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Bipartite edge partitions and the former Alon-Saks-Seymour conjecture

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Abstract

A famous result of Graham and Pollak states that the complete graph with n vertices can be edge partitioned into $n-1$, but no fewer, complete bipartite graphs. This result has led to the study of the relationship between the chromatic and biclique partition numbers of graphs. It has become even more exciting with recent connections to the clique versus stable set problem, communication protocols and constraint satisfaction and homomorphism problems. By defining an extended hypercube we construct a framework that provides much structural information regarding the relationship between these two parameters and a third, the induced bipartite edge partition number. Using this we show that the minimum counterexample to the former Alon-Saks-Seymour conjecture must have biclique partition number at least 10. Finally we identify a family of graphs to investigate for a smaller counterexample to the former Alon-Saks-Seymour conjecture.

1 Introduction

Given a graph G the *biclique partition number* of G, denoted by $bp(G)$ is the minimum number of complete bipartite graphs to which edges of G can be partitioned. It is a famous theorem of Graham and Pollak that $bp(K_n) = n - 1$ [13, 14] where K_n is the complete graph on n vertices. The early proofs of this simple statement were based on linear algebra. There have been some movement towards more combinatorial proofs [19]. The search for a simple combinatorial proof has resulted in development of several extensions of the theorem. For example it is proved in [4] that if the edges of a complete graph are coloured such that each colour class induces a complete bipartite graph, then there is a spanning tree whose edges have all distinct colours. Other extensions exist to hypergraphs [1, 7] and biclique coverings [2, 9].

Noting that $\chi(K_n) = n$, one could consider extending the theorem to the question of bounding $\chi(G)$ by a function of $bp(G)$. To this end Alon, Saks and Seymour [16] made the following

Conjecture 1.1. *For every graph G, if $bp(G) = n$ then $\chi(G) \leq n + 1$.*

This conjecture has recently been disproved for general n by Huang and Sudakov [15], but we will show that it holds for small n. Specifically we will prove

Theorem 1.2. *Conjecture 1.1 holds for $n \leq 9$.*

We note that Theorem 1.2 is an improvement on the earlier result of Rho [18] which states that Conjecture 1.1 holds for $n \leq 4$. After developing our framework, Rho's result will follow simply by considering the degrees in digraphs with fewer than 5 vertices and their complements.

The question of what is the best upper bound for $\chi(G)$ in terms of $bp(G)$, and what is their relationship in general, remains open. These questions have recently been connected to other problems, namely: the clique versus stable set problem and the stubborn problem. The former requires finding cuts which separate cliques from stable sets and the latter requires covering the set of solutions of a constraint satisfaction problem by 2-SAT instances [6]. A polynomial upper bound for $\chi(G)$ in terms of $bp(G)$ would be equivalent to a polynomial number of cuts and equivalent to a polynomial covering. While the authors of [6] hope for a polynomial bound for $\chi(G)$ in terms of $bp(G)$, the authors of [15] conjecture that there exists a graph with $\chi(G) \geq 2^{c(\log bp(G))^2}$ for some positive constant c [15] (*throughout the paper,* *all logarithms are to the base 2*). That $2^{O((\log bp(G))^2)}$ is an upper bound was proven in [3, 17, 22]. Alon and Haviv made the connection between the communication results of Yannakakis and decomposition into bicliques [3]. Recent papers by Göös and by Fiorini et al. show that there exists graphs, G with $\chi(X)$ superpolynomially larger than $bp(G)$ [11, 12].

Another question of interest is to find the smallest counterexample to the Alon-Saks-Seymour conjecture. The disproof of the conjecture constructs a graph on 91^7 vertices with $bp(G) \leq 30 \times 91^5 < \chi(G)$. Smaller counterexamples with about 25^5 vertices were found in [8]. We hope that a smaller counterexample could be found, ideally one that is simply described.

Consider a graph with a fixed biclique partition number, $bp(G) = k$. The neighbourhood of a vertex is completely determined by which side, if any, it appears in each of the k complete bipartite graphs. Thus there are at most 3^k different vertex neighbourhoods and if G has more vertices than this, then there must be two vertices with the same neighbourhood. In any proper colouring of G, these two vertices may be given the same colour. Thus when considering the values of $\chi(G)$ when $bp(G) = k$ is fixed it is only necessary to check those G with no more than 3^k vertices, a large but finite set of graphs. However nice, in principle, it is that only a finite number of computations are necessary, this number of graphs is still very large and more reductions are needed to take advantage of this fact.

In this paper, we study this question from a graph homomorphism point of view. First we introduce another notation. An *induced bipartite subgraph* of a graph G is a bipartite subgraph B with bipartition (X, Y) such that if $x \in X$ and $y \in Y$, then $\{x, y\}$ is an edge of B if and only if it is an edge of G. Note that an induced bipartite subgraph is not necessarily an induced subgraph as G may have an edge inside X or inside Y which is not in B. We define the *induced bipartite edge–partition number*, denoted $ibp(G)$, to be the smallest number of induced bipartite subgraphs which partition the edges of G. It follows from the definition that $ibp(G) \leq bp(G)$.

We build a graph $EC(k)$ such that $ibp(G) \leq k$ if and only if G admits a homomorphism to $EC(k)$. Recall that a homomorphism is an edge-preserving mapping of vertices. We then determine a set $\mathcal{S}$ of induced subgraphs of $EC(k)$ such that $bp(G) \leq k$ if and only if there exists a homomorphism of G to one of $\mathcal{S}$. Thus finding the best bound for $\chi(G)$ in terms of $bp(G)$, is equivalent to finding the maximum chromatic number of the subgraphs in $\mathcal{S}$. However $\mathcal{S}$ is large, there are many isomorphic subgraphs in $\mathcal{S}$. Additionally the graphs in $\mathcal{S}$ are usually not minimal with respect to their chromatic number. We introduce techniques to produce a smaller set of graphs for this purpose and we propose constructions that we believe will achieve the largest chromatic number among graphs with $bp(G) = p = 4k + 1$ where p is a prime power. Finally, using some computational work on this reduced set of graphs we show that the conjecture of Alon-Saks-Seymour does hold for $bp(G) \leq 9$. This means that the smallest counterexample must have biclique partition number at least 10.

From the structure of the graph family we construct and motivated by some of the

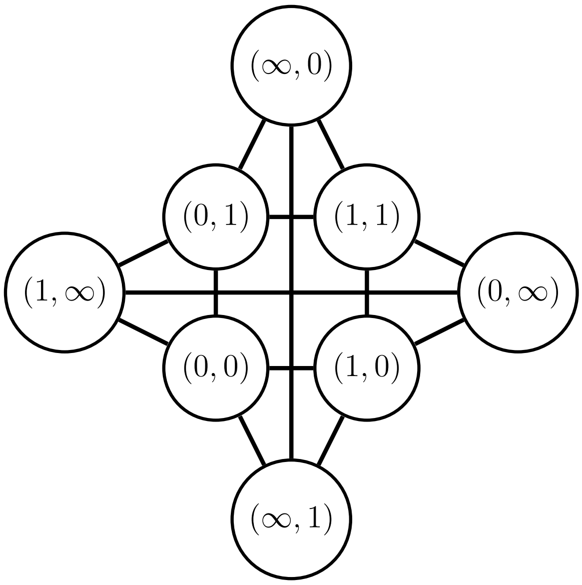

Figure 1: The extended cube $EC(2)$

computational results we propose a place to search for small counterexamples to the Alon-Saks-Seymour conjecture. These would substantially close the gap discussed above. They would also begin to tell us more about the correct relationship between $bp(G)$ and $\chi(G)$.

2 Extended cubes and homomorphisms

Let $\mathcal{M} = (\mathbb{Z}_2 \cup \{\infty\}, +, 0)$ be a monoid in which addition of elements in $\mathbb{Z}_2 = \{0, 1\}$ is done in the integers modulo 2 and $x + \infty = \infty$ for every x. Let $\mathcal{B}$ be the set of all elements of $\mathcal{M}^n$ that have a single 1 in their coordinates ($\mathcal{B}$ has $n2^{n-1}$ elements). We define $EC(n)$ (the *extended cube* of dimension n) to be the graph whose vertex set is $\mathcal{M}^n - \vec{\infty}$, $\vec{\infty} = (\infty, \infty, \ldots \infty)$, with two vertices x and y being adjacent if and only if $x + y \in \mathcal{B}$. Note that the hypercube and squashed cubes [13, 20] of dimension n are induced subgraphs of $EC(n)$ (that have no vertices whose distance apart are 0, see below). The graph $EC(2)$ is shown in Figure 1.

One can extend the definition of Hamming distance on hypercubes to $\mathcal{M}^n$ by assuming that the distance between ∞ and anything else is 0. For example in $EC(2)$ the distance between $(\infty, 0)$ and $(1, \infty)$ is 0 and there is no edge between these two vertices. The distance between $(\infty, 0)$ and $(1, 1)$ is 1 and there is an edge between these two vertices. This definition of distance is convenient although (as it is not actually a metric) it abuses the usual definition of a distance function. It was introduced by Graham and Pollak [13] for its important application to the isometric embedding of graphs in squashed cubes (as a next best possibility for graphs not embeddable in a hypercube).

With this notation, $EC(n)$ is a graph on $\mathcal{M}^n - \vec{\infty}$ with x and y being adjacent if they are at Hamming distance exactly 1. A subgraph of $EC(n)$ in which every pair of vertices are at Hamming distance at most k is called a k-*ball* subgraph. The graph $EC(n)$ has $3^n - 1$ vertices, it is connected and, moreover, it is of diameter 2 as the following proposition shows.

Proposition 2.1. *The graph $EC(n)$ is of diameter 2 for $n \geq 2$.*

Proof. Let x and y be two distinct vertices of $EC(n)$. If x and y are not adjacent then we can find coordinates i and j such that neither x_i nor y_j is ∞, i.e., they are 0 or 1. Now let a be the vertex with $a_i = 1 + x_i$ and $a_k = \infty$ for $k \neq i$. Let b the vertex with $b_j = 1 + y_j$ and $b_k = \infty$ for $k \neq j$. By construction a is adjacent to x and b is adjacent to y. If either a or b is adjacent to both x and y, then we are done. Otherwise the vertex c defined by $c_i = 1 + x_i$, $c_j = 1 + y_j$ and $c_k = \infty$ for $k \neq i, j$ will be adjacent to both. □

The next two lemmas show the first connection between $EC(n)$ and edge partitioning into certain bipartite subgraphs.

Lemma 2.2. *The edge set of $EC(n)$ can be partitioned into n induced bipartite subgraphs.*

Proof. Let B_i be the bipartite subgraph with bipartition (X_i, Y_i), where X_i consists of all vertices $x \in \mathcal{M}^n - \vec{\infty}$, with $x_i = 0$ and Y_i consists of all vertices $y \in \mathcal{M}^n - \vec{\infty}$, with $y_i = 1$. Each edge $\{x, y\}$ belongs to a unique B_i by the definition of adjacency. Thus $\{B_i\}_{i=1}^n$ partitions $E(EC(n))$ into n induced subgraphs. □

Lemma 2.3. *Each induced 1-ball subgraph of $EC(n)$ has biclique partition number at most n.*

Proof. Let M be a 1-ball subgraph of $EC(n)$. Then for each i, where $1 \leq i \leq n$, we define B_i to be a bipartite graph where X_i is the set of vertices in $V(M)$ with 0 in coordinate i and Y_i being the set of vertices in $V(M)$ with 1 in their ith coordinate. As before each edge of M belongs to a unique B_i, however the absence of any pair of vertices at Hamming distance 2 or more ensures that B_i is a complete bipartite subgraph. Hence $\{B_i\}_{i=1}^n$ is a set of edge-disjoint complete bipartite graphs partitioning $E(M)$. □

Let G be a graph with $ibp(G) = n$ and let $\{B_1, B_2, \dots B_n\}$ be a set of induced bipartite subgraphs of G partitioning $E(G)$. We define a homomorphism f of G to $EC(n)$ as follows. Given a vertex x of G, let $f(x)$ be a vertex (of $EC(n)$) whose ith coordinate is 0 if $x \in X_i$, is 1 if $x \in Y_i$, and is ∞ otherwise. To check that f is a homomorphism consider two adjacent vertices, x and y in G. Let B_i be the unique induced bipartite graph containing the edge $\{x, y\}$. By construction $f(x) + f(y) = (\infty, \infty, \dots, \infty, 1, \infty, \dots, \infty) \in \mathcal{B}$ and therefor $f(x)$ and $f(y)$ are adjacent and f is a homomorphism. Thus together with Lemma 2.2 we have:

Corollary 2.4. *The maximum possible chromatic number of a graph G with $ibp(G)$ $\leq n$ is $\chi(EC(n))$.*

Corollary 2.5. *The maximum chromatic number of a graph with $bp(G) \leq n$ is the maximum of the chromatic numbers of the 1-ball subgraphs of $EC(n)$.*

Proof. Note that each 1-ball subgraph M of $EC(n)$ satisfies $bp(M) \leq n$. On the other hand every graph G with $bp(G) \leq n$ has also $ibp(G) \leq n$ and thus admits a homomorphism to $EC(n)$. We claim that the image of G under the homomorphism is contained in a 1-ball subgraph of $EC(n)$. By contradiction, suppose that $f(x) = u$ and $f(y) = v$ are distance 2. Without loss of generality assume that $u_i = u_j = 0$ and $v_i = v_j = 1$ and for all other $l \neq i, j$ either $u_l = \infty$ or $v_l = \infty$. Thus the edge $\{x, y\}$ in G is in both bipartite graphs B_i and B_j which contradicts that the bipartite graphs decompose G. □

Without considering the isomorphisms that exist between 1-ball subgraphs, we will show that there are exactly maximal $2^{n(n-1)}$ 1-ball subgraphs. However there are many such isomorphisms and, furthermore, in many cases a 1-ball subgraph can be coloured with few colours easily. To this end in the next section we study $EC(n)$ itself.

3 Vector-domination, 1-ball subgraphs, and bases

We introduce some notation first. For a vertex $x = (x_1, x_2, \dots x_n)$ of $EC(n)$ we define $S_0(x) = \{i \mid x_i = 0\}$ and call it the 0-support of x. The 1-support, denoted by $S_1(x)$ is defined analogously. The *support* of a vertex x is $S(x) = S_0(x) \cup S_1(x)$, i.e., the set of non-infinity coordinates of x. The *rank* of a vertex is the cardinality of its support; a vertex of rank n, that is a vertex with no ∞ coordinate, is called a *full–rank* vertex. Given a pair of vertices x and y, we say that x is *support-dominated* by y if $S_0(x) \subseteq S_0(y)$ and $S_1(x) \subseteq S_1(y)$.

Lemma 3.1. *The graph $EC(n)$ is 2^n-colourable and has an independent set of size $2^n - 1$.*

Proof. Any 0-ball subgraph of $EC(n)$ is an independent set of $EC(n)$. A 0-ball subgraph obtained by the set of vertices support-dominated by a fixed full rank vertex is an independent set of size $2^n - 1$. For colouring note that c, defined by $c(x) = S_1(x)$, is a proper colouring of $EC(n)$. □

We conjecture that the independence number of $EC(n)$, $\alpha(EC(N))$, is $2^n - 1$ and that each independent set of size $2^n - 1$ is support-dominated by a full-rank vertex. This conjecture for $n \leq 4$ can be easily verified. For $n = 5$ it is verified using Maple's graph theory package. If our conjecture is true, then it will imply that $\chi(EC(n)) > (\frac{3}{2})^n$ because $\chi(G) \geq |V(G)|/\alpha(G)$ holds for every graph.

Given a vertex x of $EC(n)$, let $f_i(x)$ be a vertex obtained from x by switching 0 and 1 in the ith-coordinate, or keeping it the same if the ith coordinate is ∞. Note that f_i is an automorphism of $EC(n)$. Using these automorphisms, given a subgraph M of $EC(n)$ and a vertex x of M one can always find an isomorphic copy of M in which the image of x has an empty 1-support. With this notation we give another proof of the following lemma which was originally proved in [18].

Lemma 3.2. *Any 1-ball subgraph M of $EC(n)$ with a full rank vertex is $n+1$ colourable.*

Proof. We may consider an isomorphic copy of M which contains the vertex $\mathbf{0} = (0, 0, \dots 0)$. Since no vertex of M has Hamming distance 2 or more from $\mathbf{0}$ each vertex has 1-support of size at most 1. Therefore the colouring of $EC(n)$ given above, i.e., $c(x) = S_1(x)$, is a colouring which uses at most $n+1$ colours on M. □

Observation 3.3. *Note that in a 1-ball subgraph M, if a vertex x is support-dominated by a vertex y, then every neighbour of x in M is also a neighbour of y in M.*

In finding the chromatic number of a 1-ball subgraph we may ignore the set of rank 1 vertices as the following lemma shows.

Lemma 3.4. *Let n be the smallest integer for which Conjecture 1.1 is false. Let M be a minimal 1-ball subgraph of $EC(n)$ with $\chi(M) \geq n+2$. Then each vertex of M has rank at least* 2.

Proof. For a contradiction let x be a vertex in M that is of rank 1 and let x_i be the only non-infinity coordinate of x. We may assume, without loss of generality, that $x_i = 1$. Since M is minimal, it has no vertex support-dominated by another vertex of M. Therefore, no other vertex of M has ith coordinate 1, else x would be a support-dominated vertex. Now we may delete the ith coordinate from each vertex to obtain a subgraph M' of $EC(n-1)$. The minimality of n implies that M' is n-colourable, which induces an n-colouring of $M - x$, thus proving that M is $(n+1)$-colourable. □

Let M be a maximal 1-ball subgraph of $EC(n)$, i.e., adding any other vertex results in a pair of vertices at Hamming distance greater than 1. By Corollary 2.5, to find the maximum chromatic number among graphs with $bp(G) = n$, it is enough to find the maximum chromatic number for the set of maximal 1-ball subgraphs of $EC(n)$. Below we will show that each such subgraph is uniquely determined by its set of rank 2 vertices. A rank 2 vertex x with non-infinity coordinates at i and j will be denoted by $(i, x_i)(j, x_j)$.

Note that there are total of $2n(n-1)$ rank 2 vertices in $EC(n)$, $n(n-1)$ each of two different kinds, $\{(i,0)(j,0), (i,1)(j,1) | 1 \leq i < j \leq n\}$, and $\{(i,0)(j,1), (i,1)(j,0) | 1 \leq i < j \leq n\}$. Given a maximal 1-ball subgraph M of $EC(n)$ and a pair of indices i and j, with $1 \leq i < j \leq n$, exactly one of the two rank 2 vertices $a = (i,1)(j,1)$ and $b = (i,0)(j,0)$ can be in $V(M)$. To see that at least one of them must be in $V(M)$ note that if a is at Hamming distance 2 from a vertex a' in $V(M)$, then a' must agree with b on the ith and the jth coordinates. If b is also at Hamming distance 2 from a vertex b' in $V(M)$, then b' must agree with a on the ith and the jth coordinates. But then a' and b' are at Hamming distance at least 2, which contradicts the choice of M. Now since M is a maximal 1-ball subgraph of $EC(n)$, one of a or b must be in $V(M)$. Similarly, exactly one of $(i,1)(j,0)$ and $(i,0)(j,1)$ is in $V(M)$. Thus every maximal 1-ball set of vertices has exactly $n(n-1)$ rank 2 vertices.

On the other hand, any set B of $n(n-1)$ rank 2 vertices, pairwise at Hamming distance at most 1, determines a maximal 1-ball subgraph. To obtain such a maximal 1-ball subgraph from B we define M_B to be set of all vertices in $EC(n)$ that are at Hamming distance at most 1 from vertices in B. We claim M_B forms a maximal 1-ball subgraph.

To prove this claim, first note that the size of B forces us to have one from each pair of rank 2 vertices at Hamming distance 2. To see that no pair of vertices of M_B are at Hamming distance 2, assume, for contradiction, that a and b are a pair of vertices in M_B with Hamming distance 2. Thus there are coordinates i and j at which a and b are both not infinity and do not agree. Hence $(i, a_i)(j, a_j)$ and $(i, b_i)(j, b_j)$ are a pair of rank 2 vertices at Hamming distance 2, and therefore exactly one of them, say $(i, a_i)(j, a_j)$, is in B. However this rank 2 vertex is also at Hamming distance 2 from b, contradicting the choice of elements in M_B.

Given a maximal 1-ball subgraph M we call B, the set of rank 2 vertices in M, the *base* of M. We have proved:

Theorem 3.5. *Any maximal 1-ball subgraph is uniquely determined by its base.*

Corollary 3.6. *There are exactly $2^{n(n-1)}$ maximal 1-ball subgraphs of $EC(n)$.*

Therefore, to find the maximum chromatic number among the graphs with $bp(G)$ $= n$, it is enough to find the chromatic number of the just $2^{n(n-1)}$ maximal 1-ball subgraphs of $EC(n)$. We should mention here that, first of all, there are many isomorphisms between distinct members of this set of subgraphs. Secondly, we do not need to check the chromatic number of the actual maximal 1-ball subgraph but rather, again by Observation 3.3, to check the chromatic number of the subgraph induced by the non-support-dominated set of vertices of the maximal 1-ball subgraph.

At this stage, it is convenient to encode each base as a 2-arc-coloured tournament T whose nodes are labelled $1, 2, \dots, n$. For every pair of vertices, i and j there is exactly one arc between them. The arc will have a colour (blue or red) which encodes 00 or 11 pairs in the base. Exactly one of $(i, 0)(j, 0)$ and $(i, 1)(j, 1)$ is in the base. If $(i, 0)(j, 0)$ is in the base, then we colour the arc between i and j B (blue); if $(i, 1)(j, 1)$ is in the base, then we colour the arc between i and j R (red). Additionally the arc will have a direction (from i to j or from j to i) which encodes the 01 or 10 pairs in the base. Exactly one of $(i, 0)(j, 1)$ and $(i, 1)(j, 0)$ is in the base. If $(i, 0)(j, 1)$ and $(i, 0)(j, 0)$ are in the base then we direct this blue arc from i to j. If $(i, 0)(j, 1)$ and $(i, 1)(j, 1)$ are in the base then we direct this red arc from j to i. Similarly if $(i, 1)(j, 0)$ and $(i, 0)(j, 0)$ are in the base then we direct this blue arc from j to i. If $(i, 1)(j, 0)$ and $(i, 1)(j, 1)$ are in the base then we direct this red arc from i to j. That is, we direct blue arcs from the 0 towards the 1 and red arcs from the 1 towards the 0. This gives a one-to-one correspondence between the bases and the 2-arc-coloured tournaments. In fact there are a few ways of encoding a base in a 2-arc-coloured tournament. Our choice has the simplest description of the action of an automorphism of $EC(n)$ on the encoded 1-ball subgraph. This description is given at the end of next section, after we determine all the automorphisms of $EC(n)$.

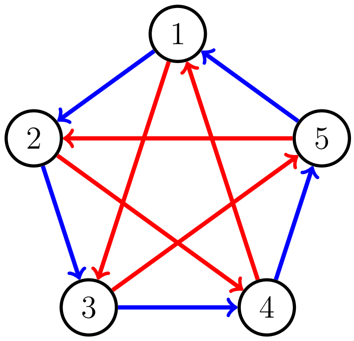

Figure 2: 2-arc-coloured tournament from Example 3.7.

Example 3.7. *Consider the 1-ball subgraph of $EC(5)$ consisting of the following vertices.*

Rank 2 vertices:
$(0, 0, \infty, \infty, \infty), (\infty, 0, 0, \infty, \infty), (\infty, \infty, 0, 0, \infty), (\infty, \infty, \infty, 0, 0), (0, \infty, \infty, \infty, 0),$
$(0, 1, \infty, \infty, \infty), (\infty, 0, 1, \infty, \infty), (\infty, \infty, 0, 1, \infty), (\infty, \infty, \infty, 0, 1), (1, \infty, \infty, \infty, 0),$
$(1, \infty, 1, \infty, \infty), (\infty, 1, \infty, 1, \infty), (\infty, \infty, 1, \infty, 1), (1, \infty, \infty, 1, \infty), (\infty, 1, \infty, \infty, 1),$
$(1, \infty, 0, \infty, \infty), (\infty, 1, \infty, 0, \infty), (\infty, \infty, 1, \infty, 0), (0, \infty, \infty, 1, \infty), (\infty, 0, \infty, \infty, 1).$

Rank 3 vertices:
$(0, 1, \infty, 1, \infty), (\infty, 0, 1, \infty, 1), (1, \infty, 0, 1, \infty), (\infty, 1, \infty, 0, 1), (1, \infty, 1, \infty, 0).$
The base of this 1-ball subgraph is the set of its rank 2 vertices. The corresponding 2-arc-coloured tournament is show in Figure 2

Another way to think about finding the chromatic number of 1-ball subgraphs comes from the following observation about their independent sets. Although we do not have a characterization of all maximal independent sets of $EC(n)$, there is an easy characterization of independent sets in each 1-ball subgraph. Since there are no two vertices at Hamming distance 2 or more in a 1-ball subgraph and since every pair of vertices at Hamming distance exactly 1 are adjacent, the only independent sets in a 1-ball subgraph are 0-ball subgraphs. It is not hard to check that the set of vertices in any 0-ball subgraph is support-dominated by a full rank vertex. Thus determining the chromatic number of any given 1-ball subgraphs is equivalent to finding the minimum set of full rank vertices (which need not be in the 1-ball subgraph) which support-dominate every vertex in the 1-ball subgraph.

4 Automorphisms of $EC(n)$

Let a be a permutation of the n coordinates of the vertices of $EC(n)$; obviously a is an automorphism of $EC(n)$. Let f_i be the function which exchanges values of 0 and 1 in the ith coordinate of each vertex of $EC(n)$, leaving it the same if the ith coordinate is ∞. As we already mentioned it is not hard to see that f_i is an automorphism of $EC(n)$. Composition of any number of these automorphisms is also an automorphism; let $\mathcal{A}(n)$ be the set of all these automorphisms. In this section we show that $\mathcal{A}(n)$ is the full automorphism group of $EC(n)$. We note that $\mathcal{A}(n)$ is isomorphic to $S_2(S_n)$, the wreath product of S_2 and S_n. For discussion of wreath products and their properties we refer the reader to [10].

Let x be a vertex of rank i in $EC(n)$. Any vertex y adjacent to x can be formed from x by exchanging 0 and 1 in one entry (i choices), then replacing some subset of the other entries in $S(x)$ by ∞ (2^{i-1} choices), then setting each entry outside $S(x)$ arbitrarily (3^{n-i} choices). Hence, a vertex of rank i has degree $d_i = i2^{i-1}3^{n-i}$. Using prime factorization, it is straightforward, although not elegant, to check that $d_i = d_j$ implies $i = j$ or $i = 2, j = 3$. Since a vertex of rank i can be mapped to any other vertex of rank i using automorphisms from $\mathcal{A}(n)$, the set of vertices of rank i ($i \neq 2, 3$) forms an orbit under the action of the automorphism group of $EC(n)$ on vertices of $EC(n)$.

Let ϕ be any automorphism of $EC(n)$. Then ϕ must map a vertex of rank 1 to a vertex of rank 1. Let (i, x_i) be a vertex of rank 1 with $x_i \neq \infty$. Then the action of ϕ on $(i, 0)$ or $(i, 1)$ completely determines how ϕ acts on the ith coordinate of each vertex. Since $\phi((i, 0))$ is a vertex of rank 1, let l be its non-infinity coordinate. We first investigate the effect of ϕ on the value, x_i. If $\phi((i, 0))$ has a 1 in its non-infinity coordinate, then ϕ must exchange every 0 and 1 in the ith coordinate of each vector in order to preserve adjacency. If the non-infinity coordinate of $\phi((i, 0))$ is a 0, then ϕ should not make any change to the ith coordinate of any vector. Note that in either case if the ith coordinate of a vertex z is ∞, then ϕ should keep it as ∞, as otherwise $\phi(z)$ will be adjacent to either $\phi((i, 0))$ or $\phi((i, 1))$, whereas z was *not* adjacent to $(i, 0)$ nor $(i, 1)$.

Now we investigate the positional effect of ϕ on x_i. If the lth coordinate of $\phi((i, 0))$ is the non-infinity coordinate, then ϕ must take the ith coordinate of each vertex to the lth coordinate (after the change from previous part). Moreover, this correspondence of i to l must be one-to-one as we are not allowed to create new adjacency. Hence we have proved:

Proposition 4.1. *The set $\mathcal{A}(n)$ of automorphisms, generated by permutations of coordinates and exchanges of* 1 *and* 0 *in a fixed coordinate, is the full automorphism group of $EC(n)$.*

Two bases are isomorphic if there is an automorphism of $EC(n)$ which maps one base to the other. We note that two bases are isomorphic if and only if their corresponding maximal 1-ball subgraphs are isomorphic. Isomorphism between two bases can also be described using their corresponding 2-arc-coloured tournaments, as follows: We note that a permutation of the coordinates of the vertices in $EC(n)$ corresponds precisely to the same permutation acting on the nodes of the tournaments. The automorphism f_i, i.e., switching 0 and 1 at coordinate i, does not change the nodes nor the directions of the arcs of the tournament, it simply switches the colour of every arc leaving node i while keeping the colour of every other arc unchanged. This observation will be used in Section 5 to verify the former Alon-Saks-Seymour Conjecture for small values of n.

5 The rank of 1-ball subgraphs

As Lemma 3.2 indicates, the existence of a vertex of full rank in a 1-ball subgraph leads to a small chromatic number. This leads to the following natural question: what is the minimum value of the maximum rank of a vertex over all maximal 1-ball subgraphs? For this purpose, we define the rank of a maximal 1-ball subgraph to be the maximum rank of its vertices, and let $\rho(n)$ denote the minimum rank over all maximal 1-ball subgraphs of $EC(n)$. We now prove

Proposition 5.1. *$\rho(n) = \Theta(\log n)$, that is, there are positive constants c_1 and c_2 such that $c_1 \log n \leq \rho(n) \leq c_2 \log n$.*

Proof. This problem of finding $\rho(n)$ is closely related to the Ramsey number $R(m, m)$. The Ramsey number $R(m, m)$ is defined to be the least positive integer t such that every blue-red edge colouring of the complete graph K_t contains either a blue clique or a red clique on m vertices [5]. Let $C(n)$ be the largest integer such that every 2-arc-coloured K_n contains a monochromatic clique of order $C(n)$ and there is a 2-arc-coloured K_n which does not contain a monochromatic clique of order $C(n) + 1$. We note that

$$R(C(n), C(n)) \leq n < R(C(n) + 1, C(n) + 1).$$

Since $c_1 2^{m/2} \leq R(m, m) \leq 4^{m-1}$ for some positive constants c_1 (see, e.g., [20], pp. 360-361), we have

$$c_1 2^{C(n)/2} \leq n \leq 4^{C(n)},$$

which implies

$$C(n) = \Theta(\log n).$$

Next we will show that $C(n) \leq \rho(n) \leq 2C(n)$ for every positive integer n. This immediately implies Proposition 5.1.

Let M be a maximal 1-ball subgraph of $EC(n)$ and T be the corresponding 2-arc-coloured tournament. A vertex x of M, with 1-support $S_1(x)$ and 0-support $S_0(x)$, corresponds to an ordered pair (T_0, T_1) of sub-tournaments of T, where T_0 is induced by $S_0(x)$ whose arcs all have colour B, T_1 is induced by $S_1(x)$ whose arcs all have colour R, all arcs in T from $S_0(x)$ to $S_1(x)$ have colour B and all arcs in T from $S_1(x)$ to $S_0(x)$ have colour R. This observation will also be useful for generating all the vertices in M of rank greater than 2.

Now let M be a maximal 1-ball subgraph of $EC(n)$ whose rank is $\rho(n)$. Since the corresponding 2-arc-coloured tournament T contains a monochromatic sub-tournament of size $C(n)$, we have $\rho(n) \geq C(n)$.

On the other hand, let T be a 2-arc-coloured tournament of order n such that it does not contain a monochromatic sub-tournament of order $C(n) + 1$. Let v be a vertex with maximum rank in the corresponding maximal 1-ball subgraph. Then

$$\rho(n) \leq |S_0(v)| + |S_1(v)| \leq 2C(n).$$

This completes the proof of Proposition 5.1. □

6 Proof of Theorem 1.2

With the terminology from previous sections we make further reductions, and using these reductions we can prove the claim of the former conjecture for small values of n and thus increase the lower bound on the minimum counterexample. If T is a 2-arc-coloured tournament we define the term *monochromatic subgraph* of T to mean the directed graph on the same node-set as T containing every arc of just one of the colours from T. The union of the two monochromatic subgraphs of T is T.

Let M be a maximal 1-ball subgraph of $EC(n)$ and $k < n$ be an integer. Let J be a k-subset of $\{1, 2, \ldots, n\}$. A set U of vertices in M is called J-independent if $|S_0(x) \cap S_1(y) \cap J| + |S_1(x) \cap S_0(y) \cap J| = 0$ for any two vertices x and y in U. We note that, if U is J-independent, then the subgraph $M[U]$ of M induced by U admits a homomorphism to a maximal 1-ball subgraph of $EC(n-k)$. Let $M[\bar{U}]$ be the subgraph of M induced by the complement of U, then we have

$$\chi(M) \leq \chi(M[U]) + \chi(M[\bar{U}]).$$

Suppose that the stated result of the former Alon-Saks-Seymour Conjecture is true for all $k \leq n-1$ and there is a subset U of $V(M)$ which is J-independent for some J such that $\chi(M[\bar{U}]) \leq k$. Then we have

$$\chi(M) \leq k + (n-k+1) = n+1.$$

This observation is used to derive the following.

Proposition 6.1. *Suppose that for all graphs G with $bp(G) < n$, we have $\chi(G) \leq bp(G)+1$. Let T be a 2-arc-coloured tournament with n nodes such that one (or both) of the two monochromatic subgraphs of T contains a node with outdegree 0. Then the corresponding maximal 1-ball subgraph is $(n+1)$-colourable.*

Proof. We only need to consider the case when the monochromatic subgraph of T of colour R contains a node with outdegree 0, since the other case can be converted to the first case by switching 0 and 1 at every coordinate. Let T_R be the monochromatic subgraph of T of colour R and suppose that node i has outdegree 0 in T_R. Let M be the maximal 1-ball subgraph corresponding to T, and U be the subset of $V(M)$ consisting of all the vertices whose ith coordinate is either 0 or ∞. Then $\bar{U}$, the complement of U, must form an independent set in M. If $\bar{U}$ contains two adjacent vertices $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, then there is a coordinate $j \neq i$ such that $\{x_j, y_j\} = \{0, 1\}$. This implies that T_R contains the arc directed from i to j, contradicting the assumption that node i has outdegree 0 in T_R. By the above observation, $\chi(M) \leq 1 + n$. □

Thus we only need to consider 2-arc-coloured tournaments T such that at every node there is a leaving arc of colour B and a leaving arc of colour R. In particular, we only need to consider those tournaments in which every node has outdegree at least 2. This observation itself immediately gives the following.

Theorem 6.2. *Conjecture 1.1 is true for $n \leq 5$.*

Proof. We note that $EC(1)$ is isomorphic to K_2, and hence the result is true for $n = 1$. For $n \leq 4$, every tournament contains a node whose outdegree is less than 2 (otherwise it needs at least $2n$ arcs), thus we may apply Proposition 6.1 and induction. We also note that the case $n \leq 4$ was previously proved by Rho [18].

Next we consider the case $n = 5$. In this case either Proposition 6.1 will apply again or each node has exactly 2 out-going arcs, one of each colour. We claim that all such 2-arc-coloured tournaments are basis of a 1-ball subgraph isomorphic to the one obtained from two monochromatic 5-cycles. To see this, suppose T is a 2-arc-coloured tournament on 5 nodes with each node having exactly 2 out-going arcs one of each colour. If each node also has an incoming arc of each colour then T must be union of two directed monochromatic 5-cycles and we are done. Else there is a node v_1 with two incoming blue arcs. Let v_2v_1, v_3v_1, v_1v_4 be the blue arcs incident to v_1, thus v_1v_5 is a red arc. By the automorphism f_1 we have symmetry between v_4 and v_5 so we may assume v_5v_4 is the arc connecting v_4 and v_5. Without loss of generality we may also assume the other out-going arc at v_5 is v_5v_3. The automorphism, f_5 allows us to choose the colours of these two out-going arcs. The arc connecting v_2 and v_5 must be directed towards v_5 and thus must be coloured red. Now applying f_2 will change the coloured indegree of v_5 and v_1, giving them both one in-coming arc of each colour. The out-going arcs from v_4 connect to v_2 and v_3 and using automorphism, f_4 we may assume they are colours blue and red respectively. This finally forces arc v_3v_2 to be colour red. Thus, up to automorphism, each vertex has one incoming arc of each colour.

We use the blue directed 5-cycle to assign a distance to every red arc, equivalent to the oriented distance from its tail to its head. It is easy to see that the distance must be either 2 or 3. If it is 3 then we apply all the automorphisms, f_i for $1 \leq i \leq 5$, switching colours and now the red arcs will all have distance 2 with respect to the blue 5-cycle. This shows that the orientations of the two directed monochromatic 5-cycles can be chosen arbitrarily. Thus, without loss of generality, the 1-ball subgraph will have the following 10 vertices as the only non-support-dominated vertices.

Rank 2 vertices:
$(0, 0, \infty, \infty, \infty), (\infty, 0, 0, \infty, \infty), (\infty, \infty, 0, 0, \infty), (\infty, \infty, \infty, 0, 0), (0, \infty, \infty, \infty, 0)$.
Rank 3 vertices:
$(0, 1, \infty, 1, \infty), (\infty, 0, 1, \infty, 1), (1, \infty, 0, 1, \infty), (\infty, 1, \infty, 0, 1), (1, \infty, 1, \infty, 0)$.

A 6-colouring of the corresponding 1-ball subgraph is easily obtained by colouring all the rank 2 vertices with one colour and assigning each rank 3 vertex a new distinct colour. Note that rank 2 vertices form an independent set as they are support-dominated by $(0, 0, 0, 0, 0)$. We will now show that there cannot be any vertices of rank 4 or 5. In every element of a base (all the rank 2 vertices) a 1 in position i implies an ∞ in position $i+1 \pmod 5$. A vertex of rank at least 4 must have at least three pairs of consecutive $\pmod 5$ positions that do not contain ∞. Thus a vertex of rank at least 4 will contain at least three 0's. No mater how these are arranged in the 5 positions, it is incompatible with the rank 2 vertices that contain two 1's. So there are no vertices of rank 4 or 5. □

Proof of Theorem 1.2. With the aid of a long computation we were able to prove Theorem 1.2. The use of Proposition 6.1 to prove the case for n depends on its validity for all $k < n$. In the reminder of this section we describe the computational methods for $n = 9$, this computation depends on $n = 6, 7, 8$ which were checked with the same methods.

Note that for $n = 9$, the number of 1-ball subgraphs needed to be checked is 2^{72} and, with today's computational power, it is practically impossible to even produce all these graphs. However there are many isomorphisms among maximal 1-ball subgraphs and the task is tractable once most isomorphic copies are removed. To this end we first start with a list of the 191536 non-isomorphic tournaments on 9 nodes.

Now, given a fixed tournament T on 9 nodes, we have 2^{36} possible 2-arc-colourings, each corresponding to a maximal 1-ball subgraph. However there are still many isomorphisms among these maximal 1-ball subgraphs. Such an isomorphism may be induced by an automorphism f_i of $EC(n)$ for $i = 1, 2, \ldots 9$, by an automorphism of T itself, or by any combination of these. We note that the actions of f_i on a maximal 1-ball subgraph is equivalent to exchanging the colour of the out-going arcs from node i in the corresponding 2-arc-coloured tournament. If we pick an out-going arc for each node in T and then fix a colour for that arc, we still will produce all the non-isomorphic maximal 1-ball subgraphs obtained form T, but with far fewer isomorphic copies. Those colourings for which all the arcs leaving some node have the same colour can be discarded, using Proposition 6.1.

Our next step was to produce and then colour the 1-ball subgraph from a given 2-arc coloured tournament. To make the graphs smaller, and so easier to colour, we discarded the support-dominated vertices as we discussed earlier. To produce the set of all non-support-dominated vertices from a given 2-arc-coloured tournament we introduced the concept of bident-free subtournament. A *bident* in a 2-arc-coloured tournament is a subgraph consisting of 2 arcs with different colours and common beginning node. In Figure 2, the nodes 1, 2, 3, blue arc $(1, 2)$ and red arc $(1, 3)$ are a bident. A *subtournament* of a given tournament is an induced sub-digraph on any subset of nodes of the tournament. A *bident-free subtournament* in a 2-arc-coloured tournament is a subtournament that has no bident. In Figure 2 nodes 1, 3 and 4 induce a bident-free subtournament.

If a maximal 1-ball subgraph M is encoded by a 2-arc-coloured tournament T_M, then each vertex v of M will correspond to a subtournament T_v which is the union of a blue clique T_b (nodes corresponding to 0 coordinates) and a red clique T_r (nodes corresponding to 1 coordinates) with the extra property that each out-going arc from T_b is blue and each out-going arc from T_r is red. It is now clear that T_v is bident-free. On the other hand given a bident-free subtournament T' we can construct one or two vertices corresponding to T'. To this end, for each coordinate x, if x is not a node of T' choose ∞ for x, if x is a node of T' with an out-going arc of colour B then choose 0 for this coordinate, if x has an out-going arc of colour R choose 1 for this coordinate, if x has no out-going arc then x can be either 0 or 1, in which case we will have two vertices of M corresponding to T'. There can be at most one such x; supposing that there were two, there would have to be an arc between them contradicting the fact that they each have no out-going arcs.

Note that of the two vertices obtained in the last case, it is possible to have a non-support-dominated vertex and a vector-dominated vertex. Thus it is not necessarily true that the bident-free tournament corresponding to a non-support-dominated vertex is maximal with respect to being bident-free, but this is almost true. We use the set of maximal bident-free subtournaments to produce our set of non-support-dominated vertices as follows.

We first find all the maximal bident-free subtournaments. This was achieved using the following iteration. For disjoint $C, L, F \subseteq \{1, 2, \ldots, n\}$, let $\langle C, L, F\rangle$ denote the set of all bident-free sets X such that $C \subseteq X \subseteq C \cup L$ and such that $X + x$ is not bident-free for any $x \in (L \setminus X) \cup F$. The set of all maximal bident-free subsets is $\langle \emptyset, \{1, 2, \ldots, 9\}, \emptyset\rangle$. We determine the elements of this set by repeatedly applying the refinement $\langle C, L, F\rangle = \langle C{+}x, L{-}x, F\rangle \cup \langle C, L{-}x, F{+}x\rangle$ for some $x \in L$. At any step we can apply the following simplification rules:

1. $\langle C, L, F\rangle = \langle C, L{-}x, F\rangle$ for any $x \in L$ such that $C{+}x$ is not bident-free.
2. $\langle C, L, F\rangle = \langle C, L, F{-}x\rangle$ for any $x \in L$ such that $C{+}x$ is not bident-free.
3. $\langle C, L, F\rangle = \emptyset$ if $C \cup L{+}x$ is bident-free for any $x \in F$.
4. $\langle C, L, F\rangle = \{C \cup L\}$ if rules 1–3 have been applied and $C \cup L$ is bident-free.

The set of non-support-dominated vertices are now built as follows. First for each maximal bident-free subtournament we construct one or two vertices corresponding to it. Then for each node x of a maximal bident-free subtournament T', we remove all the out-going arcs together with their end nodes to obtain a bident-free subtournament T'_x in which x is a sink. Then we add to L the two vertices made from T'_x if they are not support-dominated. Finally, we remove duplicate vertices by sorting; usually there were very few.

For each graph M produced by this method, we applied the standard greedy colouring algorithm with random vertex labellings until a colouring with 10 colours was found. This was successful in all cases.

Of 191536 nonisomorphic tournaments on 9 vertices, 6880 have minimum outdegree 0 and 50816 have minimum outdegree 1. There are 105916 tournaments of minimum outdegree 2, 27909 tournaments of minimum outdegree 3, and 15 of minimum outdegree 4.

Our computation produced 2709401599952 graphs with 21 being the number of vertices of the smallest graph and 66 being the number of vertices of the largest graph. Table 6 shows how many graphs there were of each order. The total time for the computation was 11 GHz–years. Source code for the search is available for interested readers at `http://users.cecs.anu.edu.au/~bdm/data/jbr.tar.gz`.

$\|V(M)\|$	Number of graphs	$\|V(M)\|$	Number of graphs
21	1	44	125901710465
22	604	45	81267251957
23	22546	46	48642558565
24	335917	47	27104914998
25	3009042	48	14110182128
26	19453912	49	6883192715
27	99243002	50	3150968376
28	414422072	51	1356263082
29	1449107249	52	548904802
30	4319529292	53	209052332
31	11117985007	54	74877323
32	24970401600	55	25204691
33	49387718809	56	7956173
34	86792501140	57	2363278
35	136716885660	58	652342
36	194566281907	59	167827
37	251704504252	60	40896
38	297176474277	61	8294
39	320821496578	62	1722
40	316875725672	63	207
41	286424162481	64	34
42	237128311702	65	1
43	180127755021	66	1
Total	2709401599952		

Table 1. Colouring 1-ball subgraphs of $EC(9)$

7 Concluding Remarks

It is natural to think that the more non-support-dominated vertices we have in a maximal 1-ball subgraph the more chances we have for a higher chromatic number with a fixed *bp*. In this regard we would like to ask what is the largest size of a 1-ball subgraph of $EC(n)$ with no support-dominated vertices. For $n = 5$ and $n = 9$ we have observed that such a largest graph is obtained when the 2-arc colouring in the corresponding tournament comes from a quadratic residue graph, i.e., $\{x, y\}$ is coloured B if $x - y$ is a square in $\mathbb{Z}_n$ and R otherwise. We note that for these orders this is also a Ramsey colouring; i.e., a colouring in which the largest monochromatic subgraph is as small as possible. The arcs in these tournaments are directed from x to $x + i \pmod n$ for $i < n/2$. We do not know if this is a general pattern. But we propose that perhaps answering the following question will provide a much smaller counterexample to the Alon-Saks-Seymour Conjecture than is already known:

Problem 7.1. *Given $n = 4p + 1$ where p is a prime power, what is the largest chromatic number of a 1-ball subgraph of $EC(n)$ corresponding to a 2-arc coloured Eulerian tournament whose colour classes correspond to the quadratic residue graph?*

We note that the origin of Graham Pollak theorem comes from studying the possibility of embedding a graph G in $EC(n)$ for some n. By proving that the largest clique size of $EC(n)$ is equal to $n+1$, they showed that the smallest n might be as big as $|V(G)| - 1$. P. Winkler, proving the Graham-Pollak conjecture [21], showed that $n = |V(G)| - 1$ works for every graph. Perhaps studying $EC(n)$ would help to find a combinatorial proof of the Graham-Pollak theorem. So we propose the following question:

Problem 7.2. *What is $\alpha(EC(n))$?.*

We believe that it is $2^n - 1$.

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