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A degree condition for graphs to be fractional ID-$[a, b]$-factor-critical

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Abstract

Let G be a graph of sufficiently large order n, and let a and b be integers with $1 \le a \le b$. Let $h : E(G) \to [0, 1]$ be a function. If $a \le \sum_{x \in e} h(e) \le b$ holds for any $x \in V(G)$, then $G[F_h]$ is called a fractional $[a, b]$-factor of G with indicator function h, where $F_h = \{e \in E(G) \mid h(e) > 0\}$. A graph G is fractional independent-set-deletable $[a, b]$-factor-critical (simply, fractional ID-$[a, b]$-factor-critical) if $G - I$ includes a fractional $[a, b]$-factor for every independent set I of G. In this paper, we prove that G is fractional ID-$[a, b]$-factor-critical if $\delta(G) \ge \frac{bn}{a+2b} + a$ and $\max\{\deg_G(x), \deg_G(y)\} \ge \frac{(a+b)n}{a+2b}$ for any two nonadjacent vertices $x, y \in V(G)$. This result is best possible in some sense.

1 Introduction

In this paper, we consider only finite, simple and undirected graphs. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. For $x \in V(G)$, we denote by $\deg_G(x)$ the degree of x in G, and by $N_G(x)$ the set of vertices adjacent to x in G, and $N_G[x]$ for $N_G(x) \cup \{x\}$. The minimum degree of G is denoted by $\delta(G)$. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and by $G - S$ the subgraph obtained from G by deleting vertices in S together with the edges incident to vertices in S; $G - S = G[V(G) - S]$. A vertex set S of G is called an independent set if $G[S]$ has no edges.

Let a and b be two integers with $1 \le a \le b$. A spanning subgraph F of G is called an $[a, b]$-factor if $a \le \deg_F(x) \le b$ for each $x \in V(G)$. If $a = b = k$, then an $[a, b]$-factor is called a k-factor. Let $h : E(G) \to [0, 1]$ be a function. Then we call $G[F_h]$ a fractional $[a, b]$-factor of G with indicator function h if $a \le \sum_{x \in e} h(e) \le b$ holds for any vertex $x \in V(G)$, where $F_h = \{e \in E(G) \mid h(e) > 0\}$. A graph G

is fractional ID-$[a, b]$-factor-critical if $G - I$ has a fractional $[a, b]$-factor for every independent set I of G. Other notation and terminology are the same as those in [1].

The following results on k-factors, $[a, b]$-factors and fractional ID-$[a, b]$-factor-critical graphs are already known.

Theorem A (Egawa and Enomoto [2]; Katerinis [3]) *Let k be a positive integer, and let G be a graph of order $n \geq 4k-5$, and suppose that $\delta(G) \geq k$, kn is even and $\delta(G) \geq \frac{n}{2}$. Then G has a k-factor.*

Theorem B (Nishimura [6]) *Let k be an integer with $k \geq 3$, and let G be a connected graph of order $n \geq 4k-3$, and suppose that $\delta(G) \geq k$, kn is even and $\max\{\deg_G(x), \deg_G(y)\} \geq \frac{n}{2}$ for any two nonadjacent vertices $x, y \in V(G)$. Then G has a k-factor.*

Theorem C (Li and Cai [4]) *Let a and b be integers with $1 \leq a \leq b$, let G be a graph of order $n \geq 2a + b + \frac{a^2-a}{b}$, and suppose that $\delta(G) \geq a$ and $\max\{\deg_G(x), \deg_G(y)\} \geq \frac{an}{a+b}$ for any two nonadjacent vertices $x, y \in V(G)$. Then G has an $[a, b]$-factor.*

Theorem D (Zhou, Sun and Liu [7]) *Let a and b be integers with $1 \leq a \leq b$. Let G be a graph of order $n \geq \frac{(a+2b)(a+b-2)}{b}$, and suppose that $\delta(G) \geq \frac{(a+b)n}{a+2b}$. Then G is fractional ID-$[a, b]$-factor-critical.*

In this paper, we prove the following theorem for graphs to be fractional ID-$[a, b]$-factor-critical, which is a extension of Theorem D in the same way that Theorem B implies Theorem A.

Theorem 1.1 *Let a and b be integers with $1 \leq a \leq b$, and let G be a graph of order $n \geq \frac{(a+2b)(2a+b+1)}{b}$, and suppose that $\delta(G) \geq \frac{bn}{a+2b} + a$ and $\max\{\deg_G(x), \deg_G(y)\} \geq \frac{(a+b)n}{a+2b}$ for any two nonadjacent vertices $x, y \in V(G)$. Then G is fractional ID-$[a, b]$-factor-critical.*

We prove Theorem 1.1 in next section. In the rest of this section, we show examples concerning two sharpnesses in Theorem 1.1.

The lower bound of the degree condition in Theorem 1.1 is sharp in some sense. We construct examples which show that we cannot replace $\frac{(a+b)n}{a+2b}$ by $\frac{(a+b)n}{a+2b} - 1$. Let t be any sufficiently large positive integer. We define a graph G by

$$\begin{aligned} V(G) &= \{v_i \mid 1 \leq i \leq bt+1\} \cup \{w_j \mid 1 \leq j \leq at\} \cup \{x_k \mid 1 \leq k \leq bt\}, \\ E(G) &= \{v_i w_j \mid 1 \leq i \leq bt+1, 1 \leq j \leq at\} \cup \{v_i x_k \mid 1 \leq i \leq bt+1, 1 \leq k \leq bt\} \\ &\quad \cup \{w_j x_k \mid 1 \leq j \leq at, 1 \leq k \leq bt\}. \end{aligned}$$

Then it is easily seen that the order of G is $n = (a+2b)t+1$. Furthermore, it follows that

$$\begin{aligned}\frac{(a+b)n}{a+2b} &> \max\{\deg_G(x), \deg_G(y)\} = (a+b)\cdot\frac{n-1}{a+2b}\\ &= \frac{(a+b)n}{a+2b} - \frac{a+b}{a+2b} > \frac{(a+b)n}{a+2b} - 1\end{aligned}$$

for any two nonadjacent vertices $x, y \in \{v_i \mid 1 \le i \le bt+1\}$. However, G cannot be fractional ID-$[a,b]$-factor-critical. Set $I = \{v_i \mid 1 \le i \le bt+1\}$, and hence I is an independent set of G. Application of Fractional Factor Theorem due to Liu and Zhang with $S = \{w_j \mid 1 \le j \le at\}$ and $T = \{x_k \mid 1 \le k \le bt\}$ proves that $G - I$ has no fractional $[a,b]$-factor (we will show the Fractional Factor Theorem in next section). Hence, G is not fractional ID-$[a,b]$-factor-critical.

Moreover, we construct examples which show that the lower bound $\frac{bn}{a+2b} + a$ on $\delta(G)$ is sharp. Let t be a sufficiently large positive integer, and bt even. We define a graph G by

$$\begin{aligned}V(G) &= \{v\} \cup \{w_i \mid 1 \le i \le bt\} \cup \{x_j \mid 1 \le j \le at-1\} \cup \left\{y_k \mid 1 \le k \le \frac{bt}{2}\right\}\\ &\quad \cup \left\{z_l \mid 1 \le l \le \frac{bt}{2}\right\},\\ E(G) &= \{vw_i \mid 1 \le i \le bt\} \cup \{vx_j \mid 1 \le j \le a-1\} \cup \left\{y_k z_l \mid 1 \le k = l \le \frac{bt}{2}\right\}\\ &\quad \cup \left\{w_i y_k, x_j y_k \mid 1 \le i \le bt, 1 \le j \le at-1, 1 \le k \le \frac{bt}{2}\right\}\\ &\quad \cup \left\{w_j z_l, x_j z_l \mid 1 \le i \le bt, 1 \le j \le at-1, 1 \le l \le \frac{bt}{2}\right\},\end{aligned}$$

and set $I = \{w_i \mid 1 \le i \le bt\}$. Then it is easily seen that the order of G is $n = (a+2b)t$, $\delta(G) = \deg_G(v) = \frac{bn}{a+2b} + a - 1$, and $\max\{\deg_G(y), \deg_G(y')\} \ge \frac{(a+b)n}{a+2b}$ for any two nonadjacent vertices $y, y' \in V(G)$. In particular, the degree condition takes the minimum value for any two vertices $y, y' \in V(\left\{y_k \mid 1 \le k \le \frac{bt}{2}\right\})$. However, $G - I$ has no fractional $[a,b]$-factor because $\delta(G-I) = \deg_{G-I}(v) = a - 1$. Thus G is not fractional ID-$[a,b]$-factor-critical.

2 Proof of Theorem 1.1

In our proof, we use the following theorem for the existence of a fractional $[a,b]$-factor.

Theorem E (Liu and Zhang [5]; Fractional Factor Theorem) *Let a and b be integers with $1 \le a \le b$. Let G be a graph. Then G has a fractional $[a,b]$-factor if and only if for every subset $S \subseteq V(G)$,*

$$b|S| + \sum_{x \in T} \deg_{G-S}(x) - a|T| \ge 0,$$

where $T = \{x \in V(G) - S \mid \deg_{G-S}(x) \le a\}$.

Let a, b, G and n be as in Theorem 1.1. Let I be an independent set of G and $H = G - I$. In order to prove Theprem 1.1, it suffices to show that H has a fractional $[a, b]$-factor. By way of contradiction, we suppose that H has no fractional $[a, b]$-factors. Then by Theorem E, there exists some subset S of $V(G)$ such that

$$\theta(S, T) := b|S| + \sum_{x \in T} \deg_{H-S}(x) - a|T| \leq -1, \tag{2.1}$$

where $T = \{x \in V(G) - S \mid \deg_{H-S}(x) \leq a\}$.

We start with the following claims.

Claim 2.1 $|I| \leq \frac{bn}{a+2b}$.

Proof. For $0 \leq |I| \leq 1$, the inequality holds because $n \geq \frac{(a+2b)(2a+b+1)}{b}$. Thus we may assume that $|I| \geq 2$. By the degree condition of Theorem 1.1 and I is an independent set, there exist two vertices $x, y \in I$ such that $\max\{\deg_G(x), \deg_G(y)\} \geq \frac{(a+b)n}{a+2b}$. Since I is an independent set, $I \cap (N_G(x) \cup N_G(y)) = \emptyset$. Hence, we obtain

$$|I| + \frac{(a+b)n}{a+2b} \leq |I| + \max\{\deg_G(x), \deg_G(y)\} \leq n,$$

that is,

$$|I| \leq n - \frac{(a+b)n}{a+2b} = \frac{bn}{a+2b}.$$

□

Claim 2.2 $\delta(H) \geq a$.

Proof. Since $H = G - I$, it follows from Claim 2.1 that

$$\delta(H) \geq \delta(G) - |I| \geq \left(\frac{bn}{a+2b} + a\right) - \frac{bn}{a+2b} = a.$$

□

Claim 2.3 $|T| \geq b + 1$.

Proof. Suppose that $|T| \leq b$. Then it follows from Claim 2.2 and $|S| + \deg_{H-S}(x) \geq \deg_H(x) \geq \delta(H) \geq a$ for each $x \in T$ that

$$\begin{aligned} \theta(S, T) &= b|S| + \sum_{x \in T} \deg_{H-S}(x) - a|T| \\ &\geq |T||S| + \sum_{x \in T} \deg_{H-S}(x) - a|T| \\ &= \sum_{x \in T} \Big(|S| + \deg_{H-S}(x) - a\Big) \\ &\geq \sum_{x \in T} \Big(\delta(H) - a\Big) \geq 0, \end{aligned}$$

which contradicts (2.1). □

Claim 2.4 $a|T| > b|S|$.

Proof. Suppose that $a|T| \le b|S|$. Then by (2.1) we obtain

$$-1 \ge \theta(S,T) = b|S| + \sum_{x\in T} \deg_{H-S}(x) - a|T| \ge b|S| - a|T| \ge 0,$$

a contradiction. □

Claim 2.5 $|S| + |I| < \frac{(a+b)n}{a+2b}$.

Proof. By Claims 2.1 and 2.4 and $|S| + |T| + |I| \le n$, we have

$$\begin{aligned} an &\ge a|S| + a|T| + a|I| \ge a|S| + b|S| + a|I| + 1 = (a+b)(|S| + |I|) - b|I| + 1 \\ &\ge (a+b)(|S| + |I|) - \frac{b^2 n}{a+2b} + 1, \end{aligned}$$

that is,

$$|S| + |I| < \frac{(a+b)n}{a+2b}.$$

□

By Claim 2.3, we have $T \ne \emptyset$.

Now we define

$$h_1 = \min\{\deg_{H-S}(x) \mid x \in T\},$$

and let x_1 be a vertex such that $\deg_{H-S}(x_1) = h_1$. If $T - N_{H[T]}[x_1] \ne \emptyset$, we define

$$h_2 = \min\{\deg_{H-S}(x) \mid x \in T - N_{H[T]}[x_1]\},$$

and let x_2 be a vertex such that $\deg_{H-S}(x_2) = h_2$. Furthermore, we define

$$T_0 = \{x \in T \mid \deg_{H-S}(x) = 0\}.$$

Note that $0 \le h_1 \le h_2 \le a$ by the definitions of h_1, h_2 and T.

We divide the proof into three cases.

Case 1: $|T_0| \ge 2$.

Then, there exist at least two nonadjacent vertices $x, y \in T_0$ such that $\deg_{H-S}(x) = \deg_{H-S}(y) = 0$. By Claim 2.5 and the degree condition of this theorem, we obtain

$$\begin{aligned} \frac{(a+b)n}{a+2b} &\le \max\{\deg_G(x), \deg_G(y)\} \le \max\{\deg_H(x), \deg_H(y)\} + |I| \\ &\le \max\{\deg_{H-S}(x) + |S|, \deg_{H-S}(y) + |S|\} + |I| = |S| + |I| < \frac{(a+b)n}{a+2b}, \end{aligned}$$

which is a contradiction.

Case 2: $|T_0| = 1$.

Then we obtain $h_1 = 0$ and $T_0 = \{x_1\}$. In view of Claim 2.3, we have $T - N_{H[T]}[x_1] \neq \emptyset$. We now take a vertex $x_2 \in T - N_{H[T]}[x_1]$ such that $\deg_{H-S}(x_2) = h_2$, and hence $x_1x_2 \notin E(G)$ and $1 \leq h_2 \leq a$ hold. By the degree condition of this theorem, we obtain

$$\begin{aligned} \frac{(a+b)n}{a+2b} &\leq \max\{\deg_G(x_1), \deg_G(x_2)\} \\ &\leq \max\{\deg_{H-S}(x_1) + |S| + |I|, \deg_{H-S}(x_2) + |S| + |I|\} \\ &\leq h_2 + |S| + |I|, \end{aligned}$$

that is,

$$|S| \geq \frac{(a+b)n}{a+2b} - h_2 - |I|. \tag{2.2}$$

Also, it is easy to see that $|T - N_{H[T]}[x_1]| = |T| - 1$. Consequently, it follows from Claim 2.1, (2.2), $|S| + |T| + |I| \leq n$, $b \geq a \geq 1$, $1 \leq h_2 \leq a$ and $n \geq \frac{(a+2b)(2a+b+1)}{b}$ that

$$\begin{aligned} \theta(S,T) &= b|S| + \sum_{x \in T} \deg_{H-S}(x) - a|T| \\ &= b|S| + \sum_{x \in N_{H[T]}[x_1]} \deg_{H-S}(x) + \sum_{x \in T - N_{H[T]}[x_1]} \deg_{H-S}(x) - a|T| \\ &= b|S| + \sum_{x \in T - N_{H[T]}[x_1]} \deg_{H-S}(x) - a|T| \\ &\geq b|S| + h_2(|T| - 1) - a|T| \\ &= b|S| - (a - h_2)|T| - h_2 \\ &\geq b|S| - (a - h_2)(n - |S| - |I|) - h_2 \\ &= (a + b - h_2)|S| - (a - h_2)n + (a - h_2)|I| - h_2 \\ &\geq (a + b - h_2)\left(\frac{(a+b)n}{a+2b} - h_2 - |I|\right) - (a - h_2)n + (a - h_2)|I| - h_2 \\ &= (a + b - h_2)\left(\frac{(a+b)n}{a+2b} - h_2\right) - (a - h_2)n - b|I| - h_2 \\ &\geq (a + b - h_2)\left(\frac{(a+b)n}{a+2b} - h_2\right) - (a - h_2)n - \frac{b^2 n}{a+2b} - h_2 \\ &= h_2^2 + \left(\frac{bn}{a+2b} - a - b - 1\right)h_2 \\ &\geq h_2^2 + ah_2 \\ &= (a + h_2)h_2 \ \geq \ 0, \end{aligned}$$

which contradicts (2.1).

Case 3: $|T_0| = 0$.
If $h_1 = a$, then by (2.1) we obtain

$$\begin{aligned} -1 \geq \theta(S,T) &= b|S| + \sum_{x \in T} \deg_{H-S}(x) - a|T| \\ &\geq b|S| + h_1|T| - a|T| = b|S| \geq 0, \end{aligned}$$

a contradiction. Thus, we obtain

$$1 \leq h_1 \leq a - 1. \tag{2.3}$$

Next, we prove the following claim.

Claim 2.6 $T - N_{H[T]}[x_1] \neq \emptyset$.

Proof. Otherwise, it is easy to see $T = N_{H[T]}[x_1]$. Then, it follows from (2.3) that

$$|T| = |N_{H[T]}[x_1]| \leq |N_{H-S}[x_1]| = \deg_{H-S}(x_1) + 1 = h_1 + 1 \leq a,$$

which contradicts Claim 2.3. □

In view of Claim 2.6, there exists $x_2 \in T - N_{H[T]}[x_1]$ such that $\deg_{H-S}(x_2) = h_2$, and $x_1x_2 \notin E(G)$ holds. By the condition of this theorem, we obtain

$$\begin{aligned} \frac{(a+b)n}{a+2b} &\leq \max\{\deg_G(x_1), \deg_G(x_2)\} \\ &\leq \max\{\deg_{H-S}(x_1) + |S| + |I|, \deg_{H-S}(x_2) + |S| + |I|\} \\ &\leq h_2 + |S| + |I|, \end{aligned}$$

that is,

$$|S| \geq \frac{(a+b)n}{a+2b} - h_2 - |I|. \tag{2.4}$$

Also, it follows that

$$|N_{H[T]}[x_1]| \leq |N_{H-S}[x_1]| = \deg_{H-S}(x_1) + 1 = h_1 + 1. \tag{2.5}$$

Hence, it follows from: $1 \leq h_1 \leq h_2 \leq a$; $|S|+|T|+|I| \leq n$; $n \geq \dfrac{(a+2b)(2a+b+1)}{b}$; and Claims 2.1, (2.4) and (2.5), that

$$\begin{aligned} \theta(S,T) &= b|S| + \sum_{x \in T} \deg_{H-S}(x) - a|T| \\ &= b|S| + \sum_{x \in N_{H[T]}[x_1]} \deg_{H-S}(x) + \sum_{x \in T - N_{H[T]}[x_1]} \deg_{H-S}(x) - a|T| \end{aligned}$$

$$
\begin{aligned}
&\geq b|S| + h_1|N_{H[T]}[x_1]| + h_2(|T| - |N_{H[T]}[x_1]|) - a|T| \\
&= b|S| - (h_2 - h_1)|N_{H[T]}[x_1]| - (a - h_2)|T| \\
&\geq b|S| - (h_2 - h_1)(h_1 + 1) - (a - h_2)(n - |S| - |I|) \\
&= (a + b - h_2)|S| - (h_2 - h_1)(h_1 + 1) - (a - h_2)n + (a - h_2)|I| \\
&\geq (a + b - h_1)\left(\frac{(a+b)n}{a+2b} - h_2 - |I|\right) - (h_2 - h_1)(h_1 + 1) - (a - h_2)n \\
&\qquad + (a - h_2)|I| \\
&= (a + b - h_2)\left(\frac{(a+b)n}{a+2b} - h_2\right) - (h_2 - h_1)(h_1 + 1) - (a - h_2)n - b|I| \\
&\geq (a + b - h_2)\left(\frac{(a+b)n}{a+2b} - h_2\right) - (h_2 - h_1)(h_1 + 1) - (a - h_2)n - \frac{b^2 n}{a+2b} \\
&= \frac{bn}{a+2b} \cdot h_2 - (a + b - h_2)h_2 - (h_2 - h_1)(h_1 + 1) \\
&= h_2^2 + \left(\frac{bn}{a+2b} - a - b - 1 - h_1\right)h_2 + (h_1 + 1)h_1 \\
&\geq h_2^2 + (a - h_2)h_2 + (h_1 + 1)h_1 \geq 0,
\end{aligned}
$$

which contradicts (2.1).

This completes the proof of Theorem 1.1.

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