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The Möbius function of the small Ree groups

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Abstract

The Möbius function for a group G was introduced in 1936 by Hall in order to count ordered generating sets of G. In this paper we determine the Möbius function of the simple small Ree groups, $R(q) = {}^2G_2(q)$ where $q = 3^{2m+1}$ for $m > 0$, using their 2-transitive permutation representation of degree $q^3 + 1$. We also describe their maximal subgroups in terms of this representation. We use this to enumerate smooth epimorphisms from Γ to G for various finitely presented groups Γ, such as F_2 and the modular group $PSL_2(\mathbb{Z})$. We then highlight applications of these enumerations to Grothendieck's theory of dessins d'enfants as well as probabilistic generation of the small Ree groups.

1 Introduction

The Möbius function of a finite group has its origins in the generalised enumeration principle due to Weisner [42] first and shortly followed by Hall's independent discovery in [17]. Whereas Weisner considered the problem in more generality, Hall was primarily concerned with Möbius inversion in the lattice of subgroups of a finite group and so we mostly refer to Hall's work. The motivating problem of [17] was to enumerate the number of ordered tuples of elements of a finite group G which also generate G. We begin with the following definition.

Definition 1.1. *Let G be a finite group and $H \leqslant G$ a subgroup of G. Let $X = \{x_1, \ldots, x_n\}$ be an ordered subset of elements of G of size n, satisfying a finite, possibly empty, family of relations, $f_i(X) = 1$, and let $\Gamma = \langle X \mid f_i(X)\rangle$. We call a* ***summatory function of*** *H the function $\sigma_\Gamma(H)$ which counts the number of subsets $X \subset H$ satisfying the relations $f_i(X)$ and an* ***Eulerian function of*** *H $\phi_\Gamma(H)$ the function counting the number of such X where $\langle X\rangle = H$.*

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Remark 1.2. *In the case where X as in the above definition has size n and there are no other relations, i.e. when $\Gamma \cong F_n$ the free group on n generators, we write $\sigma_n(H)$ and $\phi_n(G)$ for our summatory and Eulerian functions respectively. We also note that for certain considerations [17, Section 1.4] the ordering of the n elements is necessary to consider.*

The principle Hall uses is as follows. If G is a finite group and X is an ordered n-tuple of elements of G, then X will generate some subgroup $H \leqslant G$, not necessarily equal to G. From this we can write the following

$$\sigma_\Gamma(G) = \sum_{H \leqslant G} \phi_\Gamma(H).$$

Since these are two functions defined on a lattice and taking values in an abelian group, we are able to use Möbius inversion to give

$$\phi_\Gamma(G) = \sum_{H \leqslant G} \sigma_\Gamma(H)\mu_G(H)$$

where the Möbius function $\mu_G(H)$ is given by the formula

$$\sum_{K \geqslant H} \mu_G(K) = \begin{cases} 1 & \text{if } H = G \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.3. *The function $\mu_G(H)$ for $H \leqslant G$ is called the* ***Möbius function of*** *H. We refer to the collection of $\mu_G(H)$ for all $H \leqslant G$ as the* ***Möbius function of*** *G and $\mu_G(1)$ as the* ***Möbius number*** *of G.*

Remark 1.4. *In the case that G is a cyclic group, $\phi_1(G)$ is precisely the Euler totient function $\phi(|G|)$. We denote this as usual by $\phi(n)$ for a positive integer n. The Möbius function of $H \leqslant G$ is then $\mu_G(H) = \mu(|G|/|H|)$ where $\mu(n)$ is the classical Möbius function for a natural number $n \geq 1$.*

A priori, it seems as though we might have to work through the entire subgroup lattice of G. But, since it is clear that $\mu_G(H_1) = \mu_G(H_2)$ if H_1 and H_2 are conjugate in G, we need only determine $\mu_G(H)$ on a set of conjugacy class representatives of subgroups. In fact, due to the following theorem of Hall [17, Theorem 2.3], we need only determine $\mu_G(H)$ on a set of conjugacy class representatives of subgroups which occur as the intersection of maximal subgroups.

Theorem 1.5 (Hall, 1936). *If $H \leqslant G$ then $\mu_G(H) = 0$ unless $H = G$ or H is an intersection of maximal subgroups of G.*

The theory of Möbius functions and enumeration in a general poset was later developed extensively by Rota in [35] and this was shortly followed by a short paper due to Crapo [7] which extends Rota's work by introducing the use of complements. In the specific case of the Möbius function of a finite group we also draw the reader's attention to the works of Kratzer and Thévenaz [22], Hawkes, Isaacs and Özaydin [18] and Pahlings [29].

In general, determining the Möbius function of a finite group is a lengthy process and one must have a large amount of information about the subgroup structure of G including knowledge of its classes of maximal subgroups. However, a number of results are known which facilitate its determination. The following, which can already be found in Weisner [43, Theorem 1], is an immediate consequence of the fact that if N is a normal subgroup of G, the subgroup lattice of the quotient G/N is in bijective correspondence with the lattice of subgroups of G containing N.

Theorem 1.6 (Weisner, 1935). *Let G be a group and let $N \lhd G$ be a normal subgroup of G. Then*

$$\mu_G(N) = \mu_{G/N}(1).$$

From Theorems 1.5 and 1.6 it is immediate that if H does not contain the Frattini subgroup of G, then $\mu_G(H) = 0$. Hall already makes the point [17, Paragraph 3.7] that given the Möbius functions of A_4, S_4 and A_5, the Möbius functions of their double covers $2.A_4$, $2.S_4$ and $2.A_5$, respectively, can be "written down at once from that of the corresponding factor group". This immediacy extends to the Eulerian function of a group. The following is an immediate corollary of a result due to Pahlings [29, Lemma 1] for which a proof can be found in the author's PhD thesis [31].

Corollary 1.7. *Let G be a finite group. Then*

$$\phi_n(G) = \phi_n(G/\Phi(G))|\Phi(G)|^n.$$

Remark 1.8. *In principle this corollary can be generalised to arbitrary Eulerian functions of G. However, the relationship between $\sigma_\Gamma(H)$ and $\sigma_\Gamma(H/\Phi(G))$ becomes more delicate for arbitrary Γ.*

In the case that G is a soluble group, Kratzer and Thévenaz take these ideas to their extreme conclusion by relating $\mu_G(H)$ to the complements of factors of a fixed chief series of G [22, Theorem 2.6]. In the case of nilpotent groups specifically, a combination of results due to Weisner [43, Section 3] and Hall [17, Sections 2.7 and 2.8] essentially gives the Möbius function of any nilpotent group. These results seem to have been reproved independently by Kratzer and Thévenaz in [22, Proposition 2.4], generalising the work of Delsarte [8].

Remark 1.9. *Kratzer and Thévenaz cite Rota and Delsarte in their paper, but neither Kratzer and Thévenaz nor Delsarte make mention of the work of Weisner or Hall.*

Kratzer and Thévenaz also prove the following result which has implications for the Möbius number of G [22, Theorem 3.1].

Theorem 1.10. *If G is a group and $H \leqslant G$, then*

$$\mu_G(H)\frac{[G:G']_0}{[N_G(H):H]} \in \mathbb{Z}$$

where, for a positive integer n, n_0 is the largest positive divisor of n without square factors. In particular, $\mu_G(1)$ is a multiple of $|G|/[G:G']_0$.

However, as they point out at the end of their paper: "It results from Theorem 3.1 that $\mu_G(1)$ is a multiple of $|G|$ if G is perfect. For example, $\mu_{A_5}(1) = 60 = |A_5|$, $\mu_{A_6}(1) = 720 = 2|A_6|$, but $\mu_{L_2(7)}(1) = 0$. Thus, contrary to the case of soluble groups, the behaviour of the Möbius function of simple groups seems more difficult to comprehend." Their interest in Möbius numbers stems from two sources: idempotents in the Burnside ring and their relation with certain homology groups, however, that is not to say the two are not connected cf. the work of Bouc [1]. We note that the connection between Möbius numbers and Lefschetz numbers is also considered in Shareshian's thesis [36] to which we direct the interested reader, particularly, the reader who does not read French.

The connection to the Burnside ring of a group, G, is related via the table of marks of G, originally introduced by Burnside [2]. As one might expect, there is a deep connection between the Möbius function of G and the table of marks of G [29, 30]. This relationship then extends to properties of the Burnside ring of G for which we direct the interested reader to the aforementioned paper of Kratzer and Thévenaz [22] and Solomon [37]. Their relation to the homology and homotopy comes from considering the lattice of subgroups of a finite group, G, as a simplicial complex. For more on the algebraic topological considerations we direct the reader to the aforementioned papers and the references therein.

1.1 Applications of the Eulerian functions of a group

The Eulerian functions of a group are of natural interest to group theorists since they can be used to answer questions of generation of G. However, the scope of this function was first broadened, as far as the author is aware, through the work of Downs and Jones [10, 11, 12, 13, 14] in their application of it to other categories. Another way of interpreting $\phi_\Gamma(G)$ is that it enumerates epimorphisms from Γ to G, hence $d_\Gamma(G) = \phi_\Gamma(G)/|\mathrm{Aut}(G)|$ is equal to the number of normal subgroups $N \lhd \Gamma$ such that $\Gamma/N \cong G$ [17, Theorem 1.4].

Following this line of reasoning, Downs and Jones observed that if the normal subgroups of Γ were in one-to-one correspondence with the regular objects of some category $\mathfrak{K}$ then $d_\Gamma(G)$ could be used to count the number of distinct regular objects in that category whose automorphism group is isomorphic to G. For example, if X is a topological space with covering space $\tilde{X}$ and fundamental group $\pi_1(X) \cong \Gamma$, then d_Γ is the number of distinct regular covers of X having covering group isomorphic to G [14].

One important case is when X is the thrice-punctured Riemann sphere which has $\pi_1(X) \cong F_2$ and which, through Grothendieck's dessins d'enfants programme

[16], is also related to the absolute Galois group. The quantity $d_2(G)$ then counts the number of distinct regular dessins having automorphism group isomorphic to G. A number of other categories of maps are considered in the aforementioned work of Downs and Jones which we explore in Section 4.

1.2 The small Ree groups

The existence of the small Ree groups was first announced in 1960 by Ree [32] who constructed them shortly after in [34]. Ree observed that Suzuki's original construction [38] of the Suzuki groups $Sz(2^{2m+1}) = {}^2B_2(2^{2m+1})$ for $m > 0$ could be interpreted in terms of Lie theory and applied to the Chevalley groups of types G_2 [32] and F_4 [33] in certain characteristics. In the case of G_2 in characteristic 3 the groups which arise are known as the small Ree groups and are denoted ${}^2G_2(q) = R(q)$ where $q = 3^{2m+1}$ and $m \geq 0$.

The small Ree groups $R(q)$ can naturally be considered as subquotients of the matrix groups $SL_7(q)$ as in [23] or of $\Omega_8^+(q)$ as in [21]. For the purpose of determining all possible intersections of maximal subgroups in $R(q)$ this is quite unwieldy. Thankfully, Tits [39] determined the existence of a natural 2-transitive permutation representation of $R(q)$ of degree $q^3 + 1$, where $R(q)$ can be seen as the group of automorphisms of a certain 6-dimensional projective variety defined over $\mathbb{F}_q$ and consisting of $q^3 + 1$ points. Tits' construction, however, still relies on the Lie theory. A construction of the small Ree groups that is Lie-free is due to recent work by Wilson [44, 45, 46]. In addition to these constructions, the small Ree groups have an interpretation as the automorphism groups of finite generalized hexagons for which we direct the reader to [40, Section 7.7] and as the automorphism group of a $2-(q^3+1, q+1, 1)$ design [27].

As far as the author is aware, the only families of finite simple groups for which the Möbius function is known are as follows. The Möbius function of the simple groups $L_2(p)$, for $p \geq 5$, were originally determined by Hall [17]. This was extended to the Möbius function of $L_2(q)$ and $PGL_2(q)$, for all prime powers $q \geq 5$, by Downs [10]. Recently, Downs and Jones [14] have determined the Möbius function for the simple Suzuki groups $Sz(2^{2m+1})$, where $m > 0$. It seems natural to then determine the Möbius function of the simple small Ree groups.

The following is our main result.

Theorem 1.11. *Let $G = R(3^n)$ be a simple small Ree group for a positive odd integer $n > 1$. If $H \leqslant G$, then $\mu_G(H) = 0$ unless H belongs to one of the following classes of subgroups of G.*

Isomorphism type of $H \leqslant G$	*for* $h \mid n$ *and s.t.*	$[G\colon N_G(H)]$	$\mu_G(H)$
$R(3^h)$	–	$\lvert G\rvert/3^{3h}(3^{3h}+1)(3^h-1)$	$\mu(n/h)$
$3^h+\sqrt{3^{h+1}}+1\colon 6$	–	$\lvert G\rvert/6(3^h+\sqrt{3^{h+1}}+1)$	$-\mu(n/h)$
$3^h-\sqrt{3^{h+1}}+1\colon 6$	$h>1$	$\lvert G\rvert/6(3^h-\sqrt{3^{h+1}}+1)$	$-\mu(n/h)$
$(3^h)^{1+1+1}\colon (3^h-1)$	–	$\lvert G\rvert/3^{3h}(3^h-1)$	$-\mu(n/h)$
$2\times L_2(3^h)$	$h>1$	$\lvert G\rvert/3^h(3^{2h}-1)$	$-\mu(n/h)$
$2\times(3^h\colon \frac{3^h-1}{2})$	$h>1$	$\lvert G\rvert/3^h(3^h-1)$	$\mu(n/h)$
$(2^2\times D_{(3^h+1)/2})\colon 3$	$h>1$	$\lvert G\rvert/6(3^h+1)$	$-\mu(n/h)$
$2^2\times D_{(3^h+1)/2}$	$h>1$	$\lvert G\rvert/6(3^h+1)$	$3\mu(n/h)$
$2\times L_2(3)$	–	$\lvert G\rvert/24$	$-2\mu(n)$
2^3	–	$\lvert G\rvert/168$	$21\mu(n)$

The structure of this paper is as follows. In Section 2 we describe the structure of the simple small Ree groups. In Section 3 we determine how maximal subgroups of $R(q)$ can intersect and use these results to determine the Möbius function of $R(q)$. Finally, in Section 4, we use the Möbius function of $R(q)$ to determine a number of Eulerian functions associated to the simple small Ree groups. In addition, we use these to prove a number of results on their generation and asymptotic generation as well as applying these results to a number of other categories. The results of this paper formed part of the author's thesis [31] in which analogous results for $R(3)$ are included. Since the Möbius function of $R(3)$ can be found in GAP [15], we do not include this content here. We use the ATLAS [6] notation throughout.

2 The structure of the simple small Ree groups

We turn now to the simple small Ree groups. Unless otherwise specified we let $G = R(q)$ be a simple small Ree group for $q = 3^n$ where $n > 1$ is a positive odd integer and Ω is a set of size q^3+1. We consider the natural 2-transitive permutation representation of G on Ω whose action we now describe.

2.1 Conjugacy classes and centralisers of elements in $R(q)$

We begin by describing the conjugacy classes of elements of G and in particular the action of their elements on Ω. We assemble the necessary results from the character table of $R(q)$, due to Ward [41], as well as results from Levchuk and Nuzhin [23] and the summary given by Jones in [19].

We begin with the notation for the conjugacy classes of elements of orders 2, 3, 6 and 9. These are summarised in Table 1. For an element $g \in G$, we denote the set of points in Ω stabilised by g as Ω^g.

The Sylow 2-subgroups of G are elementary abelian of order 8 and the normaliser of $S \in \mathrm{Syl}_2(G)$ in G has shape $2^3\colon 7\colon 3 \cong A\Gamma L_1(8)$. An involution in G is represented by t and fixes $q+1$ points in Ω which we refer to as the **block** of t. The centraliser in G of t has shape $2\times L_2(q)$ and acts 2-transitively on the block of t [27]. Any two

Conjugacy class of g	Order of g	$\|C_G(g)\|$	$\|\Omega^g\|$
$\mathcal{C}_2$	2	$q(q^2-1)$	$q+1$
$\mathcal{C}_3^0$	3	q^3	1
$\mathcal{C}_3^+, \mathcal{C}_3^-$	3	$2q^2$	1
$\mathcal{C}_6^+, \mathcal{C}_6^-$	6	$2q$	1
$\mathcal{C}_9^0, \mathcal{C}_9^+, \mathcal{C}_9^-$	9	$3q$	1

Table 1: Conjugacy classes in G of elements of orders 2, 3, 6 and 9.

distinct blocks can intersect in at most one point and any two points belong to a unique block.

The Sylow 3-subgroups of G have order q^3, exponent 9 and have trivial intersection with one another. Let $B \in \mathrm{Syl}_3(G)$. The centre of B is elementary abelian of order q and nontrivial elements of B belong to the conjugacy class $\mathcal{C}_3^0$. There is an elementary abelian normal subgroup E of B, of order q^2, such that $Z(B) \leqslant E \leqslant B$. The elements of $E \setminus Z(B)$ belong to $\mathcal{C}_3^* = \mathcal{C}_3^+ \cup \mathcal{C}_3^-$ and the elements of $B \setminus E$ have order 9. Elements of $\mathcal{C}_3^0$ and $\mathcal{C}_9^0$ are conjugate to their inverse whereas the inverses of elements of $\mathcal{C}_3^+$ belong to $\mathcal{C}_3^-$. Similarly for elements of the classes $\mathcal{C}_6^+$ and $\mathcal{C}_9^+$. Elements of $\mathcal{C}_3^+$ are denoted u and elements of $\mathcal{C}_6^+$ are the product of an involution t with an element conjugate to u where $tu = ut$. If $g \in G$ is any element of order 9, then $g^3 \in \mathcal{C}_3^0$[41].

The remaining elements of G are all semisimple and are conjugate to a power of an element appearing in Table 2. Where we write tr we mean an involution t commuting with an element r of order $(q-1)/2$. Similarly for ts.

Representative element $g \in G$	$o(g)$	$\|C_G(g)\|$	$\|\Omega^g\|$
$tr = rt$	$q-1$	$q-1$	2
$ts = st$	$(q+1)/2$	$q+1$	0
w	$q-\sqrt{3q}+1$	$q-\sqrt{3q}+1$	0
v	$q+\sqrt{3q}+1$	$q+\sqrt{3q}+1$	0

Table 2: Representatives of non-involution semisimple elements in G.

The Hall subgroups of G are denoted A_i for $i = 0, 1, 2, 3$. They are all cyclic and have pairwise trivial intersection. We introduce the following notation to denote their orders.

Definition 2.1. *Let $q = 3^n$ be an odd power of 3. For a positive divisor l of n we define*

$$a_0(l) = (3^l-1)/2, \quad a_1(l) = (3^l+1)/4, \quad a_2(l) = 3^l - 3^{\frac{l+1}{2}} + 1, \quad a_3(l) = 3^l + 3^{\frac{l+1}{2}} + 1.$$

We may simply write a_i when $l = n$ and if no confusion can arise.

A Hall subgroup of G conjugate to A_i has order a_i for $i = 0, 1, 2, 3$. Note that $a_1a_2a_3$ is always congruent to 0 modulo 7 and elements of order 7 are all conjugate in G. The order of G can then be written

$$|G| = 2^3 q^3 a_0 a_1 a_2 a_3.$$

2.2 Maximal subgroups of $R(q)$

The maximal subgroups of the simple small Ree groups were determined by Levchuk and Nuzhin [23] and independently by Kleidman [21]. They are conjugate to one of those listed in Table 3. In order to determine their possible mutual intersections we describe the action of the maximal subgroups on Ω.

Group	Description
$R(q^{1/p})$, p prime	Maximal subfield subgroups
$q^{1+1+1}:(q-1)$	Parabolic subgroups
$2 \times L_2(q)$	Involution centralisers
$(2^2 \times D_{(q+1)/2}):3$	Four-group normalisers
$q - \sqrt{3q} + 1:6$	Normalisers of a Hall subgroup A_2
$q + \sqrt{3q} + 1:6$	Normalisers of a Hall subgroup A_3

Table 3: Conjugacy classes of maximal subgroups of the simple small Ree groups $R(q)$.

2.2.1 Subfield subgroups

The subfield subgroups of G are denoted $G_l \cong R(3^l)$ for $l \geq 1$ dividing n and they are maximal when n/l is prime. There are $3^{3l} + 1$ Sylow 3-subgroups in G_l, each stabilising a distinct point in Ω. We denote the union of these points by $\Omega(l)$, on which G_l acts 2-transitively. If $g \in G_l$ fixes 1 or 2 points in Ω, then they again belong to $\Omega(l)$. The blocks of involutions in G_l stabilise $q^l + 1$ points in $\Omega(l)$, with the remaining $3^n - 3^l$ points in $\Omega \setminus \Omega(l)$.

2.2.2 Parabolic subgroups

The parabolic subgroups of G are the normalisers of the Sylow 3-subgroups. The have shape $q^{1+1+1}:(q-1)$ and consist of all elements fixing a point $\omega \in \Omega$. As such we also refer to them as point stabilisers and denote the stabiliser in G of ω by P_ω. The elements of the Sylow 3-subgroup B have been discussed, all remaining elements of $P_\omega \setminus B$ have order 6 or order dividing $q - 1$.

2.2.3 Involution centralisers

Let $t \in G$ be an involution, $C = C_G(t)$ its centraliser in G and Ω^t the block of t stabilised by C. Elements of C of order 3 belong to $\mathcal{C}_3^*$ and fix a point in Ω^t, elements of order dividing $q - 1$ fix two points in Ω^t and elements of order dividing $(q + 1)/2$

do not fix any points in Ω^t. It follows that any pair of commuting involutions in G have disjoint blocks. We can also prove the following extension.

Lemma 2.2. *Let $t_1 \neq t_2$ be involutions in C. Then $\Omega^{t_1} \cap \Omega^{t_2} = \varnothing$.*

Proof. Since the blocks of two distinct involutions in G can intersect in at most one point, assume for a contradiction that $|\Omega^{t_1} \cap \Omega^{t_2}| = 1$. The dihedral subgroup $D = \langle t_1, t_2 \rangle$ is then contained in a point stabiliser and so either $D \cong D_6$ or D_{18}. From the list of maximal subgroups of $L_2(q)$ [9], neither of these are possible subgroups of C and so $\Omega^{t_1} \cap \Omega^{t_2} = \varnothing$. □

The $q^2 - q + 1$ involutions in $2 \times L_2(q)$ fall into the following three C-conjugacy classes of involutions:

1. $\{t\}$, the central involution,
2. the $q(q-1)/2$ involutions in C', and
3. the $q(q-1)/2$ involutions in the coset tC'.

As a corollary of this along with the previous lemma we have that the blocks of the involutions in C form a disjoint partition of Ω. That is to say, each $\omega \in \Omega$ belongs to the block of one and only one involution in C.

2.2.4 Four-group normalisers

The four-group normalisers of G can be built in two different ways.

- Let $t_1 \neq t_2$ be commuting involutions in G with $t_3 = t_1 t_2$. The four-group $V = \langle t_1, t_2 \rangle$ is centralised in G by a dihedral subgroup of shape $D_{(q+1)/2}$ and normalised by an element $u \in \mathcal{C}_3^*$ such that $\langle t_1, t_2, u \rangle \cong L_2(3)$. The normaliser in G of V is then $N = N_G(V) \cong (2^2 \times D_{(q+1)/2}){:}3$.
- Alternatively, let $\langle s \rangle$ be a Hall subgroup conjugate to A_1. The centraliser of $\langle s \rangle$ in G is a unique four-group V and $V \times \langle s \rangle$ is normalised by an element tu of order 6, where t commutes with V and u normalises $\langle s \rangle$.

A counting argument shows that $\langle s \rangle$ belongs to a unique four-group normaliser, whereas a four-group belongs to $1 + 3(q+1)/2$ four-group normalisers. To avoid confusion with the normalisers of the other Hall subgroups, we refer to groups conjugate to N in G as four-group normalisers.

There are four subgroups of N isomorphic to $D_{(q+1)/2}$. One of them is normal in N, which we denote by D_t, the other three are conjugate in N and we denote a representative by $D_{t'}$. The three $N_G(V)$-conjugacy classes of involutions in $N_G(V)$ are then the following:

1. the 3 involutions in V, namely t_1, t_2 and t_3,
2. the $(q+1)/4$ involutions in D_t, whose representative we denote by t_d, and;

3. the $3(q+1)/4$ involutions in the conjugates of $D_{t'}$ in N, whose representative we denote by t'_d.

The centraliser of t_d in N is $C_N(t_d) \cong 2 \times L_2(3)$; conversely, if L is a subgroup of N isomorphic to $2 \times L_2(3)$, then its central involution is conjugate to t_d since the only $N_G(V)$-conjugacy class of involutions whose order is not divisible by 3 is that of t_d. The centraliser of t'_d in N is a Sylow 2-subgroup of N which is a Sylow 2-subgroup of G and is thus elementary abelian of order 8. If $V_0 \neq V$ is a four-group in N, then one and only one of its nontrivial involutions belongs to V, at most one of its involutions is conjugate to t_d and at most one of its involutions is conjugate to t'_d since the order of D_t is not divisible by 4. Thus, V_0 is conjugate in N to either $\langle t_i, t_i t_d \rangle$ or $\langle t_i, t_j t_d \rangle$ where $i \neq j$, $1 \leq i, j \leq 3$ and $t_i t_d$ and $t_j t_d$ are both conjugate in N to t'_d.

The geometric interpretation of N is then as follows, much of which follows from the fact that $C_G(V) \leqslant C_G(t)$ for each $t \in V$. Since the involutions in N are contained in an involution centraliser, their blocks are all pairwise disjoint. Furthermore, the action of s stabilises the blocks of only the involutions in V, since $s \in C_G(V)$. An involution in N has centraliser order in N divisible by 3 if and only if it belongs to D_t. Hence, the fixed point in Ω of an element in N of order 3 or 6 belongs to the block of an involution in D_t.

2.2.5 Normalisers of Hall subgroups A_2, A_3

The cyclic Hall subgroups, A_2, A_3, are normalised by cyclic subgroups of order 6. Let $A = \langle a \rangle$ be conjugate to a Hall subgroup A_2 or A_3, and let N be its normaliser in G; the geometric picture of $N_G(A_2)$ is analogous to that of $N_G(A_3)$. Since nontrivial elements of A do not fix any points in Ω the action of A partitions Ω into $(q^3+1)/|A|$ subsets of size $|A|$. If $u \in \mathcal{C}_3^*$ normalises A, then there are $|A|$ conjugates of u in N and the fixed points of elements conjugate to u belong to a unique subset of this partition. For each conjugate of u there is an involution t with which it commutes and so the fixed point of u belongs to the block of t. The remaining elements in the block of t each belong to a distinct orbit of a, since if an orbit of a contained more than one element of Ω^t then t would commute with a. As with the four-group normalisers, elements conjugate to tu behave similarly to the elements conjugate to u.

2.3 Conjugacy classes and normalisers of subgroups in $R(q)$

In order to facilitate the determination of the Möbius function of G we would like to restrict ourselves to a small number of classes of subgroups of G by proving that any subgroup H lying outside these classes has $\mu_G(H) = 0$. We begin by determining the subgroups of G that occur as intersections of maximal subgroups. However, as we shall see, there are various classes of subgroups which can exist as intersections of maximal subgroups that also have $\mu_G(H) = 0$. In anticipation we define the union of the classes of subgroups of G appearing in Table 4 as **MaxInt**.

Remark 2.3. *Our use of the notation for the classes appearing in Table 4 is as follows. Where we include the* (l)*, for example* $\boldsymbol{P}(l)$*, we mean the union over all*

Class	Isomorphism type	Description
$\boldsymbol{R}(l)$	$R(3^l)$	Subfield subgroups
$\boldsymbol{P}(l)$	$(3^l)^{1+1+1}:(3^l-1)$	Parabolic subgroups of $R(3^l)$
$\boldsymbol{C}_t(l)$	$2 \times L_2(3^l)$	Involution centralisers in $R(3^l)$, $l > 1$
$\boldsymbol{C}_t^\omega(l)$	$2 \times (3^l : \frac{3^l-1}{2})$	Point stabilisers of elements of $\boldsymbol{C}_t(l)$, $l > 1$
$\boldsymbol{F}(l)$	3^l	Sylow 3-subgroups of elements of $\boldsymbol{C}_t(l)$, $l > 1$
$\boldsymbol{C}_0(l)$	$3^l - 1$	Centralisers of Hall subgroups A_0 in $R(3^l)$, $l > 1$
$\boldsymbol{N}_V(l)$	$(2^2 \times D_{(3^l+1)/2}):3$	Four-group normalisers in $R(3^l)$, $l > 1$
$\boldsymbol{N}_2(l)$	$a_2(l):6$	Normalisers of Hall subgroups A_2 in $R(3^l)$, $l > 1$
$\boldsymbol{N}_3(l)$	$a_3(l):6$	Normalisers of Hall subgroups A_3 in $R(3^l)$
$\boldsymbol{C}_V(l)$	$2^2 \times D_{(3^l+1)/2}$	Four-group centralisers in $R(3^l)$, $l > 1$
$\boldsymbol{D}_2(l)$	$D_{2a_2(l)}$	Normal dihedral subgroups of elements of $\boldsymbol{N}_V(l)$, $l > 1$
$\boldsymbol{D}_3(l)$	$D_{2a_3(l)}$	Normal dihedral subgroups of elements of $\boldsymbol{N}_V(l)$
$\boldsymbol{C}_t(1)$	$2 \times L_2(3)$	Involution centralisers in $R(3)$
$\boldsymbol{E}$	2^3	Sylow 2-subgroups of G
$\boldsymbol{V}$	2^2	Four-groups
$\boldsymbol{C}_6^*$	6	Cyclic subgroups of order 6 generated by $tu \in \mathcal{C}_6^+$
$\boldsymbol{C}_3^*$	3	Cyclic subgroups of order 3 generated by $u \in \mathcal{C}_3^+$
$\boldsymbol{C}_2$	2	Cyclic subgroups of order 2 generated by $t \in \mathcal{C}_2$
$\boldsymbol{I}$	1	The identity subgroup

Table 4: The disjoint subsets of **MaxInt**. Each subset consists of subgroups of G for all l dividing n unless otherwise stated.

divisors l of n of parabolic subgroups of subfield subgroups conjugate to $R(3^l)$. Where we omit the (l) by writing, for example $\boldsymbol{P}$, we mean those elements of $\boldsymbol{P}(l)$ for which $l = n$ or, in the case of $\boldsymbol{R}$, the maximal subfield subgroups. In certain classes we have made exclusions to avoid the following repetitions

$$\boldsymbol{N}_V(1) = \boldsymbol{C}_t(1), \quad \boldsymbol{C}_V(1) = \boldsymbol{E}, \quad \boldsymbol{C}_t^\omega(1) = \boldsymbol{C}_6^*, \quad \boldsymbol{F}(1) = \boldsymbol{C}_3^*, \quad \boldsymbol{D}_2(1) = \boldsymbol{C}_0(1) = \boldsymbol{C}_2.$$

In particular, the list is ordered so that no element of a class appears in more than one class. Furthermore, no element of any class of **MaxInt** *is a subgroup of any element of a successive class in the stated ordering with the possible exceptions of elements of $\boldsymbol{N}_2(l)$ being subgroups of elements of $\boldsymbol{N}_3(l)$ and elements of $\boldsymbol{D}_2(l)$ being subgroups of elements of $\boldsymbol{D}_3(l)$.*

Our aim is then to prove the following lemma.

Lemma 2.4. *Let G be a simple small Ree group and let $H \leqslant G$. If $\mu_G(H) \neq 0$, then $H \in$* **MaxInt**.

An important step in determining the inversion formula of a group is to determine the conjugacy classes of contributing subgroups along with their sizes. The following results are also necessary in enumerating containments between subgroups in **MaxInt**. Since they are logically independent from determining the Möbius function of G and will be used along the way to proving Lemma 2.4, we state them first.

Lemma 2.5. *Elements of* $\boldsymbol{R}(l) \cup \boldsymbol{P}(l) \cup \boldsymbol{C}_t(l) \cup \boldsymbol{C}_t(1) \cup \boldsymbol{C}_t^{\omega}(l) \cup \boldsymbol{N}_V(l) \cup \boldsymbol{N}_2(l) \cup \boldsymbol{N}_3(l)$ *are self-normalising in* G.

Proof. If $H \in \boldsymbol{R}(l)$, then H is contained only in larger subfield subgroups, all of which are simple, hence $N_G(H) = H$. Let $H \cong (3^h)^{1+1+1}:(3^h-1) \in \boldsymbol{P}(l)$ and let P denote the unique parabolic subgroup of G containing H, then the normaliser in G of H is contained in P [23, Lemma 1]. Let S denote the Sylow 3-subgroup of P, let S_0 denote the Sylow 3-subgroup of H and let $A \cong 3^h - 1$ denote a complement to S_0 in H. Since the normaliser in P of A has order $q-1$, there are $|S|$ conjugates of A in P and since $C_G(A) \cap S = 1$, the elements of S permute the conjugates of A as a regular permutation group. Since by a similar argument there are $|S_0|$ conjugates of A in H being permuted regularly by S_0, we have that $N_G(H) \cap S = S_0$. Now, the centraliser in G of A also acts regularly on the $q-1$ trivial elements of the centre of S [41, Section III.4]. Since nothing in $C_G(A) \setminus A$ normalises $Z(S_0)$, it follows that $N_G(H) \cap C_G(A) = A$. Hence $N_G(H) = S_0:A = H$.

If $H \cong 2 \times L_2(3^h) \in \boldsymbol{C}_t(l) \cup \boldsymbol{C}_t(1)$ or $H \cong 2 \times (3^h:\frac{3^h-1}{2}) \in \boldsymbol{C}_t^{\omega}$, then the normaliser of H in G must fix its unique central involution and so $N_G(H) \leqslant 2 \times L_2(q)$. Since subfield subgroups are self-normalising in $L_2(q)$ and since subgroups of $L_2(q)$ isomorphic to $3^h:\frac{3^h-1}{2}$ are also self-normalising in $L_2(q)$, we have that $N_G(H) = H$ in each case.

If $H \in \boldsymbol{N}_V(l) \cup \boldsymbol{N}_2(l) \cup \boldsymbol{N}_3(l)$, then H contains a characteristic subgroup A of shape $2^2 \times (3^h+1)/4$, $3^h - \sqrt{3^{h+1}} + 1$ or $3^h + \sqrt{3^{h+1}} + 1$ as appropriate. This characteristic subgroup is centralised in G by a Hall subgroup conjugate to $2^2 \times A_1$, A_2 or A_3 and so $N_G(A)$ is either a four-group normaliser of Hall subgroup normaliser. Let $\langle tu \rangle$ be a cyclic subgroup of order 6 in H. Since $\langle tu \rangle$ is self-normalising in $N_G(A)$, there are $4|A_1|$, $|A_2|$ or $|A_3|$ conjguates of $\langle tu \rangle$ in $N_G(A)$, as appropriate. Since $C_G(tu) \cap C_G(A)$ is trivial these conjugates are permuted regularly by the elements of $C_G(A)$. It follows then that A cannot grow in $N_G(H)$, otherwise the orbit of its $|A|$ conjugates of tu would not be preserved. □

Lemma 2.6. *Elements of* $\boldsymbol{C}_V(l)$ *are normalised in* G *by elements of* $\boldsymbol{N}_V(l)$.

Proof. If $H \cong 2^2 \times D_{(3^h+1)/2} \in \boldsymbol{C}_V(l)$, then $N_G(H)$ contains the subgroup $H:3 \in \boldsymbol{N}_V(l)$ in which H is normal. Since $H:3 \leqslant N_G(H) \leqslant N_G(V)$ where V is the characteristic normal four-group in H, the only way N can grow is by a power of an element s of order $(q+1)/4$ which centralises the characteristic normal cyclic subgroup A of order $(3^h+1)/4$ in H. Since $\langle s \rangle$ acts regularly on the involutions which normalise but do not centralise A, A does not grow in $N_G(H)$ and we have that $H:3$ is the full normaliser of G in H. □

Lemma 2.7. *Elements of* $\boldsymbol{D}_i(l)$ *are normalised in* G *by a subgroup isomorphic to* $(2^2 \times D_{2a_j(l)}):3$*, where* $i, j \in \{2,3\}$ *as appropriate.*

Proof. If $D \in \boldsymbol{D}_2(l) \cup \boldsymbol{D}_3(l)$ is isomorphic to $D_{2a_2(h)}$ or $D_{2a_3(h)}$ then D is contained in the normal dihedral subgroup of order $(q+1)/2$ in a four-group normaliser, N. Hence, the normal subgroup of D of order $a_2(h)$ or $a_3(h)$, as appropriate, is characteristic in N, and is normalised in N by an element of order 3. The normaliser in G of H is then isomorphic to $(2^2 \times D_{2a_j(l)}):3$, as claimed. □

If $H \in \boldsymbol{E} \cup \boldsymbol{V} \cup \boldsymbol{C}_2 \cup \boldsymbol{I}$, then the normaliser of H in G is clear or has already been established. This leaves the following lemma to prove.

Lemma 2.8. *Let $H \leqslant R(q)$, where $q = 3^n$, and $H \in \boldsymbol{F}(l) \cup \boldsymbol{C}_0(l) \cup \boldsymbol{C}_6^* \cup \boldsymbol{C}_3^*$.*

1. *If $H \cong 3^h \in \boldsymbol{F}(l) \cup \boldsymbol{C}_3^*$, then $N_G(H) \cong q^{1+1} : (3^h - 1)$.*
2. *If $H \cong 3^h - 1 \in \boldsymbol{C}_0(l)$, then $N_G(H) \cong D_{2(q-1)}$.*
3. *If $H \cong 6 \in \boldsymbol{C}_6^*$, then $N_G(H) \cong 2 \times q$.*

Proof. We determine the normaliser in G of H by beginning with its centraliser in G.

1. If $H \cong 3^h \in \boldsymbol{F}(l)$, then the nontrivial elements of H belong to $\mathcal{C}_3^*$ with $|H \cap \mathcal{C}_3^+| = |H \cap \mathcal{C}_3^-| = (3^h - 1)/2$. Let $S \in \mathrm{Syl}_3(G)$ be the unique Sylow 3-subgroup to which H belongs and let $h \in H$ be nontrivial. The centraliser in G of H is contained in $C_G(h)$ which has order $2q^2$. Since H belongs to the elementary abelian normal subgroup of order q^2 in S and since H belongs to an involution centraliser, we have $|C_G(H)| = 2q^2$. The elements normalising but not centralising H in G are of order $(3^h - 1)/2$ and belong to the subgroup of G isomorphic to $L_2(3^h)$ containing H. Hence the full normaliser in G of H has size $q^2(q-1)$.

2. If $H \cong 3^h - 1 \in \boldsymbol{C}_0(l)$, then the normaliser of H in G must fix the unique central involution in H and so $N_G(H) \leqslant 2 \times L_2(q)$. The normaliser in $L_2(q)$ of an element of order $(q-1)/2$ is dihedral of order $q-1$ from which is follows that $N_G(H) \cong 2 \times D_{q-1} \cong D_{2(q-1)}$.

3. If $H \cong 6 \in \boldsymbol{C}_6^*$, then as in the previous case, the normaliser in G of H must fix the unique involution of H and so $N_G(H) \leqslant 2 \times L_2(q)$. Since the normaliser in $L_2(q)$ of an element of order 3 is its Sylow 3-subgroup, we have $N_G(H) \cong 2 \times q$.

This completes the proof. □

The following result [23, Lemma 4] will aid us in determining the conjugacy classes of subgroups in **MaxInt**.

Lemma 2.9. *Let G be a simple small Ree group and let $\boldsymbol{R}(l)$ be the set of subfield subgroups of G. If $G_m, G_k \in \boldsymbol{R}(l)$ are isomorphic, then they are conjugate in G.*

Lemma 2.10. *Isomorphic elements of* **MaxInt** *are conjugate in G.*

Proof. By Lemma 2.9 isomorphic elements of $\boldsymbol{R}(l)$ are conjugate in G. Since maximal subgroups of G are conjugate in G if they are isomorphic it follows that isomorphic elements of $\boldsymbol{P}(l) \cup \boldsymbol{C}_t(l) \cup \boldsymbol{C}_t(1) \cup \boldsymbol{N}_V(l) \cup \boldsymbol{N}_2(l) \cup \boldsymbol{N}_3(l)$ are also conjugate in G. The conjugacy of isomorphic elements of $\boldsymbol{E} \cup \boldsymbol{V} \cup \boldsymbol{C}_2$ is immediate from their conjugacy within the normaliser of a Sylow 2-subgroup of G [41] and from the preceding statements it follows that isomorphic elements of $\boldsymbol{C}_V(l) \cup \boldsymbol{D}_2(l) \cup \boldsymbol{D}_3(l)$ are conjugate.

Isomorphic elements of $\boldsymbol{C}_0(l) \cup \boldsymbol{C}_6^* \cup \boldsymbol{C}_3^* \cup \boldsymbol{I}$ are generated by conjugate elements in G and so isomorphic subgroups belonging to these classes are conjugate in G. Elements of $\boldsymbol{C}_t^\omega(l)$ are involution centralisers of elements in $\boldsymbol{P}(l)$ and since involutions are conjugate in each element of $\boldsymbol{P}(l)$, isomorphic elements of $\boldsymbol{C}_t^\omega(l)$ are conjugate in G. Finally, since elements of $\boldsymbol{F}(l)$ are the Sylow 3-subgroups of conjugate elements of $\boldsymbol{C}_t(l)$, we have that isomorphic elements of $\boldsymbol{F}(l)$ are conjugate in G. This completes the proof. $\square$

3 The Möbius function of the simple small Ree groups

Throughout this section $G = R(q)$ denotes a simple small Ree group acting 2-transitively on Ω, a set of size $q^3 + 1$, as described in the previous section. We let $\omega \in \Omega$ and $P_\omega \in \boldsymbol{P}$ denote the stabiliser of ω in G. We let $t \in G$ denote an involution, $C = C_G(t) \in \boldsymbol{C}_t$ be its centraliser in G and Ω^t the points in Ω fixed by t. A subfield subgroup is denoted by $G_m \in \boldsymbol{R}(l)$, where m divides n, and $\Omega(m)$ denotes the $3^{3m} + 1$ points in Ω stabilised by the Sylow 3-subgroups of G_m. A four-group of G is denoted by V and the normaliser in G of V is denoted by $N = N_G(V) \in \boldsymbol{N}_V$.

We follow closely the style used by Downs [11] in order to calculate $\mu_G(H)$ for a subgroup $H \leqslant G$ of a group G. In order to enumerate overgroups conjugate to K in G of a fixed subgroup $H \leqslant G$ we take care since conjugacy in G is not necessarily preserved in K. The following definition will be necessary.

Definition 3.1. *Let $H \leqslant K$ be subgroups of G. We denote by $\nu_K(H)$ the number of subgroups conjugate to K in G that contain H. This is enumerated using the formula*

$$\sum_{i=1}^{n} \frac{[G : N_G(K)][K : N_K(H_i)]}{[G : N_G(H)]} = \sum_{i=1}^{n} \frac{|K||N_G(H_i)|}{|N_G(K)||N_K(H_i)|}$$

where $\{H_1, \ldots, H_n\}$ is a set of representatives from each conjugacy class in K of subgroups conjugate to H in G.

We also recall the definition and an important property of the classical Möbius function from number theory since they will be necessary for our calculations. For a positive integer n we define

$$\mu(n) = \begin{cases} (-1)^d & \text{if } n \text{ is the product of } d \text{ distinct primes} \\ 0 & \text{if } n > 1 \text{ and has a square factor greater than 1.} \end{cases}$$

If $n > 0$ is a positive integer, then

$$\sum_{l|n} \mu(l) = \begin{cases} 1 & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n > 1 \end{cases}$$

where l sums over all positive divisors of n.

3.1 Intersections with parabolic subgroups and maximal subfield subgroups

From our discussion on the action of elements of G on Ω, intersections with parabolic subgroups are relatively straightforward to determine.

Lemma 3.2. *Let $P_\omega \in \boldsymbol{P}$, let $M \neq P_\omega$ be a maximal subgroup of G and let $H = M \cap P_\omega$.*

1. *If $M \in \boldsymbol{R}$ is a maximal subfield subgroup, then $H \in \boldsymbol{P}(l) \cup \boldsymbol{C}_2 \cup \boldsymbol{I}$.*
2. *If $M \in \boldsymbol{P} \setminus \{P_\omega\}$, then $H \in \boldsymbol{C}_0$.*
3. *If $M \in \boldsymbol{C}_t$, then $H \in \boldsymbol{C}_t^\omega \cup \boldsymbol{C}_2$.*
4. *If $M \in \boldsymbol{N}_V \cup \boldsymbol{N}_2 \cup \boldsymbol{N}_3$, then $H \in \boldsymbol{C}_6^* \cup \boldsymbol{C}_3^* \cup \boldsymbol{C}_2 \cup \boldsymbol{I}$.*

Proof. (1) Let $G_m \in \boldsymbol{R}$ be a maximal subfield subgroup. If $\omega \in \Omega(m)$ then the intersection of G_m with P_ω is the stabiliser of ω in G_m, belonging to $\boldsymbol{P}(l)$. If $\omega \notin \Omega(m)$ and $H \notin \boldsymbol{I}$, then ω lies in the block of a unique involution in G_m, in which case $H \in \boldsymbol{C}_2$.
(2) The Sylow 3-subgroups of G have trivial intersection and so H consists of all elements which pointwise fix two points, hence $H \in \boldsymbol{C}_0$.
(3) If $\omega \in \Omega^t$ then H is isomorphic to the direct product of $\langle t \rangle$ with a point stabiliser in $L_2(q)$, hence $H \cong 2 \times (q : \frac{q-1}{2}) \in \boldsymbol{C}_t^\omega$. Otherwise, since ω belongs to the block of exactly one involution of M, if $\omega \notin \Omega^t$, then $H \in \boldsymbol{C}_2$.
(4) This follows from comparison of the orders of these groups. ☐

In the case of the maximal subfield subgroups, their pairwise intersection is a little less well-behaved in certain cases. From analysis using GAP it can be shown that when $G = R(27)$ a number of unexpected possibilities arise for the intersection of two subgroups isomorphic to $R(3)$ including subgroups of shape 3, 3^2, 9 and $3 \times S_3$. In order not to have to deal with these cases we prove the following lemmas which allow us to immediately rule out a large class of subgroups $H \leqslant G$ which occur as the intersection of maximal subgroups but have $\mu_G(H) = 0$. In order to determine them, we use the preceding lemmas in this section to determine the Möbius function of a number of classes of subgroups in **MaxInt**. We first prove the following partial result on the intersection of maximal subfield subgroups.

Lemma 3.3. *Let $G_{m_1}, G_{m_2} \in \boldsymbol{R}$ be maximal subfield subgroups of G. If $|\Omega(m_1) \cap \Omega(m_2)| \geq 3$, then $G_{m_1} \cap G_{m_2} \in \boldsymbol{R}(l)$.*

Proof. Let $\Omega(m_1, m_2) = \Omega(m_1) \cap \Omega(m_2)$, let ω_1, ω_2 and ω_3 be three distinct elements of $\Omega(m_1, m_2)$ and let t_i be the unique involution fixing ω_j and ω_k pointwise where $1 \leq i, j, k \leq 3$ are pairwise distinct. The subgroup $T = \langle t_1, t_2, t_3 \rangle$ is not contained in a parabolic subgroup of G and furthermore, since any pair of involutions contained in an involution centraliser or a four-group or Hall subgroup normaliser have disjoint blocks, we have that $L_2(8) \leqslant T \leqslant G_{m_0}$ where m_0 divides $\gcd(m_1, m_2)$. Since subgroups isomorphic to $L_2(8)$ are contained in a unique subgroup isomorphic to $R(3)$, which is a subgroup of both G_{m_1} and G_{m_2}, we have that $H \in \boldsymbol{R}(l)$. ☐

In the subsequent lemmas we summarise the calculation of each $\mu_G(H)$ in a table where we record the overgroups $K \geqslant H$ contributing to $\mu_G(H)$ according to their isomorphism type. These correspond to the classes of **MaxInt**. The subgroups H occur for each positive divisor h of n and their overgroups occur for k dividing n such that h divides k. Any extra conditions are recorded in the table. We record the normaliser in K of H in order to aid computation of $\nu_K(H)$, the number of overgroups of H conjugate to K in G.

Lemma 3.4. *If* $H \cong R(3^h) \in \boldsymbol{R}(l)$*, then* $\mu_G(H) = \mu(n/h)$*.*

Proof. Let H be as in the hypotheses. If M is a maximal subgroup of G containing H, then M is a maximal subfield subgroup. A counting argument then shows that for a subfield subgroup $R(3^h)$, the subfield subgroups which contain it are in one-to-one correspondence with the elements of the lattice of positive divisors of n/h. This is summarised in Table 5 from which we see that $\mu_G(H) = \mu(n/h)$. ☐

Isomorphism type of overgroup K	for $h\|k\|n$ and s.t.	$N_K(H)$	$\nu_K(H)$	$\mu_G(K)$
$R(3^k)$	–	$R(3^k)$	1	$\mu(n/k)$

Table 5: $H \cong R(3^h) \in \boldsymbol{R}(l)$

Lemma 3.5. *If* $H \cong (3^h)^{1+1+1}\!:(3^h-1) \in \boldsymbol{P}(l)$*, then* $\mu_G(H) = -\mu(n/h)$*.*

Proof. Let H be as in the hypotheses. Since H contains elements from the conjugacy classes $\mathcal{C}_0$, the only maximal subgroups containing H are maximal subfield subgroups or a unique parabolic subgroup. By Lemma 3.2 and since H is self normalising in G, for each positive number k such that $h|k|n$ the only subgroups of G containing H are a unique element in $\boldsymbol{R}(l)$ and a unique element in $\boldsymbol{P}(l)$. We present this in Table 6 from which we see that $\mu_G(H) = -\mu(n/h)$. ☐

Isomorphism type of overgroup K	for $h\|k\|n$ and s.t.	$N_K(H)$	$\nu_K(H)$	$\mu_G(K)$
$R(3^k)$	–	$(3^h)^{1+1+1}\!:(3^h-1)$	1	$\mu(n/k)$
$(3^k)^{1+1+1}\!:(3^k-1)$	$k > h$	$(3^h)^{1+1+1}\!:(3^h-1)$	1	$-\mu(n/k)$

Table 6: $H \cong (3^h)^{1+1+1}\!:(3^h-1) \in \boldsymbol{P}(l)$

Lemma 3.6. *If* $H \leqslant P_\omega$ *and* $H \cap \mathcal{C}_3^0 \neq \varnothing$*, then* $\mu_G(H) \neq 0$ *if and only if* $H \in \boldsymbol{P}(l)$*.*

Proof. Let H be as in the hypotheses. By Lemma 3.5 we can assume that $H \notin \boldsymbol{P}(l)$. Note that if $M \neq P_\omega$ is any other maximal subgroup of G containing H, then M is a maximal subfield subgroup. Also note that if H is contained in any subfield subgroup G_m not necessarily maximal, then the normaliser of H in G_m is equal to the normaliser of H in $G_m \cap P_\omega \in \boldsymbol{P}(l)$. We proceed by induction on H. Suppose that H is contained in P_ω, but no other element of $\boldsymbol{P}(l)$. This implies that H is not contained in any element of $\boldsymbol{R}(l) \setminus \{G\}$ and the only contributions to the Möbius function of H are those of G and P_ω, which cancel, and so $\mu_G(H) = 0$. Now, suppose H is as in our hypothesis and maximal so that our hypothesis is true for all overgroups of H. A counting argument shows that for each divisor k of n, the number of subgroups of G, conjugate to G_k, that contain H is equal to the number of subgroups $(3^k)^{1+1+1}{:}(3^k - 1) \in \boldsymbol{P}(l)$ that contain H. As such, the Möbius function of H cancels at each divisor and we have $\mu_G(H) = 0$. This completes the induction step. □

Before proving the following lemma we make an important observation. Let G_m be a subfield subgroup of G. A Hall subgroup of G conjugate to A_i, where $1 \leq i \leq 3$, is not necessarily contained in a Hall subgroup of G_m of order $a_i(m)$. We have more to say on this below, for now consider the particular case when $G = R(3^{3m})$. A subfield subgroup $G_m \leqslant G$ contains elements of Hall subgroups of G_m of orders $a_1(l)$, $a_2(l)$ or $a_3(l)$, but each of these elements is contained in some Hall subgroup of G conjugate to A_1 of order $(3^{3m}+1)/4$. The centraliser in G_m of such an element will then either be cyclic of order 6 or conjugate to $2 \times L_2(3)$ depending on whether $i = 1, 2$ or 3.

Lemma 3.7. *The intersection of a maximal subfield subgroup and an involution centraliser belongs to* $\boldsymbol{C}_t(l) \cup \boldsymbol{C}_t(1) \cup \boldsymbol{F}(l) \cup \boldsymbol{D}_2(l) \cup \boldsymbol{D}_3(l) \cup \boldsymbol{V} \cup \boldsymbol{C}_3^* \cup \boldsymbol{C}_2 \cup \boldsymbol{I}$.

Proof. Let $G_m \in \boldsymbol{R}$ and let $H = G_m \cap C$. Recall that if $g \in H$ fixes a point $\omega \in \Omega$, then $\omega \in \Omega^t \cap \Omega(m)$. If $|\Omega^t \cap \Omega(m)| \geq 2$, then $t \in G_m$ and H is the centraliser in G_m of t, hence $H \in \boldsymbol{C}_t(l) \cup \boldsymbol{C}_t(1)$. If $\Omega^t \cap \Omega(m) = \{\omega\}$, then $H \in \boldsymbol{F}(l) \cup \boldsymbol{C}_3^* \cup \boldsymbol{I}$.

Now suppose that $\Omega^t \cap \Omega(m) = \varnothing$. Then $t \notin G_m$ and H is isomorphic to a subgroup of $C' \cong L_2(q)$ not containing elements of order 3, or dividing $(q-1)/2$, hence H is isomorphic to a subgroup of D_{q+1} [9]. If H does not contain elements of order $k > 2$ dividing $(q+1)/4$ then H is a subgroup of a Sylow 2-subgroup of C. Since every Sylow 2-subgroup of C contains t, we have $H \leqslant V$ for some $V \in \boldsymbol{V}$ and belongs to our list. If there exists $s \in H$ of order k, then k divides $a_1(l)$ or $a_2(l/3)a_3(l/3)$ depending on whether 3 divides m or not. Let V be the unique four-group centralising $\langle s \rangle$ in G and let t' be an involution of C' normalising but not centralising $\langle s \rangle$ in G. If k divides $a_1(l)$, then $V \leqslant G_m$, contradicting our assumption, so $s \notin H$. If k divides $a_2(l/3)a_3(l/3)$, then $V \cap G_m = 1$, hence $H \cong D_{2a_2(l/3)}$ or $D_{2a_3(l/3)}$. Furthermore, since $\langle s, t' \rangle$ is centralised by V, H is contained in the normal dihedral subgroup of order $(q+1)/2$ of a four-group normaliser. □

Lemma 3.8. *If* $H \cong 2 \times L_2(3^h) \in \boldsymbol{C}_t(l)$, *then* $\mu_G(H) = -\mu(n/h)$.

Proof. Let H be as in the hypothesis. The only maximal subgroups of G containing H are those in $\boldsymbol{R}(l)$ and in $\boldsymbol{C}_t(l)$. Since elements of $\boldsymbol{C}_t(l)$ are self-normalising, for each divisor $h|k|n$ there is a unique element in $\boldsymbol{R}(l)$ and in $\boldsymbol{C}_t(l)$ containing H. This is presented in Table 7 and from this we have that $\mu_G(H) = -\mu(n/h)$. □

Isomorphism type of overgroup K	for $h\|k\|n$ and s.t.	$N_K(H)$	$\nu_K(H)$	$\mu_G(K)$
$R(3^k)$	–	$2\times L_2(3^h)$	1	$\mu(n/k)$
$2\times L_2(3^k)$	$k>h$	$2\times L_2(3^h)$	1	$-\mu(n/k)$

Table 7: $H\cong 2\times L_2(3^h)\in \boldsymbol{C}_t(l)$

Lemma 3.9. *If* $H\cong 2\times(3^h : \frac{3^h-1}{2})\in \boldsymbol{C}_t^{\omega}(l)$, *then* $\mu_G(H)=\mu(n/h)$.

Proof. Let H be as in the hypotheses. Since H contains a unique central involution and since the order of H is divisible by 9 the only maximal subgroups of G containing H are maximal subfield subgroups, a unique parabolic subgroup and a unique involution centraliser. By Lemmas 3.2, 3.3 and 3.7 if $K\in$ **MaxInt** contains H, then $K\in \boldsymbol{R}(l)\cup \boldsymbol{P}(l)\cup \boldsymbol{C}_t(l)\cup \boldsymbol{C}_t^{\omega}$. Since H is self-normalising in each subgroup which contains it, the enumeration of overgroups of H contributing to its Möbius function is as given in Table 8 from which we deduce that $\mu_G(H)=\mu(n/h)$. □

Isomorphism type of overgroup K	for $h\|k\|n$ and s.t.	$N_K(H)$	$\nu_K(H)$	$\mu_G(K)$
$R(3^k)$	–	$2\times(3^h : \frac{3^h-1}{2})$	1	$\mu(n/k)$
$(3^k)^{1+1+1} : (3^k-1)$	–	$2\times(3^h : \frac{3^h-1}{2})$	1	$-\mu(n/k)$
$2\times L_2(3^k)$	–	$2\times(3^h : \frac{3^h-1}{2})$	1	$-\mu(n/k)$
$2\times(3^k : \frac{3^k-1}{2})$	$k>h$	$2\times(3^h : \frac{3^h-1}{2})$	1	$\mu(n/k)$

Table 8: $H\cong 2\times(3^h : \frac{3^h-1}{2})\in \boldsymbol{C}_t^{\omega}(l)$

Lemma 3.10. *If* $3^2\leqslant H<2\times(3^k : \frac{3^k-1}{2})\in \boldsymbol{C}_t^{\omega}(l)$, *then* $\mu_G(H)=0$.

Proof. Let $H\leqslant R(3^n)$ be as in the hypotheses. The Sylow 3-subgroup of H has order 3^h where $2\leq h\leq k$ for h not necessarily dividing k and its non-trivial elements belong to $\mathcal{C}_3^*$. If M is a maximal subgroup of G containing H, then M is a maximal subfield subgroup, an involution centraliser or a unique parabolic subgroup. By Lemmas 3.2, 3.3 and 3.7, the subgroups which contribute to the Möbius function of H belong to $\boldsymbol{R}(l)\cup \boldsymbol{P}(l)\cup \boldsymbol{C}_t(l)\cup \boldsymbol{C}_t^{\omega}$. In analogy with the proof of Lemma 3.6, if $P\in \boldsymbol{P}(l)$ and $G_m\in \boldsymbol{R}$ are such that $H\leqslant P\leqslant G_m$, then $N_P(H)=N_{G_m}(H)$. Since $\mu_G(P)=-\mu_G(G_m)$, the contribution from each of these such groups cancel. A similar argument applies to elements of $\boldsymbol{C}_t$ and $\boldsymbol{C}_t^{\omega}$. From this it follows that $\mu_G(H)=0$. □

The preceding lemmas give the following corollary which allows us to complete our analysis of the potential intersections between maximal subfield subgroups.

Corollary 3.11. *If* $H\leqslant P_\omega\in \boldsymbol{P}$ *and* $H\notin \boldsymbol{P}(l)\cup \boldsymbol{C}_t^{\omega}(l)\cup \boldsymbol{C}_0(l)\cup \boldsymbol{C}_6^*\cup \boldsymbol{C}_3^*\cup \boldsymbol{C}_2\cup \boldsymbol{I}$, *then* $\mu_G(H)=0$.

Lemma 3.12. *If $H \leqslant G$ is equal to the intersection of two distinct maximal subfield subgroups and $\mu_G(H) \neq 0$, then $H \in$* **MaxInt**.

Proof. Let $G_m \neq G_k$ be maximal subfield subgroups of G and let $d = \gcd(m, k)$. Let $H = G_m \cap G_k$ and let $\Omega(m, k)$ denote the intersection $\Omega(m) \cap \Omega(k)$. We suppose that $H \notin \boldsymbol{I}$ and determine possible intersections according to $|\Omega(m, k)|$. By Lemma 3.3 it remains to prove the case when $|\Omega(m, k)| \leq 2$. If $\Omega(m, k) = \varnothing$, then the order of any nontrivial elements of H is 2 or $k > 2$ where k divides $q^3 + 1$. If $h \in H$ has order k, then h is normalised in H by an element of order 6 whose unique fixed point must belong to $\Omega(m, k)$, a contradiction. Hence any nontrivial element of H is an involution and H is a subgroup of an element of $\boldsymbol{E}$, all of which belong to **MaxInt**. We can now assume that $\Omega(m, k) \neq \varnothing$. If $\Omega(m, k) = \{\omega\}$, then $H \leqslant P_\omega$ and by Corollary 3.11 $H \in$ **MaxInt**.

Now suppose that $|\Omega(m, k)| = 2$. There is a unique Hall subgroup conjugate to A_0 stabilising $\Omega(m, k)$ pointwise and containing H. Note that H does not contain elements which interchange the points in $\Omega(m, k)$ since otherwise H would contain a dihedral subgroup of order $2(3^{d_0} - 1)$ where d_0 divides d. Such subgroups are contained only in subfield subgroups, involution centralisers or four-group normalisers, and in either case we would have $|\Omega(m, k)| > 2$. We then have that $H \cong 3^d - 1 \in \boldsymbol{C}_0(l) \cup \boldsymbol{C}_2 \subset$ **MaxInt**. □

3.2 Intersections with involution centralisers, four-group and Hall subgroup normalisers

We now determine the intersections between the remaining possible pairs of maximal subgroups.

Lemma 3.13. *The intersection of two distinct involution centralisers belongs to $\boldsymbol{C}_V \cup \boldsymbol{F} \cup \boldsymbol{V} \cup \boldsymbol{C}_2 \cup \boldsymbol{I}$.*

Proof. Let $t' \neq t$ be an involution in G. The intersection $H = C \cap C_G(t')$ is the the centraliser $C_C(t')$ of t' in C. If $t' \in C$, then $H = C_G(\langle t, t' \rangle) \in \boldsymbol{C}_V$.

Now suppose that $t' \notin C$. If there exists $\omega \in \Omega$ such that $\omega \in \Omega^t \cap \Omega^{t'}$, then $H \leqslant P_\omega$ and nontrivial elements of H belong to cannot have order dividing $(q+1)/2$ or order dividing $q - 1$. Hence, if $h \in H$ is nontrivial, then $h \in \mathcal{C}_3^*$ and belongs to the Sylow 3-subgroup of C stabilising ω and so $H \in \boldsymbol{F} \cup \boldsymbol{I}$. If there is no point in Ω fixed by both t and t', then any nontrivial element of H has order dividing $(q + 1)/2$. If $s \in H$ is an element of order $k > 2$ dividing $(q + 1)/4$, then a counting argument shows there is a unique four-group centralising it in G, implying $[t, t'] = 1$, a contradiction. Hence any nontrivial element of H has order 2 and is a subgroup of a Sylow 2-subgroup of G. Since t is contained in every Sylow 2-subgroup of C, and similarly for t', H must be a strict subgroup and so $H \in \boldsymbol{V} \cup \boldsymbol{C}_2 \cup \boldsymbol{I}$. □

Lemma 3.14. *The intersection of an involution centraliser with a four-group normaliser belongs to $\boldsymbol{C}_V \cup \boldsymbol{C}_t(1) \cup \boldsymbol{E} \cup \boldsymbol{C}_6^* \cup \boldsymbol{C}_3^* \cup \boldsymbol{C}_2 \cup \boldsymbol{I}$.*

Proof. The intersection $H = N \cap C$ is equal to the centraliser $C_N(t)$ of t in N. We classify possible intersections according to whether t belongs to one of the three N-conjugacy classes of involutions or whether $t \notin N$. The involution centralisers of N are described in Section 2 and belong either to $\boldsymbol{C}_V$, $\boldsymbol{C}_t(1)$ or $\boldsymbol{E}$. Now suppose that $t \notin N$. Let A denote the Hall subgroup of G contained in N. A counting argument shows that the centraliser $C_G(A) \cong 2^2 \times A$ is contained in a unique four-group normaliser, hence H is isomorphic to a subgroup of $N/C_G(A) \cong 6$. □

We are now in a position to prove the following.

Lemma 3.15. *If $H \cong 3^h - 1 \in \boldsymbol{C}_0(l)$, then $\mu_G(H) = 0$.*

Proof. Let H be as in the hypotheses. If M is a maximal subgroup containing H, then M is a maximal subfield subgroup, unique for each divisor k such that $h|k|n$, one of two parabolic subgroups or a unique involution centraliser. By Lemmas 3.2, 3.7, 3.12 and 3.13, the subgroups which contribute to the Möbius function of H are as they appear in Table 9. We see that for each k the contributions from the first pair of classes cancel with one another, as do the contributions from the second pair of classes, giving $\mu_G(H) = 0$. □

Isomorphism type of overgroup K	for $h\|k\|n$ and s.t.	$N_K(H)$	$\nu_K(H)$	$\mu_G(K)$
$R(3^k)$	–	$D_{2(3^k-1)}$	1	$\mu(n/k)$
$2 \times L_2(3^k)$	–	$D_{2(3^k-1)}$	1	$-\mu(n/k)$
$(3^k)^{1+1+1}{:}3^k - 1$	–	$3^k - 1$	2	$-\mu(n/k)$
$2 \times (3^k{:}\frac{3^k-1}{2})$	–	$3^k - 1$	2	$\mu(n/k)$
$3^k - 1$	$k > h$	$3^k - 1$	1	0

Table 9: $H \cong 3^h - 1 \in \boldsymbol{C}_0(l)$

We now determine containments between Hall subgroup normalisers of subfield subgroups. Since $\langle h \rangle$ is cyclic we need only prove the following number theoretic lemma in order to aid the accurate determination of the overgroups of such an intersection.

Lemma 3.16. *Let l be a positive factor of $n > 1$ an odd natural number. Then $a_i(l)$ divides one and only one of a_1, a_2 or a_3 for each $i = 1, 2, 3$.*

Proof. Let l and n be as in the hypothesis and $1 \leq i, j \leq 3$. It is clear that for a fixed l we have $\gcd(a_i(l), a_j(l)) = 1$ for $i \neq j$ and so the $a_i(l)$ divide at most one of the a_i. Also, $a_1(l)$ divides a_1 and if 3 divides n/l then a_1 is divisible by $a_1(l)a_2(l)a_3(l)$, so assume that $i = 2$ or 3 and that $n/l \equiv \pm 1 \bmod 3$. Consider the values of $a_2(l)$ and $a_3(l)$ modulo $a_{2,3}(l) := a_2(l)a_3(l) = 3^{2l} - 3^l + 1$. We have that $a_1(l)a_{2,3}(l) = 3^{3l} + 1$ and so $3^{3l} \equiv -1 \bmod a_{2,3}(l)$ which gives us the following chain of congruences

$$3^n \equiv (-1)3^{n-3l} \equiv (-1)^2 3^{n-6l} \equiv \cdots \equiv (-1)^k 3^{n-3kl} \bmod a_{2,3}(l)$$

where $0 \leq n - 3kl < 3l$, from which it follows that

$$3^n \equiv \begin{cases} (-1)^{\frac{n-l}{3l}} 3^l = 3^l & \mod a_{2,3}(l), \quad \text{if } (n/l) \equiv 1 \bmod 3 \\ (-1)^{\frac{n-2l}{3l}} 3^{2l} = -3^{2l} & \mod a_{2,3}(l), \quad \text{if } (n/l) \equiv -1 \bmod 3. \end{cases}$$

Similarly, we have

$$3^{\frac{n+1}{2}} \equiv \cdots \equiv (-1)^k 3^{\frac{n+1}{2} - 3kl} \mod a_{2,3}(l)$$

where this time $0 \leq \frac{n+1}{2} - 3kl < 3l$. Eventually we find

$$3^{\frac{n+1}{2}} \equiv \begin{cases} (-1)^{\frac{n-l}{6l}} 3^{\frac{l+1}{2}} & \mod a_{2,3}(l), \quad \text{if } (n/l) \equiv 1 \bmod 3 \\ (-1)^{\frac{n-5l}{6l}} 3^{\frac{5l+1}{2}} & \mod a_{2,3}(l), \quad \text{if } (n/l) \equiv -1 \bmod 3. \end{cases}$$

It can then be easily verified that

$$\frac{n-l}{6l} \equiv \frac{n-5l}{6l} \equiv \begin{cases} 0 \bmod 2 & \text{if } (n/l) \equiv 1 \bmod 4 \\ 1 \bmod 2 & \text{if } (n/l) \equiv 3 \bmod 4. \end{cases}$$

Assembling these results, along with the observation that

$$\mp 3^{\frac{5l+1}{2}} - 3^{2l} + 1 = (3^l + 1 \pm 3^{\frac{l+1}{2}})(3^{l+1} - 3^l + 1 \mp (3^{\frac{3l+1}{2}} + 3^{\frac{l+3}{2}} - 2.3^{\frac{l+1}{2}})),$$

we finally arrive at the following

$$3^n \pm 3^{\frac{n+1}{2}} + 1 \equiv \begin{cases} 3^l \pm 3^{\frac{l+1}{2}} + 1 & \mod a_{2,3}(l) \quad \text{if } (n/l) \equiv \pm 1 \bmod 12 \\ 3^l \mp 3^{\frac{l+1}{2}} + 1 & \mod a_{2,3}(l) \quad \text{if } (n/l) \equiv \pm 5 \bmod 12. \end{cases}$$

This completes the proof. □

Lemma 3.17. *The intersection of an element of* $\boldsymbol{N}_V \cup \boldsymbol{N}_2 \cup \boldsymbol{N}_3$ *with a maximal subfield subgroup belongs to* $\boldsymbol{N}_V(l) \cup \boldsymbol{N}_2(l) \cup \boldsymbol{N}_3(l) \cup \boldsymbol{C}_6^* \cup \boldsymbol{C}_3^* \cup \boldsymbol{C}_2 \cup \boldsymbol{I}$.

Proof. Let G_m be a maximal subfield subgroup, let $N \in \boldsymbol{N}_V \cup \boldsymbol{N}_2 \cup \boldsymbol{N}_3$ and let $H = G_m \cap N$. Let a generate any Hall subgroup conjugate to A_i, where $i = 1, 2, 3$. If $a \in G_m$, then H is equal to the normaliser in G_m of a which belongs to $\boldsymbol{N}_V(l) \cup \boldsymbol{N}_2(l) \cup \boldsymbol{N}_3(l)$. A counting argument can be used to show that the centraliser in G_m of $\langle a \rangle$ is contained in a unique subgroup of G conjugate to G_m. Hence, if $a \notin G_m$, then H is isomorphic to a subgroup of $N/C_N(a) \cong 6$ and so $H \in \boldsymbol{C}_6^* \cup \boldsymbol{C}_3^* \cup \boldsymbol{C}_2 \cup \boldsymbol{I}$. □

Lemma 3.18. *The intersection of two distinct four-group normalisers belongs to* $\boldsymbol{C}_t(1) \cup \boldsymbol{E} \cup \boldsymbol{C}_6^* \cup \boldsymbol{C}_3^* \cup \boldsymbol{C}_2 \cup \boldsymbol{I}$.

Proof. Recall that the normaliser of a four-group is equal to the normaliser of the unique Hall subgroup conjugate to A_1 with which it commutes and that this Hall subgroup belongs to a unique four-group normaliser. The quotient of a four-group normaliser by its normal Hall subgroup is isomorphic to $2 \times L_2(3)$ and so the intersection of two distinct four-group normalisers is isomorphic to a subgroup of $2 \times L_2(3)$.

Let N be the normaliser of a four-group V in G and let $V' \neq V$ be a four-group in G. If $V \leqslant N$ then $N \cap N_G(V')$ is the normaliser of a four-group in N and isomorphic to 2^3 or $2 \times L_2(3)$. If V is not contained in N then the intersection $N \cap N_G(V')$ is isomorphic to a subgroup of $L_2(3)$ not containing a four-group and is hence a subgroup of a cyclic group of order 6. □

Lemma 3.19. *Let $N \in \boldsymbol{N}_2 \cup \boldsymbol{N}_3$. If $M \in \boldsymbol{C}_t \cup \boldsymbol{N}_V \cup \boldsymbol{N}_2 \cup \boldsymbol{N}_3$, then $N \cap M \in \boldsymbol{C}_6^* \cup \boldsymbol{C}_3^* \cup \boldsymbol{C}_2 \cup \boldsymbol{I}$.*

Proof. This follows from comparison of the orders of the various groups and since distinct cyclic Hall subgroups have trivial intersection and belong to a unique Hall subgroup normaliser in G. □

We have now proved the following.

Lemma 3.20. *If $H \leqslant G$ is equal to the intersection of a pair of maximal subgroups of G and $\mu_G(H) \neq 0$, then $H \in$ **MaxInt**.*

3.3 The proof of Lemma 2.4 and the Möbius function of the remaining subgroups

We now show that arbitrary intersections of maximal subgroups of G do not yield new subgroups by proving Lemma 2.4.

Proof of Lemma 2.4. Let $H \notin$ **MaxInt** be a subgroup of G that occurs as the intersection of a number of maximal subgroups of G, let $\mu_G(H) \neq 0$ and let $\mathbf{M}$ be the set of maximal subgroups containing H. From the preceding lemmas we can assume $|\mathbf{M}| > 2$ and by Corollary 3.11 we can assume that H is not contained in a parabolic subgroup of G and so $\mathbf{M} \cap \boldsymbol{P} = \varnothing$.

If $\mathbf{M}$ contains more than two elements from $\boldsymbol{N}_V \cup \boldsymbol{N}_2 \cup \boldsymbol{N}_3$ then, by Lemmas 3.18 and 3.19, H is isomorphic to a subgroup of $2 \times L_2(3)$ and the only such subgroups not already contained in **MaxInt** are isomorphic to $L_2(3)$. Hence we can assume that $\mathbf{M} \cap (\boldsymbol{N}_2 \cup \boldsymbol{N}_3) = \varnothing$. To show that subgroups isomorphic to $L_2(3)$ do not appear on our list, suppose that M is maximal and contains $H \cong L_2(3)$. Then $M \in \mathbf{M} \subset \boldsymbol{R} \cup \boldsymbol{C}_t \cup \boldsymbol{N}_V$. By the argument in the proof of Lemma 3.3, if $\mathbf{M}$ contains at least two maximal subfield subgroups, then their intersection must be an element of $\boldsymbol{R}(l)$ and so we can assume that $\mathbf{M} \cap \boldsymbol{R}$ consists of a single subfield subgroup isomorphic to $R(3)$. By Lemma 3.13 we can assume that $\mathbf{M}$ contains at most one involution centraliser. By Lemma 3.7 we may assume that H is equal to the intersection of $M_0 \cong 2 \times L_2(3)$ with a number of elements from $\boldsymbol{N}_V$. Since the normaliser of a four-group contained in M_0 is either M_0 or is isomorphic to its elementary abelian Sylow 2-group of order 8 we have that $H \notin$ **MaxInt**.

If $\mathbf{M} \subset \boldsymbol{R} \cup \boldsymbol{C}_t \cup \boldsymbol{N}_2$ or $\mathbf{M} \subset \boldsymbol{R} \cup \boldsymbol{C}_t \cup \boldsymbol{N}_3$, then by Lemmas 3.7, 3.17 and 3.19 $H \in$ **MaxInt**. Hence, we can assume that $\mathbf{M} \subset \boldsymbol{R} \cup \boldsymbol{C}_t \cup \boldsymbol{N}_V$ contains at most one element from $\boldsymbol{R}$ and at most one element from $\boldsymbol{N}_V$. Moreover, by Lemma 3.17 again we can assume $\mathbf{M} \subset \boldsymbol{C}_t \cup \boldsymbol{N}_V(l)$. Finally, by Lemmas 3.13 and 3.14, $H \in \mathbf{MaxInt}$, a contradiction. This completes the proof. □

It remains to determine the Möbius function for elements of the remaining classes.

Lemma 3.21. *If $H \cong (2^2 \times D_{(3^h+1)/2}):3 \in \boldsymbol{N}_V(l)$, then $\mu_G(H) = -\mu(n/h)$.*

Proof. Let H be as in the hypothesis. The only maximal subgroups of G containing H are maximal subfield subgroups, and the normaliser of the normal four-group in H. From the calculations in Table 10 we find that $\mu_G(H) = -\mu(n/k)$. □

Isomorphism type of overgroup K	for $h\|k\|n$ and s.t.	$N_K(H)$	$\nu_K(H)$	$\mu_G(K)$
$R(3^k)$	–	$(2^2 \times D_{(3^h+1)/2}):3$	1	$\mu(n/k)$
$(2^2 \times D_{(3^k+1)/2}):3$	$k > h$	$(2^2 \times D_{(3^h+1)/2}):3$	1	$-\mu(n/k)$

Table 10: $H \cong (2^2 \times D_{(3^h+1)/2}):3 \in \boldsymbol{N}_V(l)$

Lemma 3.22. *If $H \cong 3^h - 3^{\frac{h+1}{2}} + 1:6$ or $3^h + 3^{\frac{h+1}{2}} + 1:6 \in \boldsymbol{N}_2(l) \cup \boldsymbol{N}_3(l)$, then $\mu_G(H) = -\mu(n/h)$.*

Proof. Let H be as in the hypothesis. By Lemma 3.16, for each divisor k such that $h|k|n$ there is a unique element from $\boldsymbol{N}_V(l) \cup \boldsymbol{N}_2(l) \cup \boldsymbol{N}_3(l)$ containing H. Similarly, there is a unique element from $\boldsymbol{R}(l)$ for each such k. These contributions cancel and we present the calculations for $H \in \boldsymbol{N}_2(l)$ in Table 11, the calculations for $H \in \boldsymbol{N}_3(l)$ are similar. We are then left with $\mu_G(H) = -\mu_G(R(3^h)) = -\mu(n/h)$. □

Isomorphism type of overgroup K	for $h\|k\|n$ and s.t.	$N_K(H)$	$\nu_K(H)$	$\mu_G(K)$
$R(3^k)$	–	H	1	$\mu(n/k)$
$(2^2 \times D_{(3^k+1)/2}):3$	$\frac{k}{h} \equiv 0 \bmod 3$	H	1	$-\mu(n/k)$
$3^k + \sqrt{3^{k+1}} + 1:6$	$\frac{k}{h} \equiv \pm 5 \bmod 12$	H	1	$-\mu(n/k)$
$3^k - \sqrt{3^{k+1}} + 1:6$	$k > h, \frac{k}{h} \equiv \pm 1 \bmod 12$	H	1	$-\mu(n/k)$

Table 11: $H \cong 3^h - 3^{\frac{h+1}{2}} + 1:6 \in \boldsymbol{N}_2(l)$

Lemma 3.23. *If $H \cong 2^2 \times D_{(3^h+1)/2} \in \boldsymbol{C}_V(l)$, then $\mu_G(H) = 3\mu(n/h)$.*

Proof. Let H be as in the hypotheses. For each divisor k such that $h|k|n$, H belongs to a unique element of $\boldsymbol{R}(l)$ and to a unique element of $\boldsymbol{N}_v(l)$. The contributions from each of these groups cancel, as shown in Table 12, and the remaining contributions from the involution centralisers give $\mu_G(H) = 3\mu(n/h)$. □

Lemma 3.24. *If $H \cong D_{2a_2(h)}$ or $D_{2a_3(h)} \in \boldsymbol{D}_2(l) \cup \boldsymbol{D}_3(l)$, then $\mu_G(H) = 0$.*

Isomorphism type of overgroup K	for $h\|k\|n$ and s.t.	$N_K(H)$	$\nu_K(H)$	$\mu_G(K)$
$R(3^k)$	–	$(2^2 \times D_{(3^h+1)/2}){:}3$	1	$\mu(n/k)$
$(2^2 \times D_{(3^k+1)/2}){:}3$	–	$(2^2 \times D_{(3^h+1)/2}){:}3$	1	$-\mu(n/k)$
$2 \times L_2(3^k)$	–	$2^2 \times D_{(3^h+1)/2}$	3	$-\mu(n/k)$
$2^2 \times D_{(3^k+1)/2}$	$k > h$	$2^2 \times D_{(3^h+1)/2}$	1	$3\mu(n/k)$

Table 12: $H \cong 2^2 \times D_{(3^h+1)/2} \in \boldsymbol{C}_V(l)$

Proof. Let H be as in the hypotheses and note that these subgroups arise when h is such that $3h|n$. The overgroups of H for a divisor k such that $h|k|n$ are dependent on the parity of $\frac{k}{h}$ modulo 3. We present the case $H \in \boldsymbol{D}_2(l)$ in Table 13, the case $H \in \boldsymbol{D}_3(l)$ is similar. From the table it is clear that for each divisor k, the contributions to $\mu_G(H)$ cancel with one another and so $\mu_G(H) = 0$. □

Isomorphism type of overgroup K	for $h\|k\|n$ and s.t.	$N_K(H)$	$\nu_K(H)$	$\mu_G(K)$
$R(3^k)$	$\frac{k}{h} \equiv 0 \bmod 3$	$(2^2 \times H){:}3$	1	$\mu(n/k)$
$(2^2 \times D_{(3^k+1)/2}){:}3$	$\frac{k}{h} \equiv 0 \bmod 3$	$(2^2 \times H){:}3$	1	$-\mu(n/k)$
$2 \times L_2(3^k)$	$\frac{k}{h} \equiv 0 \bmod 3$	$2^2 \times H$	3	$-\mu(n/k)$
$2^2 \times D_{(3^k+1)/2}$	$\frac{k}{h} \equiv 0 \bmod 3$	$2^2 \times H$	1	$3\mu(n/k)$
$R(3^k)$	$\frac{k}{h} \equiv \pm 1 \bmod 3$	$H{:}3$	4	$\mu(n/k)$
$3^k + \sqrt{3^{k+1}} + 1{:}6$	$\frac{k}{h} \equiv \pm 5 \bmod 12$	$H{:}3$	4	$-\mu(n/k)$
$3^k - \sqrt{3^{k+1}} + 1{:}6$	$\frac{k}{h} \equiv \pm 1 \bmod 12$	$H{:}3$	4	$-\mu(n/k)$

Table 13: $H \cong D_{2a_2(h)} \in \boldsymbol{D}_2(l)$

Lemma 3.25. *If $H \cong 2 \times L_2(3)$, then $\mu_G(H) = -2\mu(n)$.*

Proof. Subgroups isomorphic to H are self-normalising in G and so for each k such that k divides n belong to a unique element of each of $\boldsymbol{R}(l)$, $\boldsymbol{C}_t(l)$ and $\boldsymbol{N}_V(l)$. Since $n > 1$, the summation over the $R(3^k)$ is equal to the summation over positive divisors of k which is equal to 0. For the same reason the remainder of the remaining two classes, as shown in Table 14, give $\mu_G(H) = -2\mu(n)$. □

Isomorphism type of overgroup K	for $h\|k\|n$ and s.t.	$N_K(H)$	$\nu_K(H)$	$\mu_G(K)$
$R(3^k)$	–	$2 \times L_2(3)$	1	$\mu(n/k)$
$2 \times L_2(3^k)$	$h > 1$	$2 \times L_2(3)$	1	$-\mu(n/k)$
$(2^2 \times D_{(3^k+1)/2}){:}3$	$h > 1$	$2 \times L_2(3)$	1	$-\mu(n/k)$

Table 14: $H \cong 2 \times L_2(3) \in \boldsymbol{C}_t(1)$

Lemma 3.26. *If $H \cong 2^3 \in \boldsymbol{E}$, then $\mu_G(H) = 21\mu(n)$.*

Proof. As presented in Table 15, the summation over the $R(3^k)$ equates to 0, as does the total summation of the succeeding three lines. From the final line we then have that $\mu_G(2^3) = 21\mu(n)$. □

Isomorphism type of overgroup K	for $h\|k\|n$ and s.t.	$N_K(H)$	$\nu_K(H)$	$\mu_G(K)$
$R(3^k)$	–	$2^3{:}7{:}3$	1	$\mu(n/k)$
$2 \times L_2(3^k)$	$k > 1$	$2 \times L_2(3)$	7	$-\mu(n/k)$
$(2^2 \times D_{(3^k+1)/2}){:}3$	$k > 1$	$2 \times L_2(3)$	7	$-\mu(n/k)$
$2 \times L_2(3)$	–	$2 \times L_2(3)$	7	$-2\mu(n)$
$2^2 \times D_{(3^k+1)/2}$	$k > 1$	2^3	7	$3\mu(n/k)$

Table 15: $H \cong 2^3 \in \boldsymbol{E}$

Lemma 3.27. *If $H \cong 2^2 \in \boldsymbol{V}$, then $\mu_G(H) = 0$.*

Proof. Four-groups are conjugate in G but not necessarily conjugate in subgroups of G. Where this is the case, in the $N_K(H)$ column in Table 16 the number in parentheses denotes the number of conjugacy classes of V whose normaliser in K is of the specified isomorphism type. This quantity is incorporated into the entry in the $\nu_K(H)$ column. In order to make verification of the arithmetic a little easier, we have separated contributions from overgroups isomorphic to K according to whether the contribution depends on k or not. In the cases where there is no dependence on k the usual properties of the classical Möbius function leave us a few terms to tidy up and we eventually find that $\mu_G(2^2) = 0$. □

Isomorphism type of overgroup K	for $h\|k\|n$ and s.t.	$N_K(H)$	$\nu_K(H)$	$\mu_G(K)$
$R(3^k)$	–	$(2^2 \times D_{(3^k+1)/2}){:}3$	$(3^n+1)/(3^k+1)$	$\mu(n/k)$
$(2^2 \times D_{(3^k+1)/2}){:}3$	$k > 1$	$(2^2 \times D_{(3^k+1)/2}){:}3$	$(3^n+1)/(3^k+1)$	$-\mu(n/k)$
$2^2 \times D_{(3^k+1)/2}$	$k > 1$	$2^2 \times D_{(3^k+1)/2}$	$(3^n+1)/(3^k+1)$	$3\mu(n/k)$
$2 \times L_2(3^k)$	$k > 1$	$2^2 \times D_{(3^k+1)/2}$	$3(3^n+1)/(3^k+1)$	$-\mu(n/k)$
$(2^2 \times D_{(3^k+1)/2}){:}3$	$k > 1$	(2) 2^3	$3(3^n+1)/2$	$-\mu(n/k)$
$2^2 \times D_{(3^k+1)/2}$	$k > 1$	(6) 2^3	$3(3^n+1)/2$	$3\mu(n/k)$
$2 \times L_2(3^k)$	$k > 1$	(1) 2^3, (1) $2 \times L_2(3)$	3^n+1	$-\mu(n/k)$
$2 \times L_2(3)$	–	(2) 2^3, (1) $2 \times L_2(3)$	$7(3^n+1)/4$	$-2\mu(n)$
2^3	–	(7) 2^3	$(3^n+1)/4$	$21\mu(n)$

Table 16: $H \cong 2^2 \in \boldsymbol{V}$

Lemma 3.28. *If $H \in \boldsymbol{C}_6 \cup \boldsymbol{C}_3^* \cup \boldsymbol{C}_2 \cup \boldsymbol{I}$, then $\mu_G(H) = 0$.*

Proof. In the case $H \in \boldsymbol{C}_6^* \cup \boldsymbol{C}_3^*$ it is clear, but tedious, from the enumerations in Tables 17 and 18 that $\mu_G(H) = 0$. In the case that $H \in \boldsymbol{C}_2$, where in some subgroups the elements of order 2 split into multiple conjugacy classes, we present this in Table 19 in such a way as to make the calculations easier to check. Eventually, as in the case $H \in \boldsymbol{I}$ in Table 20. Again, after some calculation we see that $\mu_G(H) = 0$ in both of these cases. □

Isomorphism type of overgroup K	for $h\|k\|n$ and s.t.	$N_K(H)$	$\nu_K(H)$	$\mu_G(K)$
$R(3^k)$	–	2×3^k	3^{n-k}	$\mu(n/k)$
$(3^k)^{1+1+1}:3^k - 1$	–	2×3^k	3^{n-k}	$-\mu(n/k)$
$2 \times L_2(3^k)$	$k > 1$	2×3^k	3^{n-k}	$-\mu(n/k)$
$2 \times (3^k : \frac{3^k-1}{2})$	$k > 1$	2×3^k	3^{n-k}	$\mu(n/k)$
$3^k + \sqrt{3^{k+1}} + 1:6$	–	6	3^{n-1}	$-\mu(n/k)$
$3^k - \sqrt{3^{k+1}} + 1:6$	$k > 1$	6	3^{n-1}	$-\mu(n/k)$
$(2^2 \times D_{(3^k+1)/2}):3$	$k > 1$	6	3^{n-1}	$-\mu(n/k)$
$2 \times L_2(3)$	–	6	3^{n-1}	$-2\mu(n)$

Table 17: $H \cong \langle tu \rangle \in \boldsymbol{C}_6^*$

Isomorphism type of overgroup K	for $h\|k\|n$ and s.t.	$N_K(H)$	$\nu_K(H)$	$\mu_G(K)$
$R(3^k)$	–	$3^k \times (3^k:2)$	$3^{2(n-k)}$	$\mu(n/k)$
$(3^k)^{1+1+1}:(3^k - 1)$	–	$3^k \times (3^k:2)$	$3^{2(n-k)}$	$-\mu(n/k)$
$2 \times L_2(3^k)$	$k > 1$	2×3^k	3^{2n-k}	$-\mu(n/k)$
$2 \times (3^k : \frac{3^k-1}{2})$	$k > 1$	2×3^k	3^{2n-k}	$\mu(n/k)$
$3^k + \sqrt{3^{k+1}} + 1:6$	–	6	3^{2n-1}	$-\mu(n/k)$
$3^k - \sqrt{3^{k+1}} + 1:6$	$k > 1$	6	3^{2n-1}	$-\mu(n/k)$
$(2^2 \times D_{(3^k+1)/2}):3$	$k > 1$	6	3^{2n-1}	$-\mu(n/k)$
$2 \times L_2(3)$	–	6	3^{2n-1}	$-2\mu(n)$

Table 18: $H \cong \langle u \rangle \in \boldsymbol{C}_3^*$

Remark 3.29. *It follows that the Möbius number of a simple small Ree group is* 0. *This is consistent with Theorem 1.10.*

This completes the proof of Theorem 1.11. In the case when $G = R(27)$ the full subgroup lattice and Möbius function has been determined by Connor and Leemans [5] and, from personal correspondence with Leemans in October 2014, it was noted that apart from a few errors, such as their $\mu_G(2 \times (3^3:13)) = 0$, their calculations agree with ours.

Isomorphism type of overgroup K	for $h\|k\|n$ and s.t.	$N_K(H)$	$\nu_K(H)$	$\mu_G(K)$
$R(3^k)$	–	$2 \times L_2(3^k)$	$3^n(3^{2n}-1)/3^k(3^{2k}-1)$	$\mu(n/k)$
$2 \times L_2(3^k)$	$k > 1$	$2 \times L_2(3^k)$	$3^n(3^{2n}-1)/3^k(3^{2k}-1)$	$-\mu(n/k)$
$(3^k)^{1+1+1}:(3^k-1)$	–	$2 \times (3^k:\frac{3^k-1}{2})$	$3^n(3^{2n}-1)/3^k(3^k-1)$	$-\mu(n/k)$
$2 \times (3^k:\frac{3^k-1}{2})$	$k > 1$	$2 \times (3^k:\frac{3^k-1}{2})$	$3^n(3^{2n}-1)/3^k(3^k-1)$	$\mu(n/k)$
$(2^2 \times D_{(3^k+1)/2}):3$	$k > 1$	$2^2 \times D_{(3^k+1)/2}$	$3^n(3^{2n}-1)/2(3^k+1)$	$-\mu(n/k)$
$2^2 \times D_{(3^k+1)/2}$	$k > 1$	(3) $2^2 \times D_{(3^k+1)/2}$	$3^n(3^{2n}-1)/2(3^k+1)$	$3\mu(n/k)$
$2 \times L_2(3^k)$	$k > 1$	(2) $2^2 \times D_{(3^k+1)/2}$	$3^n(3^{2n}-1)/(3^k+1)$	$-\mu(n/k)$
$3^k+\sqrt{3^{k+1}}+1:6$	–	6	$3^{n-1}(3^{2n}-1)/2$	$-\mu(n/k)$
$3^k-\sqrt{3^{k+1}}+1:6$	$k > 1$	6	$3^{n-1}(3^{2n}-1)/2$	$-\mu(n/k)$
$(2^2 \times D_{(3^k+1)/2}):3$	$k > 1$	(1) 2^3, (1) $2 \times L_2(3)$	$3^{n-1}(3^{2n}-1)/2$	$-\mu(n/k)$
$2^2 \times D_{(3^k+1)/2}$	$k > 1$	(4) 2^3	$3^{n-1}(3^{2n}-1)/2$	$3\mu(n/k)$
$2 \times L_2(3)$	–	(2) 2^3, (1) $2 \times L_2(3)$	$7.3^{n-1}(3^{2n}-1)/8$	$-2\mu(n)$
2^3	–	(7) 2^3	$3^{n-1}(3^{2n}-1)/8$	$21\mu(n)$

Table 19: $H \cong \langle t \rangle \in \boldsymbol{C}_2$

Isomorphism type of overgroup K	for $h\|k\|n$ and s.t.	$\nu_K(H)$	$\mu_G(K)$
$R(3^k)$	–	$\|G\|/3^{3k}(3^{3k}+1)(3^k-1)$	$\mu(n/k)$
$3^k+\sqrt{3^{k+1}}+1:6$	–	$\|G\|/6(3^k+\sqrt{3^{k+1}}+1)$	$-\mu(n/k)$
$3^k-\sqrt{3^{k+1}}+1:6$	$k > 1$	$\|G\|/6(3^k-\sqrt{3^{k+1}}+1)$	$-\mu(n/k)$
$(3^k)^{1+1+1}:(3^k-1)$	–	$\|G\|/3^{3k}(3^k-1)$	$-\mu(n/k)$
$2 \times L_2(3^k)$	$k > 1$	$\|G\|/3^k(3^{2k}-1)$	$-\mu(n/k)$
$2 \times (3^k:\frac{3^k-1}{2})$	$k > 1$	$\|G\|/3^k(3^k-1)$	$\mu(n/k)$
$(2^2 \times D_{(3^k+1)/2}):3$	$k > 1$	$\|G\|/6(3^k+1)$	$-\mu(n/k)$
$2^2 \times D_{(3^k+1)/2}$	$k > 1$	$\|G\|/6(3^k+1)$	$3\mu(n/k)$
$2 \times L_2(3)$	–	$\|G\|/24$	$-2\mu(n)$
2^3	–	$\|G\|/168$	$21\mu(n)$

Table 20: $H \in \boldsymbol{I}$

4 Eulerian functions of the small Ree groups

In this section we determine various Eulerian functions associated with the small Ree groups and use them to prove a number of results regarding their generation and asymptotic generation. We introduce a number of summatory functions and their corresponding Eulerian functions.

Definition 4.1. *Let G be a finite group and $(k_1, \ldots, k_n)$ be an ordered n-tuple of elements from $\mathbb{N}_{>0}$. We define*

$$\sigma_{k_1,\ldots,k_n}(G) = \{(x_1, \ldots, x_n) \in G^n \mid o(x_i) = k^i \text{ for } 1 \leq i \leq n\}.$$

*By abuse of notation we may write $k_i = \infty$ to mean we do not specify the order of x_i. The corresponding Eulerian function is $\phi_{k_1,\ldots,k_n}(G)$. We say that G is $(k_1, \ldots, k_n)$-**generated** if $\phi_{k_1,\ldots,k_n} \neq 0$.*

Remark 4.2. *Let* Γ *be the group*

$$\Gamma = \langle x_1, \ldots, x_n \mid x_1^{k_1} = \cdots = x_n^{k_n} = 1 \rangle,$$

where any relation $x_i^{\infty} = 1$ *for* $1 \leq i \leq n$ *is ignored. For a finite group* G *the quantity* $\phi_{k_1,\ldots,k_n}(G)$ *corresponds to the number of smooth epimorphisms from* Γ *to* G*.*

Definition 4.3. *The Hecke group* H_n*, for* $n \in \mathbb{N}_{>0} \cup \{\infty\}$*, is the group generated by one element of order* 2*, one element of order* n *and no other relations. In particular, the Hecke group* H_3 *is isomorphic to the modular group* $PSL_2(\mathbb{Z})$*. We write* $\eta_n(G) = \phi_{2,n}(G)$ *for their corresponding Eulerian function.*

Definition 4.4. *Let* $\Gamma = C_2 * V$ *be the free product of an involution with a four-group*

$$\Gamma = \langle x_1, x_2, x_3 \mid x_1^2 = x_2^2 = x_3^2 = [x_2, x_3] = 1 \rangle.$$

We denote the Eulerian function of Γ *as* $\phi_{2,V}(G)$*.*

Definition 4.5. *We use the following to denote the number of torsion-free normal subgroups of the appropriate finitely presented group whose quotient is isomorphic to* G*.*

$$d_n(G) = \frac{\phi_n(G)}{|\mathrm{Aut}(G)|},\ d_{k_1,\ldots,k_n} = \frac{\phi_{k_1,\ldots,k_n}}{|\mathrm{Aut}(G)|},\ d_{2,V}(G) = \frac{\phi_{2,V}(G)}{|\mathrm{Aut}(G)|} \text{ and } h_n = \frac{\eta_n}{|\mathrm{Aut}(G)|}.$$

In order to determine these Eulerian functions we require the following definition.

Definition 4.6. *Let* G *be a finite group and* n *a positive integer. We write* $|G|_n$ *for the number of elements of* G *having order* n*. By abuse of notation we write* $|G|_\infty = |G|$*. We then have the relation*

$$\sigma_{k_1,\ldots,k_n}(G) = \prod_{i=1}^{n} |G|_{k_i}.$$

4.1 Some Eulerian functions of $R(q)$

In Tables 21 and 22 we present the values of $|H|_n$ for $n \in \{2, 3, 6, 7, 9\}$ and $H \leqslant G$ with $\mu_G(H) \neq 0$. These are easily determined from the conjugacy classes of G.

From these values it is routine, but tedious, to determine a number of Eulerian functions for a simple small Ree group. We present a number of such functions as the following corollary to Theorem 1.11.

Isomorphism type of $H \leqslant G$	$\lvert H\rvert_2$	$\lvert H\rvert_3$	$\lvert H\rvert_6$
$R(3^h)$	$3^{2h}(3^{2h}-3^h+1)$	$(3^{3h}+1)(3^{2h}-1)$	$3^{2h}(3^{3h}+1)(3^h-1)$
$3^h+\sqrt{3^{h+1}}+1:6$	$3^h+\sqrt{3^{h+1}}+1$	$2(3^h+\sqrt{3^{h+1}}+1)$	$2(3^h+\sqrt{3^{h+1}}+1)$
$3^h-\sqrt{3^{h+1}}+1:6$	$3^h-\sqrt{3^{h+1}}+1$	$2(3^h-\sqrt{3^{h+1}}+1)$	$2(3^h-\sqrt{3^{h+1}}+1)$
$(3^h)^{1+1+1}:3^h-1$	3^{2h}	$3^{2h}-1$	$3^{2h}(3^h-1)$
$2\times L_2(3^h)$	$3^{2h}-3^h+1$	$3^{2h}-1$	$3^{2h}-1$
$2\times(3^h:\frac{3^h-1}{2})$	1	3^h-1	3^h-1
$(2^2\times D_{(3^h+1)/2}):3$	3^h+4	$2(3^h+1)$	$2(3^h+1)$
$2^2\times D_{(3^h+1)/2}$	3^h+4	–	–
$2\times L_2(3)$	7	8	8
2^3	7	–	–

Table 21: Values of $|H|_n$ for $n=2,3$ or 6.

Isomorphism		$\lvert H\rvert_7$		
type of $H \leqslant G$	$7\mid a_1(h)$	$7\mid a_2(h)$	$7\mid a_3(h)$	$\lvert H\rvert_9$
$R(3^h)$	$\lvert H\rvert/a_1(h)$	$\lvert H\rvert/a_2(h)$	$\lvert H\rvert/a_3(h)$	$3^{2h}(3^{3h}+1)(3^h-1)$
$(3^h)^{1+1+1}:3^h-1$	–	–	–	$3^{2h}(3^h-1)$
$2\times L_2(3^h)$	$3^{h+1}(3^h-1)$	–	–	–
$(2^2\times D_{(3^h+1)/2}):3$	6	–	–	–
$2^2\times D_{(3^h+1)/2}$	6	–	–	–
$3^h-\sqrt{3^{h+1}}+1:6$	–	6	–	–
$3^h+\sqrt{3^{h+1}}+1:6$	–	–	6	–

Table 22: Values of $|H|_n$ for $n=7$ or 9.

Corollary 4.7. *Let $G=R(3^n)$ be a simple small Ree group. Then,*

$$\phi_2(G)=|G|\sum_{l|n}\mu\left(\frac{n}{l}\right)(3^l-1)(3^{6l}-3^{2l}-16),\quad \phi_{2,2,2}(G)=|G|\sum_{l|n}\mu\left(\frac{n}{l}\right)3^l(3^{2l}-3^l+1)^2,$$

$$\phi_{2,\infty}(G)=|G|\sum_{l|n}\mu\left(\frac{n}{l}\right)(3^l-1)(3^{3l}-3^l-2),\quad \phi_{3,3}(G)=|G|\sum_{l|n}\mu\left(\frac{n}{l}\right)3^l(3^{2l}+3^l-4),$$

$$\phi_{3,\infty}(G)=|G|\sum_{l|n}\mu\left(\frac{n}{l}\right)(3^l-1)(3^{4l}-3^{3l}-3^l-4),\quad \eta_3(G)=|G|\sum_{l|n}\mu\left(\frac{n}{l}\right)(3^l-1)^2,$$

$$\phi_{6,\infty}(G)=|G|\sum_{l|n}\mu\left(\frac{n}{l}\right)(3^l-1)(3^{5l}-3^l-6),\quad \eta_6(G)=|G|\sum_{l|n}\mu\left(\frac{n}{l}\right)3^l(3^{2l}-3^l-2),$$

$$\phi_{9,\infty}(G)=|G|\sum_{l|n}\mu\left(\frac{n}{l}\right)3^{5l}(3^l-1),\quad \eta_9(G)=|G|\sum_{l|n}\mu\left(\frac{n}{l}\right)3^{2l}(3^l-1),$$

for the Hecke group H_7 we have

$$\eta_7(G)=|G|\sum_{l|n}\mu\left(\frac{n}{l}\right)g(l)\quad \textit{where } g(l)=\begin{cases}3^{2l}a_2(l)-1 & \textit{if } l\equiv\pm1 \bmod 12\\ 3^{3l}-2.3^{2l}+5 & \textit{if } l\equiv\pm3 \bmod 12\\ 3^{2l}a_3(l)-1 & \textit{if } l\equiv\pm5 \bmod 12.\end{cases}$$

and, for the free product $C_2 * V$, we have

$$\phi_{2,V}(G) = |G| \sum_{l|n} \mu\left(\frac{n}{l}\right)(3^{2l}+4)(3^l-3).$$

Remark 4.8. *The automorphism group of $G = R(3^n)$ has order $n|G|$ from which the values of $d_2(G)$, etc. can easily be determined.*

Remark 4.9. *The quantity $d_2(G)$ has a number of other interpretations, a few of which we mention here.*

- *If G is simple, this is equal to the largest positive integer, d, such that G^d can be 2-generated [17].*
- *In Grothendieck's theory of dessins d'enfants [16] this is equal to the number of distinct regular dessins with automorphism group isomorphic to G.*
- *It is the number of oriented hypermaps having automorphism group isomorphic to G [13].*

We evaluate $d_2(R(3^n))$ for the first few values of n and give these in Table 23. The value $d_2(R(3))$ can be found in [31] or determined in GAP.

G	$d_2(G)$
$R(3)$	1 136
$R(3^3)$	3 357 637 312
$R(3^5)$	9 965 130 790 521 984
$R(3^7)$	34 169 987 177 353 651 660 608
$R(3^9)$	127 166 774 444 890 319 085 083 766 720

Table 23: Values of $d_2(G)$ for $R(q)$, $q \leq 3^9$.

Remark 4.10. *The quantities $d_2(G)$, $d_{2,\infty}(G)$, $d_{2,2,2}(G)$, $d_{2,V}(G)$ and $h_3(G)$ are of interest in the study of regular maps as they correspond to various classes of maps on surfaces having automorphism group isomorphic to G. We refer the reader to [13, 14] for more details.*

It is known that the simple small Ree groups are quotients of the modular group $PSL_2(\mathbb{Z})$ [19, 28]; with the Möbius function we can say a little more.

Corollary 4.11. *Let $G = R(3^n)$ be a simple small Ree group. If d is a positive integer such that*

$$d \leq h_3(G) = \frac{\eta_3(G)}{|\mathrm{Aut}(G)|} = \frac{1}{n}\sum_{l|n} \mu\left(\frac{n}{l}\right)(3^l-1)^2,$$

then G^d can be $(2,3)$-generated.

G	$h_3(G)$
$R(3)$	2
$R(3^3)$	224
$R(3^5)$	11 712
$R(3^7)$	682 656
$R(3^9)$	43 042 272

Table 24: Values of $h_3(G)$ for $R(q)$, $q \leq 3^9$.

We evaluate $h_3(R(3^n))$ for the first few values of n and give these in Table 24. The value of $h_3(R(3))$ can be found in [31] or determined in GAP.

Remark 4.12. *We note that the Möbius function can also be used to determine the number of Hurwitz triples of G, that is generating sets $\{x, y, z\}$ such that $x^2 = y^3 = z^7 = xyz = 1$. From this, the number of distinct Hurwitz curves with automorphism group isomorphic to $R(3^n)$ can also be found. Groups for which such a generating set occurs are known as Hurwitz groups and their study is well documented, see [3, 4] for Conder's surveys of this area. We shall say no more about them here since it was proven by Malle [28] and independently by Jones [19] using a restricted form of Möbius inversion that the simple small Ree groups are Hurwitz groups.*

4.2 Asymptotic results

The Möbius function can also be used to prove results on asymptotic generation of groups. In the case of probabilistic generation of finite simple groups we direct the interested reader to the recent survey by Liebeck [25]. We begin with the following definition.

Definition 4.13. *Let G be a group. We denote by $P_{a,b}(G)$ the probability that a randomly chosen element of order a and a randomly chosen element of order b generate G. More generally we define*

$$P_{k_1,\ldots,k_n}(G) = \frac{\phi_{k_1,\ldots,k_n}(G)}{\sigma_{k_1,\ldots,k_n}(G)}$$

where $k_1, \ldots, k_n \in \mathbb{N}_{>0} \cup \{\infty\}$. We define $P_{2,V}(G)$ analogously.

The following result due to Kantor and Lubotzky [20, Proposition 4] was proved using probabilistic arguments to enumerate pairs of elements which are contained in a maximal subgroup. We present an independent proof using the Möbius function.

Corollary 4.14 (Kantor–Lubotzky '90)**.** *Let $G = R(3^n)$ be a small Ree group. Then $P_{\infty,\infty} \to 1$ as $|G| \to \infty$.*

Proof. From Corollary 4.7 we have that

$$P_{\infty,\infty}(G) = \frac{\phi_2(G)}{|G|^2} = \frac{1}{|G|} \sum_{l|n} \mu\left(\frac{n}{l}\right)(3^l - 1)(3^{6l} - 3^{2l} - 16).$$

Since this tends to 1 as $|G| \to \infty$, we have the desired result. □

The following results due to Liebeck and Shalev [26, Theorems 1.1 and 1.2] can be proven using a similar argument.

Corollary 4.15 (Liebeck–Shalev, '96). *Let $G = R(3^n)$ be a simple small Ree group. Then*

1. $P_{2,\infty}(G) \to 1$ *as* $|G| \to \infty$ *and*
2. $P_{3,\infty}(G) \to 1$ *as* $|G| \to \infty$.

We can prove a number of additional results on asymptotic results using Tables 21 and 22 and the results in Corollary 4.7.

Corollary 4.16. *Let $G = R(3^n)$ be a simple small Ree group and $(k_1, \ldots, k_n)$ an n-tuple of positive integers. Then, each of*

$$P_{2,3}(G), \qquad P_{2,6}(G), \qquad P_{2,7}(G),$$

$$P_{2,9}(G), \qquad P_{2,2,2}(G), \qquad P_{2,V}(G),$$

$$P_{3,3}(G), \qquad P_{6,\infty}(G) \quad \textit{and} \quad P_{9,\infty}(G),$$

tend to 1 *as* $|G| \to \infty$.

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