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Intersecting integer partitions

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Abstract

If $a_1, a_2, \dots, a_k$ and n are positive integers such that $n = a_1 + a_2 + \cdots + a_k$, then the sum $a_1 + a_2 + \cdots + a_k$ is said to be a *partition of* n of *length* k, and $a_1, a_2, \dots, a_k$ are said to be the *parts* of the partition. Two partitions that differ only in the order of their parts are considered to be the same partition. Let P_n be the set of partitions of n, and let $P_{n,k}$ be the set of partitions of n of length k. We say that two partitions t-*intersect* if they have at least t common parts (not necessarily distinct). We call a set A of partitions t-*intersecting* if every two partitions in A t-intersect. For a set A of partitions, let $A(t)$ be the set of partitions in A that have at least t parts equal to 1. We conjecture that for $n \geq t$, $P_n(t)$ is a largest t-intersecting subset of P_n. We show that for $k > t$, $P_{n,k}(t)$ is a largest t-intersecting subset of $P_{n,k}$ if $n \leq 2k - t + 1$ or $n \geq 3tk^5$. We also demonstrate that for every $t \geq 1$, there exist n and k such that $t < k < n$ and $P_{n,k}(t)$ is not a largest t-intersecting subset of $P_{n,k}$.

1 Introduction

Unless stated otherwise, we shall use small letters such as x to denote positive integers or functions or elements of a set, capital letters such as X to denote sets, and calligraphic letters such as $\mathcal{F}$ to denote *families* (that is, sets whose elements are sets themselves). The set $\{1, 2, \dots\}$ of all positive integers is denoted by $\mathbb{N}$. For $n \geq 1$, $[n]$ denotes the set $\{1, \dots, n\}$ of the first n positive integers. We take $[0]$ to be the empty set $\emptyset$. We call a set A an r-*element set* if its size $|A|$ is r (that is, if it contains exactly r elements). For a set X, $\binom{X}{r}$ denotes the family of r-element subsets of X. It is to be assumed that arbitrary sets and families are finite.

In the literature, a sum $a_1 + a_2 + \cdots + a_k$ is said to be a *partition of* n of *length* k if $a_1, a_2, \dots, a_k$ and n are positive integers such that $n = a_1 + a_2 + \cdots + a_k$. A partition of a positive integer n is also referred to as an *integer partition* or simply as a *partition*. If $a_1 + a_2 + \cdots + a_k$ is a partition, then $a_1, a_2, \dots, a_k$ are said to be its *parts*. Two partitions that differ only in the order of their parts are considered to be the same partition. Thus, we can refine the definition of a partition as follows. We call a tuple $(a_1, \ldots, a_k)$ a *partition of* n of *length* k if $a_1, \ldots, a_k$ and n are positive integers such that $n = \sum_{i=1}^{k} a_i$ and $a_1 \leq \cdots \leq a_k$. We will be using the latter definition throughout the rest of the paper.

For any tuple $\mathbf{a} = (a_1, \ldots, a_k)$ and any $i \in [k]$, a_i is said to be the *i-th entry of* $\mathbf{a}$, and if $\mathbf{a}$ is a partition, then a_i is also said to be a *part of* $\mathbf{a}$.

Let P_n denote the set of partitions of n, and let $P_{n,k}$ denote the set of partitions of n of length k. Thus, $P_{n,k}$ is non-empty if and only if $1 \leq k \leq n$. Moreover, $P_n = \bigcup_{i=1}^{n} P_{n,i}$.

Let $p_n = |P_n|$ and $p_{n,k} = |P_{n,k}|$. These values are widely studied. To the best of the author's knowledge, no elementary closed-form expressions are known for p_n and $p_{n,k}$. For more about these values, we refer the reader to [2].

If at least one part of a partition $\mathbf{a}$ is a part of a partition $\mathbf{b}$, then we say that $\mathbf{a}$ and $\mathbf{b}$ *intersect*. We call a set A of partitions *intersecting* if for every $\mathbf{a}$ and $\mathbf{b}$ in A, $\mathbf{a}$ and $\mathbf{b}$ intersect. We make the following conjecture.

Conjecture 1.1 *For every positive integer n, the set of partitions of n that have* 1 *as a part is a largest intersecting set of partitions of n.*

We also conjecture that for $2 \leq k \leq n$ and $(n, k) \neq (8, 3)$, $\{\mathbf{a} \in P_{n,k} \colon 1 \text{ is a part of } \mathbf{a}\}$ is a largest intersecting subset of $P_{n,k}$. We will show that this is true for $n \leq 2k$ and for n sufficiently large depending on k.

For any set A of partitions, let $A(t)$ denote the set of partitions in A whose first t entries are equal to 1. Thus, for $t \leq k \leq n$,

$$P_{n,k}(t) = \{(a_1, \ldots, a_k) \in P_{n,k} \colon a_1 = \cdots = a_t = 1\} \quad \text{and} \quad P_n(t) = \bigcup_{i=t}^{n} P_{n,i}(t).$$

Note that for $t < k \leq n$, $|P_n(t)| = p_{n-t}$ and $|P_{n,k}(t)| = p_{n-t,k-t}$.

Generalising the definition of intersecting partitions, we say that two tuples $(a_1, \ldots, a_r)$ and $(b_1, \ldots, b_s)$ *t-intersect* if there are t distinct integers $i_1, \ldots, i_t$ in $[r]$ and t distinct integers $j_1, \ldots, j_t$ in $[s]$ such that $a_{i_p} = b_{j_p}$ for each $p \in [t]$. We call a set A of tuples *t-intersecting* if for every $\mathbf{a}, \mathbf{b} \in A$, $\mathbf{a}$ and $\mathbf{b}$ t-intersect. Thus, for any $A \subseteq P_n$, $A(t)$ is t-intersecting, and A is intersecting if and only if A is 1-intersecting.

We pose the following two problems, which lie in the interface between extremal set theory and partition theory.

Problem 1.2 *What is the size or the structure of a largest t-intersecting subset of P_n?*

Problem 1.3 *What is the size or the structure of a largest t-intersecting subset of $P_{n,k}$?*

This paper mainly addresses the second question. We suggest two conjectures corresponding to the two problems above and generalising the two conjectures above.

Conjecture 1.4 *For $n \geq t$, $P_n(t)$ is a t-intersecting subset of P_n of maximum size.*

Conjecture 1.1 is Conjecture 1.4 with $t = 1$.

For $2 \leq k \leq n$, the only case we discovered where $P_{n,k}(1)$ is not a largest intersecting subset of $P_{n,k}$ is $n = 8$ and $k = 3$.

Remark 1.5 We have $P_{8,3} = \{(1,1,6),(1,2,5),(1,3,4),(2,2,4),(2,3,3)\}$. Since $P_{8,3}$ is not an intersecting set, $\{(1,2,5),(1,3,4),(2,2,4),(2,3,3)\}$ is an intersecting subset of $P_{8,3}$ of maximum size $4 = |P_{8,3}(1)| + 1$. Extending this example, we have that for $t \geq 2$, $\{(1,\dots,1,a,b,c) \in P_{t+7,t+2} \colon (a,b,c) \in \{(1,2,5),(1,3,4),(2,2,4),(2,3,3)\}\}$ is a t-intersecting subset of $P_{t+7,t+2}$ of size $|P_{t+7,t+2}(t)| + 1$.

Conjecture 1.6 *For $t+1 \leq k \leq n$ with $(n,k) \neq (t+7,t+2)$, $P_{n,k}(t)$ is a t-intersecting subset of $P_{n,k}$ of maximum size.*

If $t = k < n$, then $P_{n,k}(t) = \emptyset$, $P_{n,k} \neq \emptyset$, and the non-empty t-intersecting subsets of $P_{n,k}$ are the 1-element subsets. If $k < t$, then $P_{n,k}$ has no non-empty t-intersecting subsets.

For every k and t, we leave Conjecture 1.6 open only for a finite range of values of n, namely, for $2k - t + 1 < n < 3tk^5$. We first prove it for $n \leq 2k - t + 1$.

Proposition 1.7 *Conjecture 1.6 is true for $n \leq 2k - t + 1$.*

Proof. Suppose $n \leq 2k - t + 1$. For any $\mathbf{c} = (c_1,\dots,c_k) \in P_{n,k}$, let $L_{\mathbf{c}} = \{i \in [k] \colon c_i = 1\}$ and $l_{\mathbf{c}} = |L_{\mathbf{c}}|$. We have $2k - t + 1 \geq n = \sum_{i \in L_{\mathbf{c}}} c_i + \sum_{j \in [k] \backslash L_{\mathbf{c}}} c_j \geq \sum_{i \in L_{\mathbf{c}}} 1 + \sum_{j \in [k] \backslash L_{\mathbf{c}}} 2 = l_{\mathbf{c}} + 2(k - l_{\mathbf{c}}) = 2k - l_{\mathbf{c}}$. Thus, $l_{\mathbf{c}} \geq t - 1$, and equality holds only if $n = 2k - t + 1$ and $c_j = 2$ for each $j \in [k] \backslash L_{\mathbf{c}}$. Since $c_1 \leq \cdots \leq c_k$, $L_{\mathbf{c}} = [l_{\mathbf{c}}]$.

Let A be a t-intersecting subset of $P_{n,k}$. If $l_{\mathbf{a}} \geq t$ for each $\mathbf{a} \in A$, then $A \subseteq P_{n,k}(t)$. Suppose $l_{\mathbf{a}} = t-1$ for some $\mathbf{a} = (a_1,\dots,a_k) \in A$. By the above, we have $n = 2k-t+1$, $a_i = 1$ for each $i \in [t-1]$, $a_j = 2$ for each $j \in [k] \backslash [t-1]$, and $P_{n,k} = P_{n,k}(t) \cup \{\mathbf{a}\}$. Let $\mathbf{b}$ be the partition $(b_1,\dots,b_k)$ in $P_{n,k}(t)$ with $b_k = n - k + 1 = k - t + 2$ and $b_i = 1$ for each $i \in [k-1]$. Since $k \geq t + 1$, $\mathbf{a}$ and $\mathbf{b}$ do not t-intersect, and hence $\mathbf{b} \notin A$. Thus, $|A| \leq |P_{n,k}| - 1 = |P_{n,k}(t)|$. □

In Section 3, we show that Conjecture 1.6 is also true for n sufficiently large. More precisely, we prove the following.

Theorem 1.8 *For $k \geq t + 2$ and $n \geq 3tk^5$, $P_{n,k}(t)$ is a t-intersecting subset of $P_{n,k}$ of maximum size, and uniquely so if $k \geq t + 3$.*

We actually prove the result for $n \geq \frac{8}{7}(t+1)k^5$. For this purpose, we generalise Bollobás' proof [3, pages 48–49] of the Erdős–Ko–Rado (EKR) Theorem [9], and we make some observations regarding the values $p_{n,k}$ and the structure of t-intersecting subsets of $P_{n,k}$.

Remark 1.9 Conjecture 1.6 is also true for $k = t + 1$. Indeed, if two partitions of n of length $t + 1$ have t common parts $a_1,\dots,a_t$, then the remaining part of each is $n - (a_1 + \cdots + a_t)$, and hence the partitions are the same. Thus, the non-empty

t-intersecting subsets of $P_{n,t+1}$ are the 1-element subsets. Hence $P_{n,t+1}(t)$ is a largest t-intersecting subset of $P_{n,t+1}$, but not uniquely so for $n \geq t+3$ ($\{(1, \ldots, 1, 2, n-t-1)\}$ is another one). Regarding the case $k = t + 2$, note that the size $p_{n-t,2}$ of $P_{n,t+2}(t)$ is $\lfloor (n-t)/2 \rfloor$, and that $\{\mathbf{a} \in P_{n,t+2} \colon t-1$ parts of $\mathbf{a}$ are equal to 1, 2 is a part of $\mathbf{a}\}$ is a t-intersecting subset of $P_{n,t+2}$ of size $p_{n-t-1,2} = \lfloor (n-t-1)/2 \rfloor$. Thus, if $n - t$ is odd, then $P_{n,t+2}(t)$ is not the unique t-intersecting subset of $P_{n,t+2}$ of maximum size.

We say that $(a_1, \ldots, a_r)$ and $(b_1, \ldots, b_s)$ *strongly t-intersect* if for some t-element subset T of $[\min\{r, s\}]$, $a_i = b_i$ for each $i \in T$. Following [6], we say that a set A of tuples is *strongly t-intersecting* if every two tuples in A strongly t-intersect. In [6], it is conjectured that for $t + 1 \leq k \leq n$, $P_{n,k}(t)$ is a strongly t-intersecting subset of $P_{n,k}$ of maximum size. This is verified for $t = 1$ in the same paper. Note that this conjecture is weaker than Conjecture 1.6 (for $(n, k) \neq (t + 7, t + 2)$), and that Proposition 1.7 and Theorem 1.8 imply that it is true for $n \leq 2k - t + 1$ and for $n \geq 3tk^5$.

Theorem 1.8 is an analogue of the classical EKR Theorem [9], which inspired many results in extremal set theory (see [8, 12, 10, 5, 14]). A family $\mathcal{A}$ of sets is said to be *t-intersecting* if $|A \cap B| \geq t$ for every $A, B \in \mathcal{A}$. The EKR Theorem says that if n is sufficiently larger than k, then the size of any t-intersecting subfamily of $\binom{[n]}{k}$ is at most $\binom{n-t}{k-t}$. A sequence of results [9, 11, 20, 13, 1] culminated in the complete solution, conjectured in [11], for any n, k and t; it turns out that $\{A \in \binom{[n]}{k} \colon [t] \subseteq A\}$ is a largest t-intersecting subfamily of $\binom{[n]}{k}$ if and only if $n \geq (t + 1)(k - t + 1)$. The same t-intersection problem for the family of subsets of $[n]$ was solved in [18]. These are among the most prominent results in extremal set theory.

Remark 1.10 The conjectures and results above for partitions can be rephrased in terms of t-intersecting subfamilies of a family. For any tuple $\mathbf{a} = (a_1, \ldots, a_k)$, let

$$S_{\mathbf{a}} = \{(a, i) \colon a \in \{a_1, \ldots, a_k\},\, i \in [k],\, |\{j \in [k] \colon a_j = a\}| \geq i\};$$

thus, $(a, 1), \ldots, (a, r) \in S_{\mathbf{a}}$ if and only if at least r of the entries of $\mathbf{a}$ are equal to a. For example, $S_{(2,2,5,5,5,7)} = \{(2, 1), (2, 2), (5, 1), (5, 2), (5, 3), (7, 1)\}$. Let $\mathcal{P}_n = \{S_{\mathbf{a}} \colon \mathbf{a} \in P_n\}$ and $\mathcal{P}_{n,k} = \{S_{\mathbf{a}} \colon \mathbf{a} \in P_{n,k}\}$. Let $f : P_n \to \mathcal{P}_n$ such that $f(\mathbf{a}) = S_{\mathbf{a}}$ for each $\mathbf{a} \in P_n$. Clearly, f is a bijection. Thus, $|\mathcal{P}_n| = |P_n|$ and $|\mathcal{P}_{n,k}| = |P_{n,k}|$. Note that two partitions $\mathbf{a}$ and $\mathbf{b}$ t-intersect if and only if $|S_{\mathbf{a}} \cap S_{\mathbf{b}}| \geq t$. Thus, for any $A \subseteq P_{n,k}$, A is a t-intersecting subset of $P_{n,k}$ if and only if $\{S_{\mathbf{a}} \colon \mathbf{a} \in A\}$ is a t-intersecting subfamily of $\mathcal{P}_{n,k}$.

EKR-type results have been obtained in a wide variety of contexts; many of them are outlined in [8, 12, 10, 15, 16, 5, 6, 14]. Usually the objects have symmetry properties (see [7, Section 3.2] and [19]) or enable the use of *compression* (also called *shifting*) to push t-intersecting families towards a desired form (see [12, 17, 15]). One of the main motivating factors behind this paper is that, similarly to the case of [4], although the family $\mathcal{P}_{n,k}$ does not have any of these structures and attributes, and we do not even know its size precisely, we can still determine the largest t-intersecting subfamilies for n sufficiently large.

We now start working towards proving Theorem 1.8.

2 The values $p_{n,k}$

In this section, we provide relations among the values $p_{n,k}$. The relations will be needed in the proof of Theorem 1.8.

Lemma 2.1 *If $k \leq m \leq n$, then $p_{m,k} \leq p_{n,k}$. Moreover, if $3 \leq k \leq m < n$ and $n \geq k+2$, then $p_{m,k} < p_{n,k}$.*

Proof. Let $k \leq m \leq n$. If $k = 1$, then $p_{m,k} = 1 = p_{n,k}$. Suppose $k \geq 2$. Let $f\colon P_{m,k} \to P_{n,k}$ be the function that maps $(a_1, \dots, a_k) \in P_{m,k}$ to the partition $(b_1, \dots, b_k) \in P_{n,k}$ with $b_k = a_k + n - m$ and $b_i = a_i$ for each $i \in [k-1]$. Clearly, f is one-to-one, and hence the size of its domain $P_{m,k}$ is at most the size of its co-domain $P_{n,k}$. Therefore, $p_{m,k} \leq p_{n,k}$.

Suppose $3 \leq k \leq m < n$ and $n \geq k+2$. Let $\mathbf{c} = (c_1, \dots, c_k)$ with $c_i = 1$ for each $i \in [k-3]$,

$$c_{k-2} = \begin{cases} 1 & \text{if } n-k \text{ is even} \\ 2 & \text{if } n-k \text{ is odd,} \end{cases} \quad \text{and} \quad c_{k-1} = c_k = \begin{cases} (n-k+2)/2 & \text{if } n-k \text{ is even} \\ (n-k+1)/2 & \text{if } n-k \text{ is odd.} \end{cases}$$

Then $\mathbf{c} \in P_{n,k}$. Since $m < n$, f maps $(a_1, \dots, a_k) \in P_{m,k}$ to a partition $(b_1, \dots, b_k)$ with $b_{k-1} < b_k$. Hence $\mathbf{c}$ is not in the range of f. Thus, f is not onto, and hence its domain $P_{m,k}$ is smaller than its co-domain $P_{n,k}$. Therefore, $p_{m,k} < p_{n,k}$. $\square$

Lemma 2.2 *If $k \geq 2$, $c \geq 1$, and $n \geq ck^3$, then*

$$p_{n,k} > cp_{n,k-1} \geq cp_{n-1,k-1}.$$

Proof. If $k = 2$, then $p_{n,k-1} = 1$, $p_{n,k} = \lfloor n/2 \rfloor \geq (n-1)/2 \geq ck^3/2 - 1/2 > 3c$, and hence $p_{n,k} > cp_{n,k-1}$.

Now consider $k \geq 3$. For each $i \in [ck^2]$, let

$$X_i = \{(i, a_1, \dots, a_{k-2}, a_{k-1} - i)\colon (a_1, \dots, a_{k-1}) \in P_{n,k-1}\}.$$

Let $X = \bigcup_{i=1}^{ck^2} X_i$.

For any k-tuple $\mathbf{x} = (x_1, \dots, x_k)$ of integers, let $\vec{\mathbf{x}}$ be the k-tuple obtained by putting the entries of $\mathbf{x}$ in increasing order; that is, $\vec{\mathbf{x}}$ is the k-tuple $(x_1', \dots, x_k')$ such that $x_1' \leq \cdots \leq x_k'$ and $|\{i \in [k]\colon x_i' = x\}| = |\{i \in [k]\colon x_i = x\}|$ for each $x \in \{x_1, \dots, x_k\}$.

Let $\mathbf{a}$ be a partition $(a_1, \dots, a_{k-1})$ in $P_{n,k-1}$. Since $a_1 \leq \cdots \leq a_{k-1}$ and $a_1 + \cdots + a_{k-1} = n$, we have $a_{k-1} \geq \frac{n}{k-1}$, and hence, since $n \geq ck^3$, $a_{k-1} > ck^2$. Thus, $a_{k-1} - i \geq 1$ for each $i \in [ck^2]$, meaning that the entries of each tuple in X are positive integers that add up to n. Therefore,

$$\vec{\mathbf{x}} \in P_{n,k} \text{ for each } \mathbf{x} \in X. \tag{1}$$

Let $Y = \{\mathbf{y} \in P_{n,k}\colon \mathbf{y} = \vec{\mathbf{x}} \text{ for some } \mathbf{x} \in X\}$. For each $\mathbf{y} \in Y$, let $X_{\mathbf{y}} = \{\mathbf{x} \in X\colon \vec{\mathbf{x}} = \mathbf{y}\}$. By (1), $X = \bigcup_{\mathbf{y} \in Y} X_{\mathbf{y}}$.

Consider any partition $\mathbf{y} = (y_1, \dots, y_k)$ in Y. Clearly, each element of $X_{\mathbf{y}}$ is in one of $X_{y_1}, \dots, X_{y_k}$; that is, $X_{\mathbf{y}} \subseteq \bigcup_{i=1}^{k} X_{y_i}$. Thus, $X_{\mathbf{y}} = \bigcup_{i=1}^{k} (X_{\mathbf{y}} \cap X_{y_i})$. Let $i \in [k]$ such that $X_{\mathbf{y}} \cap X_{y_i} \neq \emptyset$. Let $\mathbf{x}$ be a tuple $(x_1, \dots, x_k)$ in $X_{\mathbf{y}} \cap X_{y_i}$. By definition, $x_1 = y_i$ and $x_2 \leq \dots \leq x_{k-1}$. Thus, since $y_1 \leq \dots \leq y_k$ and $\mathbf{y} = \vec{\mathbf{x}}$, $\mathbf{x}$ is one of the $k-1$ k-tuples satisfying the following: the first entry is y_i, the k-th entry is y_j for some $j \in [k] \backslash \{i\}$, and the middle $k-2$ entries form the $(k-2)$-tuple obtained by deleting the i-th entry and the j-th entry of $\mathbf{y}$. Hence $|X_{\mathbf{y}} \cap X_{y_i}| \leq k-1$.

Therefore, we have

$$\begin{aligned} |X| &= \left| \bigcup_{\mathbf{y} \in Y} X_{\mathbf{y}} \right| \leq \sum_{\mathbf{y} \in Y} |X_{\mathbf{y}}| \\ &\leq \sum_{\mathbf{y} \in Y} \sum_{i=1}^{k} |X_{\mathbf{y}} \cap X_{y_i}| \leq \sum_{\mathbf{y} \in Y} \sum_{i=1}^{k} (k-1) \\ &= k(k-1)|Y| < k^2 |P_{n,k}|, \end{aligned}$$

and hence $p_{n,k} > \frac{|X|}{k^2}$. Now $X_1, \dots, X_{ck^2}$ are pairwise disjoint sets, each of size $p_{n,k-1}$. Thus, $|X| = ck^2 p_{n,k-1}$, and hence $p_{n,k} > cp_{n,k-1}$. By Lemma 2.1, $p_{n,k-1} \geq p_{n-1,k-1}$. Hence the result. □

In view of the result above, we pose the following problem.

Problem 2.3 *For $k \geq 2$ and $c \geq 1$, let $\rho(k, c)$ be the smallest integer m such that $p_{n,k} \geq cp_{n,k-1}$ for every $n \geq m$. What is the value of $\rho(k, c)$?*

Lemma 2.2 tells us that $\rho(k, c) \leq ck^3$. As can be seen from the proof of Theorem 1.8, an improvement of this inequality automatically yields an improved condition for n in the theorem.

3 Proof of Theorem 1.8

We now prove Theorem 1.8.

For a family $\mathcal{F}$ and a set T, let $\mathcal{F}\langle T \rangle$ denote the family $\{F \in \mathcal{F} \colon T \subseteq F\}$. If $|T| = t$, then $\mathcal{F}\langle T \rangle$ is called a *t-star of $\mathcal{F}$*. We denote the size of a largest t-star of $\mathcal{F}$ by $\tau(\mathcal{F}, t)$. A t-intersecting family $\mathcal{A}$ is said to be *trivial* if $|\bigcap_{A \in \mathcal{A}} A| \geq t$ (that is, if the sets in $\mathcal{A}$ have at least t common elements); otherwise, $\mathcal{A}$ is said to be a *non-trivial* t-intersecting family. Note that a non-empty t-star is a trivial t-intersecting family.

We call a family $\mathcal{F}$ *k-uniform* if $|F| = k$ for each $F \in \mathcal{F}$.

Generalising a theorem in [3, page 48], we obtain the following lemma.

Lemma 3.1 *If $k \geq t$, $\mathcal{A}$ is a non-trivial t-intersecting subfamily of a k-uniform family $\mathcal{F}$, and $\mathcal{A}$ is not a $(t+1)$-intersecting family, then*

$$|\mathcal{A}| \leq k\tau(\mathcal{F}, t+1) + \sum_{i=1}^{t} \binom{t}{i} \binom{k-t}{i}^2 \tau(\mathcal{F}, t+i).$$

Proof. Since $\mathcal{A}$ is t-intersecting and not $(t+1)$-intersecting, there exist $A_1, A_2 \in \mathcal{A}$ such that $|A_1 \cap A_2| = t$. Let $B = A_1 \cap A_2$. Since $\mathcal{A}$ is not a trivial t-intersecting family, there exists $A_3 \in \mathcal{A}$ such that $B \not\subseteq A_3$. For each $i \in \{0\} \cup [t]$, let $\mathcal{A}_i = \{A \in \mathcal{A} \colon |A \cap B| = t - i\}$.

Consider any $i \in [t]$. For each $A \in \mathcal{A}_i$, we have $t \leq |A \cap A_1| = |A \cap B| + |A \cap (A_1 \backslash B)| = t - i + |A \cap (A_1 \backslash B)|$, so $|A \cap (A_1 \backslash B)| \geq i$. Similarly, $|A \cap (A_2 \backslash B)| \geq i$ for each $A \in \mathcal{A}_i$. Thus,

$$\begin{aligned}\mathcal{A}_i &\subseteq \{F \in \mathcal{F} \colon |F \cap B| = t - i,\, |F \cap (A_1 \backslash B)| \geq i,\, |F \cap (A_2 \backslash B)| \geq i\} \\ &= \bigcup_{X \in \binom{B}{t-i}} \bigcup_{Y \in \binom{A_1 \backslash B}{i}} \bigcup_{Z \in \binom{A_2 \backslash B}{i}} \mathcal{F}\langle X \cup Y \cup Z \rangle,\end{aligned}$$

and hence

$$\begin{aligned}|\mathcal{A}_i| &\leq \sum_{X \in \binom{B}{t-i}} \sum_{Y \in \binom{A_1 \backslash B}{i}} \sum_{Z \in \binom{A_2 \backslash B}{i}} |\mathcal{F}\langle X \cup Y \cup Z \rangle| \\ &\leq \binom{|B|}{t-i}\binom{|A_1 \backslash B|}{i}\binom{|A_2 \backslash B|}{i}\tau(\mathcal{F}, t+i) = \binom{t}{i}\binom{k-t}{i}^2 \tau(\mathcal{F}, t+i).\end{aligned}$$

For each $A \in \mathcal{A}_0$, we have $|A \cap B| = t$ and $t \leq |A \cap A_3| = |A \cap (A_3 \cap B)| + |A \cap (A_3 \backslash B)| \leq |A_3 \cap B| + |A \cap (A_3 \backslash B)| \leq t - 1 + |A \cap (A_3 \backslash B)|$, and hence $B \subseteq A$ and $|A \cap (A_3 \backslash B)| \geq 1$. Thus,

$$\mathcal{A}_0 \subseteq \{F \in \mathcal{F} \colon B \subseteq F,\, |F \cap (A_3 \backslash B)| \geq 1\} = \bigcup_{X \in \binom{A_3 \backslash B}{1}} \mathcal{F}\langle B \cup X \rangle,$$

and hence

$$|\mathcal{A}_0| \leq \sum_{X \in \binom{A_3 \backslash B}{1}} |\mathcal{F}\langle B \cup X \rangle| \leq |A_3 \backslash B| \tau(\mathcal{F}, t+1) \leq k\tau(\mathcal{F}, t+1).$$

Since $\mathcal{A} = \bigcup_{i=0}^{t} \mathcal{A}_i$, the result follows. □

Define $\mathcal{P}_{n,k}$ and f as in Remark 1.10. Let $T_t = \{(1, i) \colon i \in [t]\}$. Note that $\{f(\mathbf{a}) \colon \mathbf{a} \in P_{n,k}(t)\} = \mathcal{P}_{n,k}\langle T_t \rangle$. Since f is a bijection, it follows that $|\mathcal{P}_{n,k}\langle T_t \rangle| = |P_{n,k}(t)|$. Therefore, $|\mathcal{P}_{n,k}\langle T_t \rangle| = p_{n-t,k-t}$ if $t < k \leq n$.

Lemma 3.2 *If $t + 1 \leq k \leq n$, then $\mathcal{P}_{n,k}\langle T_t \rangle$ is a largest t-star of $\mathcal{P}_{n,k}$, and uniquely so if $k \geq t + 3$ and $n \geq k + 2$.*

Proof. Let $\mathcal{A}$ be a t-star of $\mathcal{P}_{n,k}$, so $\mathcal{A} = \mathcal{P}_{n,k}\langle T^* \rangle$ for some t-element set T^*. Let $A = \{\mathbf{a} \in P_{n,k} \colon f(\mathbf{a}) = E \text{ for some } E \in \mathcal{A}\}$. Since f is a bijection, $|A| = |\mathcal{A}|$. Let $(e_1, i_1), \dots, (e_t, i_t)$ be the elements of T^*. By definition of A, t of the entries of each partition in A are $e_1, \dots, e_t$. Thus, $|A| \leq p_{n-q,k-t}$, where $q = \sum_{j=1}^{t} e_j \geq t$. By Lemma 2.1, we have $|A| \leq p_{n-t,k-t}$, and hence $|\mathcal{A}| \leq |\mathcal{P}_{n,k}\langle T_t \rangle|$.

Suppose $k \geq t+3$ and $n \geq k+2$. If $q > t$, then, by Lemma 2.1, we have $p_{n-q,k-t} < p_{n-t,k-t}$, and hence $|\mathcal{A}| < |\mathcal{P}_{n,k}\langle T_t \rangle|$. Suppose $q = t$. Then $e_1 = \cdots = e_t = 1$, and hence $A \subseteq P_{n,k}(t)$. Therefore, $\mathcal{A} \subseteq \mathcal{P}_{n,k}\langle T_t \rangle$. □

A t-intersecting subset A of $P_{n,k}$ is *maximal* if there is no t-intersecting subset B of $P_{n,k}$ such that A is a proper subset of B.

Lemma 3.3 *If $k \geq t$, $n > 2k^2$, and A is a maximal t-intersecting subset of $P_{n,k}$, then A is not $(t+2)$-intersecting.*

Proof. Clearly, $A \neq \emptyset$, so there exists $l \in [k]$ such that A is l-intersecting and not $(l+1)$-intersecting. Thus, there exist $\mathbf{a} = (a_1, \dots, a_k)$ and $\mathbf{b} = (b_1, \dots, b_k)$ in A such that $\mathbf{a}$ l-intersects $\mathbf{b}$ and does not $(l+1)$-intersect $\mathbf{b}$. Suppose $l \geq t+2$. Let $X = \{a_1, \dots, a_k\}$ and $Y = \{b_1, \dots, b_k\}$. Then $X \cap Y \neq \emptyset$. Let $z \in X \cap Y$. Then $z = a_j$ for some $j \in [k]$.

For any k-tuple $\mathbf{x} = (x_1, \dots, x_k)$ of integers, $\vec{\mathbf{x}}$ denotes the k-tuple obtained by putting the entries of $\mathbf{x}$ in increasing order, as in the proof of Lemma 2.2.

Suppose $a_j > 2k$. We have $k \geq l \geq t+2 \geq 3$. Let $h \in [k]\backslash\{j\}$. Let $H = \{i \in \mathbb{N} \colon a_j - i \in Y\backslash\{a_j\} \text{ or } a_h + i \in Y\}$, $I = \{i \in \mathbb{N} \colon a_j - i \in Y\backslash\{a_j\}\}$, and $J = \{i \in \mathbb{N} \colon a_h + i \in Y\}$. Since $H = I \cup J$, $|H| \leq |I| + |J| \leq |Y\backslash\{a_j\}| + |Y| \leq 2k-1$. Thus, there exists $i \in [2k]$ such that $i \notin H$, meaning that $a_j - i \notin Y\backslash\{a_j\}$ (so $a_j - i \notin Y$) and $a_h + i \notin Y$. Let $c_j = a_j - i$, $c_h = a_h + i$, and $c_r = a_r$ for each $r \in [k]\backslash\{j,h\}$. Let $\mathbf{c} = (c_1, \dots, c_k)$. Since $c_j > 0$ and $\sum_{r=1}^{k} c_r = \sum_{r=1}^{k} a_r = n$, we have $\vec{\mathbf{c}} \in P_{n,k}$. Let $B = A \cup \{\vec{\mathbf{c}}\}$. Since A is $(t+2)$-intersecting, B is a t-intersecting subset of $P_{n,k}$. Since $a_j \in Y$, $c_j, c_h \notin Y$, and $\mathbf{a}$ does not $(l+1)$-intersect $\mathbf{b}$, $\vec{\mathbf{c}}$ does not l-intersect $\mathbf{b}$. Thus, $\vec{\mathbf{c}} \notin A$ as A is l-intersecting. Thus, we have $A \subsetneq B$, which contradicts the assumption that A is a maximal t-intersecting subset of $P_{n,k}$.

Therefore, $a_j \leq 2k$. Since $n > 2k^2$, $a_j < n/k$. Since $\sum_{r=1}^{k} a_r = n$, there exists $h \in [k]\backslash\{j\}$ such that $a_h \geq n/k$. Thus, $a_h > 2k$. Let $H = \{i \in \mathbb{N} \colon a_j + i \in Y\backslash\{a_j\} \text{ or } a_h - i \in Y\}$, $I = \{i \in \mathbb{N} \colon a_j + i \in Y\backslash\{a_j\}\}$, and $J = \{i \in \mathbb{N} \colon a_h - i \in Y\}$. Since $H = I \cup J$, $|H| \leq |I| + |J| \leq 2k-1$. Thus, there exists $i \in [2k]$ such that $a_j + i \notin Y\backslash\{a_j\}$ (so $a_j + i \notin Y$) and $a_h - i \notin Y$. Let $c_j = a_j + i$, $c_h = a_h - i$, and $c_r = a_r$ for each $r \in [k]\backslash\{j,h\}$. Let $\mathbf{c} = (c_1, \dots, c_k)$. Let $B = A \cup \{\vec{\mathbf{c}}\}$. As above, we obtain that B is a t-intersecting subset of $P_{n,k}$ with $A \subsetneq B$, a contradiction.

Therefore, $l < t+2$, and hence the result. □

A closer attention to detail could improve the condition on n in the lemma above, but this alone would not strengthen Theorem 1.8. We now have all the tools needed for the proof of the theorem.

Proof of Theorem 1.8. Let $k \geq t+2$ and $n \geq 3tk^5$. Let $c_t = \frac{8}{7}(t+1)$. Then $n \geq c_t k^5$.

Let A be a largest t-intersecting subset of $P_{n,k}$. Clearly, $A \neq \emptyset$, so there exists $l \in [k]\backslash[t-1]$ such that A is l-intersecting and not $(l+1)$-intersecting. Thus, A is a largest l-intersecting subset of $P_{n,k}$. Let $\mathcal{A} = \{f(\mathbf{a}) \colon \mathbf{a} \in A\}$. Clearly, $|\mathcal{A}| = |A|$ (since f is a bijection) and $\mathcal{A}$ is k-uniform. By Remark 1.10, $\mathcal{A}$ is a largest l-intersecting subfamily of $\mathcal{P}_{n,k}$, and $\mathcal{A}$ is not $(l+1)$-intersecting. By Lemma 3.3, $l \in \{t, t+1\}$. By Lemma 3.2, $\tau(\mathcal{P}_{n,k}, i) = |\mathcal{P}_{n,k}\langle T_i \rangle| = p_{n-i,k-i}$ for each $i \in [k-1]$.

Suppose that $\mathcal{A}$ is a non-trivial l-intersecting family. Then $|\mathcal{A}| > 1$. As explained in Remark 1.9, the non-empty l-intersecting subsets of $P_{n,l+1}$ are the 1-element subsets, and hence the non-empty l-intersecting subfamilies of $\mathcal{P}_{n,l+1}$ are the subfamilies of size 1. Trivially, the same holds for l-intersecting subfamilies of $\mathcal{P}_{n,l}$. Since $\mathcal{A} \subseteq \mathcal{P}_{n,k}$ and $|\mathcal{A}| > 1$, it follows that $k \geq l+2$.

Let $m = \max\{l, k-l\}$. For each $i \in [m]$, let

$$s_i = \binom{l}{i}\binom{k-l}{i}^2 \tau(\mathcal{P}_{n,k}, l+i).$$

By Lemma 3.1, $|\mathcal{A}| \leq k\tau(\mathcal{P}_{n,k}, l+1) + \sum_{i=1}^{l} s_i$. Clearly, $\tau(\mathcal{P}_{n,k}, l+i) \neq 0$ if and only if $i \leq k-l$. Thus,

$$|\mathcal{A}| \leq k\tau(\mathcal{P}_{n,k}, l+1) + \sum_{i=1}^{k-l} s_i = kp_{n-l-1,k-l-1} + s_{k-l} + \sum_{i=1}^{k-l-1} s_i. \tag{2}$$

Consider any $i \in [k-l-1]$. Suppose $i \leq k-l-2$. If $i \geq l$, then $s_{i+1} = 0$. Suppose $i < l$. We have

$$s_{i+1} = \frac{(l-i)(k-l-i)^2 p_{n-l-i-1,k-l-i-1}}{(i+1)^3 p_{n-l-i,k-l-i}} s_i.$$

Since $n-l-i \geq c_t k^5 - l - i > c_t k^2 (k-l-i)^3 > (l-i)(k-l-i)^2(k-l-i)^3$, we have $p_{n-l-i,k-l-i} > (l-i)(k-l-i)^2 p_{n-l-i-1,k-l-i-1}$ by Lemma 2.2. Thus, $s_{i+1} < s_i/(i+1)^3$. Now suppose $i = k-l-1$. Then $\tau(\mathcal{P}_{n,k}, l+i+1) = \tau(\mathcal{P}_{n,k}, k) = 1$ and $\tau(\mathcal{P}_{n,k}, l+i) = \tau(\mathcal{P}_{n,k}, k-1) = p_{n-(k-1),k-(k-1)} = p_{n-k+1,1} = 1$. If $i \geq l$, then $s_{i+1} = 0$. If $i < l$, then

$$s_{i+1} = s_{k-l} = \frac{l-(k-l-1)}{(k-l)^3} s_{k-l-1} < \frac{l}{(k-l)^3} s_{k-l-1}.$$

We have therefore shown that $s_{i+1} \leq s_i/(i+1)^3$ for any $i \in [k-l-2]$, and that $s_{k-l} \leq (l/(k-l)^3) s_{k-l-1}$. It follows that $s_i \leq s_1/(i!)^3$ for any $i \in [k-l-1]$. Thus,

$$\begin{aligned} s_{k-l} &\leq \frac{l}{(k-l)^3} \frac{s_1}{((k-l-1)!)^3} = \frac{l^2(k-l)^2 p_{n-l-1,k-l-1}}{(k-l)^3((k-l-1)!)^3} \\ &= \frac{l^2 p_{n-l-1,k-l-1}}{(k-l)((k-l-1)!)^3} \leq \frac{l^2}{2} p_{n-l-1,k-l-1}, \end{aligned}$$

and $s_i \leq s_1/(2^{i-1})^3 = s_1/8^{i-1}$ for any $i \in [k-l-1]$. We have

$$\sum_{i=1}^{k-l-1} s_i \leq s_1 \sum_{i=1}^{k-l-1} \left(\frac{1}{8}\right)^{i-1} < s_1 \sum_{i=0}^{\infty} \left(\frac{1}{8}\right)^{i} = \frac{8}{7} s_1 = \frac{8}{7} l(k-l)^2 p_{n-l-1,k-l-1}.$$

Thus, by (2), $|\mathcal{A}| < \left(k + \frac{l^2}{2} + \frac{8}{7}l(k-l)^2\right) p_{n-l-1,k-l-1}$. Since $l \in \{t, t+1\}$ and $k \geq l+2$, we have

$$k + \frac{l^2}{2} + \frac{8}{7}l(k-l)^2 = k + \frac{8}{7}lk^2 - \frac{8}{7}l^2\left(2k - l - \frac{7}{16}\right) < k + c_t k^2 - \frac{8}{7}l^2 k < c_t k^2.$$

By Lemma 2.2, $p_{n-l,k-l} > c_t k^2 p_{n-l-1,k-l-1}$ as $n - l \geq c_t k^5 - l > c_t k^2 (k-l)^3$. Thus, we have $|\mathcal{A}| < p_{n-l,k-l} = \tau(\mathcal{P}_{n,k}, l)$, which is a contradiction as $\mathcal{A}$ is a largest l-intersecting subfamily of $\mathcal{P}_{n,k}$.

Therefore, $\mathcal{A}$ is a trivial l-intersecting family. Consequently, $\mathcal{A}$ is a largest l-star of $\mathcal{P}_{n,k}$. By Lemma 3.2, $|\mathcal{A}| = |\mathcal{P}_{n,k}\langle T_l \rangle|$. Since $n - t \geq c_t k^5 - t > (k-t)^3$, $p_{n-t,k-t} > p_{n-(t+1),k-(t+1)}$ by Lemma 2.2. Since $l \in \{t, t+1\}$ and $|A| = |\mathcal{A}| = |\mathcal{P}_{n,k}\langle T_l \rangle| = p_{n-l,k-l} \leq p_{n-t,k-t} = |P_{n,k}(t)| \leq |A|$, it follows that $l = t$ and $|A| = |P_{n,k}(t)|$. By Lemma 3.2, $A = P_{n,k}(t)$ if $k \geq t+3$. □

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