [4], can be found in Section 4 of [2].) From the remarks above (see [3] and [6]) if follows that if m_n is the asymptotic moment found in (4), then the moment generating function of m_n is, for variable $x \in \mathbb{C}$,

$$\sum_{n=0}^{\infty} m_n x^n = \tau(D(x1)),$$

where $x1 \in C[0,1]$ is the constant function x.

The next result is analogous to the combination of Lemmas 2.3, 2.8 and 2.12.

Proposition 3.1 *We have*

$$B(g) = g + \tau(A(g))g + A(g) \tag{5}$$
$$C(g) = B(g/(1-\tau(g))) \tag{6}$$
$$D(g) = 1 + C(gD(g)) \tag{7}$$

Thus, letting $F(g) = gD(g)$ and letting $F^{\langle -1 \rangle}$ be its inverse with respect to composition, we have

$$F^{\langle -1 \rangle}(h) = h\big(1 + C(h)\big)^{-1}. \tag{8}$$

Proof: For the partition $0_1 \in \mathrm{PC}(1)$, we have $\Lambda_{0_1}() = 1$, so $b_1(g) = g$. If a partition $\pi = \tilde{\sigma} \in \mathrm{PC}^+(n)$ for $\sigma \in \mathrm{PC}(n-1)$ as in Lemma 2.2, then by Lemma 4.5 of [3] we have

$$\Lambda_\pi(\underbrace{g, \ldots, g}_{n-1 \text{ times}}) = \tau\big(\Lambda_\sigma(\underbrace{g, \ldots, g}_{n-2 \text{ times}})g\big).$$

Thus, using Lemma 2.2, for $n \geq 2$ we get

$$b_n(g) = \tau(a_{n-1}(g))g + a_n(g),$$

which yields (5).

If $\pi = \Phi_n(\sigma, k_1, \ldots, k_\ell) \in \mathrm{CO}(n)$ for $\sigma \in \mathrm{PC}^+(\ell)$ and $k_j \geq 1$, $k_1 + \cdots + k_\ell = n$ as in Lemma 2.7, then by Lemma 4.4 of [3], we have

$$\Lambda_\pi(\underbrace{g, \ldots, g}_{n-1 \text{ times}}) = \Lambda_\sigma(\underbrace{g, \ldots, g}_{\ell-1 \text{ times}})\tau(g)^{n-\ell}.$$

Thus, using Lemma 2.7,

$$c_n(g) = \sum_{\ell=1}^{n} b_\ell(g) \sum_{\substack{k_1,\ldots,k_\ell \geq 1 \\ k_1+\cdots+k_\ell = n}} \prod_{j=1}^{\ell} \tau(g)^{k_j-1}$$

and we get

$$C(g) = \sum_{\ell=1}^{\infty} b_\ell(g) \left(\sum_{k=1}^{\infty} \tau(g)^{k-1} \right)^\ell = \sum_{\ell=1}^{\infty} b_\ell(g)/(1-\tau(g))^\ell.$$

By multilinearity, we have $b_\ell(gx) = b_\ell(g)x^\ell$ for all scalars x. Thus, we get (6).

Keep in mind we use the convention $\mathrm{PC}(0) = \{\emptyset\}$. Suppose $n \geq 1$, $\pi \in \mathcal{P}(n)$ and $\pi = \Theta_n(\sigma, \pi_1, \ldots, \pi_\ell)$ for $\sigma \in \mathrm{CO}(\ell)$ and $\pi_j \in \mathcal{P}(k(j))$ where $k(j) \geq 0$ and $k(1) + \cdots + k(\ell) = n - \ell$, as in Lemma 2.10. Using Lemmas 4.2 and 4.3 of [3], we get

$$\Lambda_\pi(\underbrace{g, \ldots, g}_{n-1 \text{ times}}) = \Lambda_\sigma(e_1 g, \ldots, e_{\ell-1} g)e_\ell,$$

where

$$e_j = \begin{cases} g\Lambda_{\pi_j}(\underbrace{g, \ldots, g}_{k(j)-1 \text{ times}}), & k(j) > 0 \\ 1, & k(j) = 0. \end{cases}$$

Thus, using the convention $d_0(g) = 1$, using Lemma 2.10 and using multilinearity of Λ_σ, we have

$$\begin{aligned} D(g) &= 1 + \sum_{\ell=1}^{\infty} \sum_{\sigma \in \mathrm{CO}(\ell)} \sum_{k(1),\ldots,k(\ell) \geq 0} \Lambda_\sigma\big(gd_{k(1)}(g), \ldots, gd_{k(\ell-1)}(g)\big)gd_{k(\ell)}(g) \\ &= 1 + \sum_{\ell=1}^{\infty} \sum_{\sigma \in \mathrm{CO}(\ell)} \Lambda_\sigma(gD(g), \ldots, gD(g))gD(g) = 1 + C(gD(g)). \end{aligned}$$

This proves (7). The final equality (8) follows as in the proof of Lemma 2.12. □

Acknowledgements

The author thanks Catherine Yan for helpful discussions about generating functions and anonymous referees for helpful comments.

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(Received 10 Mar 2016)