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Double-crossed chords and distance-hereditary graphs

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Abstract

An early characterization of distance-hereditary graphs is that every cycle of length 5 or more has crossing chords. A new, stronger, property is that in every cycle of length 5 or more, some chord has at least two crossing chords. This new property can be characterized by every block being complete multipartite, and also by the vertex sets of cycles of length 5 or more always inducing 3-connected subgraphs. It can also be characterized by forbidding certain induced subgraphs, as well as by requiring certain (not necessarily induced) subgraphs.

A second, even stronger property is that, in every cycle of length 5 or more in a distance-hereditary graph, every chord has at least two crossing chords. This second property has characterizations that parallel those of the first property, including by every block being complete bipartite or complete, and also by the vertex sets of cycles of length 5 or more always inducing nonplanar subgraphs.

1 Introduction

For $k \geq 3$, define a *≥k-cycle* to be a cycle of length at least k. A *chord* of a cycle (or a path) C is an edge between two vertices of C that is not an edge of C itself. Two chords x_1y_1 and x_2y_2 of C are *crossing chords* of C if their four endpoints come in the order x_1, x_2, y_1, y_2 along C. A graph G is *distance-hereditary* [1, 2, 3] if the distance between two vertices in connected induced subgraphs of G always equals the distance between them in G. One characterization of being distance-hereditary, from [3], is that every ≥5-cycle has crossing chords.

A chord x_1y_1 of a cycle C is a *double-crossed chord* of C if there are two (or more) chords x_2y_2 and x_3y_3 of C such that x_1y_1 and x_iy_i are crossing chords of C for both $i \in \{2, 3\}$; see Figure 1.

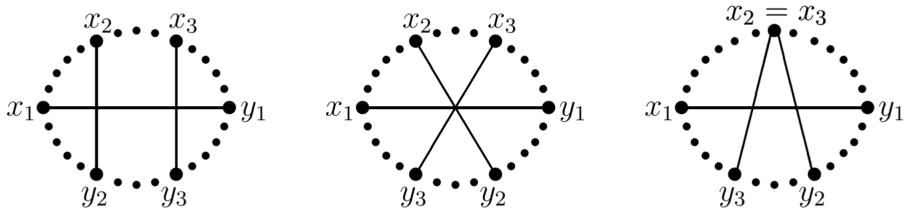

Figure 1: Three ways that x_1y_1 can be a double-crossed chord.

Section 2 below will characterize the subclass of distance-hereditary graphs in which every ≥5-cycle has a double-crossed chord—equivalently, the graphs in which every cycle that is long enough to have a double-crossed chord always does have a double-crossed chord. Section 3 will similarly characterize the even smaller subclass of distance-hereditary graphs in which every chord of every ≥5-cycle has a double-crossed chord. The manner in which the characterizations of Section 2 parallel those of Section 3 is intriguing, such as "3-connected" in Theorem 2 being replaced by "nonplanar" in Theorem 5.

2 When all ≥5-cycles have double-crossed chords

Theorem 1 will characterize the graphs in which every ≥5-cycle has a double-crossed chord. (I also mentioned the equivalence of these three conditions at the end of [4], without proof and in slightly different words, along with a fourth—every vertex of every ≥5-cycle is on a chord of the cycle—that fit into the theme of [4].) In Theorem 1, a *block* of a graph is either an edge that is in no cycle or a maximal 2-connected subgraph. A *paw* is the graph formed by a triangle with one pendant edge, and P_4 is the 4-vertex (length-3) path. Note that every complete graph is complete multipartite, since K_n is the complete n-partite graph (with n singleton partite sets). Thus, complete bipartite graphs and complete graphs are the two extremes of the spectrum of connected complete multipartite graphs.

Theorem 1 *The following are equivalent*:

(1a) *Every* ≥5*-cycle has a double-crossed chord.*

(1b) *No block contains an induced paw or* P_4*.*

(1c) *Every block is a complete multipartite graph.*

Proof. First, suppose a graph G satisfies condition (1a) and H is a block of G (toward showing that (1b) holds). The argument is by contradiction, using the two cases of H containing an induced paw or an induced P_4 (the latter with subcases for the two ways that the four vertices of P_4 can occur around a cycle).

Case 1: Suppose H contains an induced paw that consists of the triangle abc and pendant edge cd. Since H is 2-connected and every two edges of a 2-connected graph are in a common cycle, ab and cd are in a cycle C such that, without loss of generality, the vertices of the paw occur in the order a, b, c, d around C. Thus there is a ≥5-cycle C' that consists of the path a, b, c, d and a chordless a-to-d path π, where ac is a chord of C'. All the other possible chords of C' have one endpoint in $\{b, c\}$ and the other endpoint an internal vertex of π. If C' is a 5-cycle with (say) x the unique internal vertex of π, then the only possible chords that C' can have are ac, bx, and cx; thus C' would have no double-crossed chords, contradicting (1a). If C' is a ≥6-cycle, then let C'' be the ≥5-cycle that consists of the two paths a, c, d and π. But then all the possible chords of C'' have endpoint c; thus C'' would have no double-crossed chords, again contradicting (1a).

Case 2: Suppose H contains an induced P_4: a, b, c, d. Since H is 2-connected, the edges ab and cd are in a common cycle C.

Subcase 2.1: If the vertices of the P_4 occur in the order a, b, c, d around C, then there is a ≥5-cycle C' that consists of the path a, b, c, d and a chordless a-to-d path π, where all the possible chords of C' have one endpoint in $\{b, c\}$ and the other endpoint an internal vertex of π. By (1a), C' has a double-crossed chord that is, without loss of generality, cx with crossing chords by and bz where the five vertices a, x, y, z, d come in that order along π. But then there is a ≥5-cycle C'' that consists of the path x, c, d and the x-to-d subpath of π where all the possible chords of C'' have endpoint c; thus C'' would have no double-crossed chords, contradicting (1a).

Subcase 2.2: If the vertices of the P_4 occur in the order a, b, d, c around C, then there is a ≥6-cycle C' that consists of the edges ab and cd, a chordless a-to-c path π, and a chordless b-to-d path τ, where bc is a chord of C'. If C' is a 6-cycle with vertices a, b, y, d, c, x in that order, then the only possible chords that C' can have are bc, ay, bx, cy, dx, and xy; hence the only double-crossed chords that C' can have are ay, dx, and bc (with bc double-crossed only if ay or dx is also a chord). Using (1a), say ay is a chord of C', and let C'' be the 5-cycle with vertices a, y, d, c, x in that order. But the only possible chords that C'' can have are cy, dx, and xy; thus C'' would have no double-crossed chords, contradicting (1a). If C' is a ≥7-cycle, then suppose (say) π has length at least three. Let C''' be the ≥5-cycle that consists of the two paths a, b, c and π. But all the possible chords of C''' have endpoint b; thus C''' would have no double-crossed chords, again contradicting (1a).

Therefore, by cases 1 and 2, condition (1a) implies (1b).

Next, suppose G satisfies condition (1b). Since H contains no induced paw, H is triangle-free or complete multipartite (this is proved in [5]). If H is triangle-free, then every chordless cycle of H will have length 4 (since a longer chordless cycle would contain an induced P_4, contradicting (1b)), and so $H \cong K_{2,n}$ for some $n \geq 2$. Therefore, H is complete multipartite in both of the alternatives from [5], and so (1c) holds.

Finally, suppose G satisfies condition (1c) and has a ≥5-cycle C in a complete multipartite block H in which S is a maximum-size partite set. If $|S| \geq 3$ with $v_1, v_2, v_3 \in S$, then there are $v_1', v_2', v_3' \notin S$ with $v_1, v_3', v_2, v_1', v_3, v_2'$ appearing in that order around C, and so chord v_1v_1' is crossed by chords v_2v_2' and v_3v_3' of C. If $|S| = 2$ with $v_1, v_2 \in S$, then there are $x, y, z \notin S$, not all three in the same partite set, with v_1, x, v_2, y, z appearing in that order around C, and so v_1y and v_2z are crossing chords of C with one of them also crossed by a chord xy or xz of C. If $|S| = 2$, then there are v, w, x, y, z appearing in that order around C, and so the chord wy is crossed by the chords vx and xz of C. Thus, every ≥5-cycle C in every complete multipartite block has a double-crossed chord, and so (1a) holds. □

Condition (1b) characterizes the graphs in which every ≥5-cycle has a double-crossed chord in terms of forbidden subgraphs. In contrast, condition (2b) of Theorem 2 will characterize this same graph class in terms of required (but not necessarily induced) subgraphs. Let $\langle V(C)\rangle$ denote the subgraph induced by the vertex set of a cycle C; thus the edge set of $\langle V(C)\rangle$ consists of all the edges and all the chords of C.

Theorem 2 *The following are equivalent*:

(2a) *Every* ≥5*-cycle has a double-crossed chord.*

(2b) *For every* ≥5*-cycle* C*,* $\langle V(C)\rangle$ *contains a* $K_{1,2,2}$ *or* $K_{3,3}$.

(2c) *For every* ≥5*-cycle* C*,* $\langle V(C)\rangle$ *is* 3*-connected.*

Proof. First, suppose a graph G satisfies condition (2a) and C is a ≥5-cycle of G (toward showing that (2b) and (2c) hold). By Theorem 1, C is contained in a complete k-partite block H of G for some $k \geq 2$. If $k = 2$, then $\langle V(C)\rangle \cong K_{a,b}$ with $a, b \geq 3$, and so $\langle V(C)\rangle$ contains a (induced) $K_{3,3}$ subgraph. If $k = 3$, then $\langle V(C)\rangle \cong K_{a,b,c}$ with $a \leq b \leq c$ and $b \geq 2$, and so $\langle V(C)\rangle$ contains a (induced) $K_{1,2,2}$ subgraph. If $k = 4$, then $\langle V(C)\rangle \cong K_{a,b,c,d}$ with $a \leq b \leq c \leq d$ and $d \geq 2$, and so $\langle V(C)\rangle$ contains a (noninduced) $K_{1,2,2}$ subgraph. If $k \geq 5$, then $\langle V(C)\rangle$ contains an induced K_5 that in turn contains a (noninduced) $K_{1,2,2}$ subgraph. Moreover, in each of these cases $\langle V(C)\rangle$ is 3-connected. Therefore, condition (2a) implies both (2b) and (2c).

Next, suppose G satisfies condition (2b) (toward showing that (2a) holds).

Suppose for the moment that $\langle V(C)\rangle$ contains a $K_{1,2,2}$ subgraph with vertex a adjacent to each vertex b_1, b_2, c_1, c_2 and each b_i adjacent to each c_j as illustrated in Figure 2; say vertices a, b_1, b_2 partition $E(C)$ into subpaths $C[a, b_1]$, $C[b_1, b_2]$, and

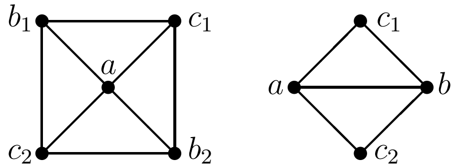

Figure 2: The complete tripartite graphs $K_{1,2,2}$ and $K_{1,1,2}$.

$C[b_2, a]$ with the indicated endpoints. If c_1, c_2 are both in $C[a, b_1]$, then the chord ab_1 will be double-crossed by chords b_2c_1 and b_2c_2. If c_1, c_2 are both in $C[b_1, b_2]$, say with the vertices b_1, c_1, c_2, b_2 coming in that order along $C[b_1, b_2]$, then the chord b_1c_2 will be double-crossed by chords ac_1 and b_2c_1. If c_1 is in $C[a, b_1]$ and c_2 is in $C[b_2, a]$, then the chord b_2c_1 will be double-crossed by chords ab_1 and b_1c_2. If c_1 is in $C[a, b_1]$ and

c_2 is in $C[b_1, b_2]$, then the chord b_2c_1 will be double-crossed by chords ab_1 and ac_1. Thus and similarly, C will always have a double-crossed chord.

Now suppose instead that $\langle V(C)\rangle$ contains a $K_{3,3}$ subgraph with each vertex a_1, a_2, a_3 adjacent to each vertex b_1, b_2, b_3; say vertices a_1, b_1, b_2, b_3 partition $E(C)$ into subpaths $C[a_1, b_1], C[b_1, b_2], C[b_2, b_3]$, and $C[b_3, a_1]$ with the indicated endpoints. If a_2 is in $C[a_1, b_1]$, then the chord a_1b_1 will be double-crossed by chords a_2b_2 and a_2b_3. If a_2 and a_3 are both in $C[b_1, b_2]$, then the chord a_1b_2 will be double-crossed by chords a_2b_3 and a_3b_3. If a_2 is in $C[b_1, b_2]$ and a_3 is in $C[b_2, b_3]$, then the chord a_1b_2 will be double-crossed by chords a_2b_3 and a_3b_1. Thus and similarly, C will always have a double-crossed chord.

Therefore, condition (2b) implies (2a).

Finally, suppose G does not satisfy condition (2a) (toward showing that (2c) fails), and so by Theorem 1, some block H of G contains an induced paw or P_4. Let cases 1 and 2 and subcases 2.1 and 2.2 be exactly as in the proof of Theorem 1 (including the cycles C, C', C'', and C''' used there).

Case 1: If C' is a 5-cycle, then $\{c, x\}$ is a *separator* of $\langle V(C')\rangle$ (in other words, deleting $\{c, x\}$ from $\langle V(C')\rangle$ will leave a disconnected graph). If C' is a ${\geq}6$-cycle and v is any internal vertex of π, then $\{c, v\}$ is a separator of $\langle V(C'')\rangle$.

Subcase 2.1: If v is the internal vertex of π that is as close as possible to a along π such that bv or cv is a chord of C', then $\{b, v\}$ is a separator of $\langle V(C')\rangle$.

Subcase 2.2: If C' is a 6-cycle, then either $ay \notin E(G)$ and $\{b, x\}$ is a separator of $\langle V(C')\rangle$ or $ay \in E(G)$ and $\{x, y\}$ is a separator of $\langle V(C'')\rangle$. If C' is a ${\geq}7$-cycle and v is any internal vertex of π, then $\{b, v\}$ is a separator of $\langle V(C''')\rangle$.

Therefore, in every case, the vertex set of some ${\geq}5$-cycle will induce a graph that is not 3-connected, so condition (2a) failing would imply (2c) failing, and so condition (2c) implies (2a). □

Theorem 3 *If every ${\geq}5$-cycle has a double-crossed chord, then, for every ${\geq}5$-cycle C, at least one chord of every pair of crossing chords of C is double-crossed in C.*

Proof. Suppose every ${\geq}5$-cycle of a graph G has a double-crossed chord. Argue by induction on the length of C that, for every ${\geq}5$-cycle C of G with crossing chords x_1y_1 and x_2y_2, at least one of x_1y_1 and x_2y_2 is double-crossed in C. For the basis step, if C is a 5-cycle with edges x_1z, zx_2, x_2y_1, y_1y_2, y_2x_1 and crossing chords x_1y_1 and x_2y_2, then the only other possible chords of C are x_1x_2, y_1z, and y_2z, and so since C has a double-crossed chord, at least one of x_1y_1 and x_2y_2 is double-crossed.

For the inductive step, suppose C is a ${\geq}6$-cycle and $E(C)$ is partitioned into four subpaths $C[x_1, x_2]$, $C[x_2, y_1]$, $C[y_1, y_2]$, and $C[y_2, x_1]$ with the indicated endpoints. Suppose further that neither x_1y_1 nor x_2y_2 is double-crossed in C (arguing by contradiction). Thus the ${\geq}6$-cycle C has a double-crossed chord (say) x_3y_3, where both x_3 and y_3 must be in the same partitioning subpath, say in $C[x_2, y_1]$ with the vertices x_2, x_3, y_3, y_1 coming in that order along $C[x_2, y_1]$. (It is possible that $x_3 = x_2$ or $y_3 = y_1$, but not both since x_3y_3 has a crossing chord that crosses neither x_1y_1 nor

x_2y_2.) Let C' be the ≥5-cycle formed from C by replacing the x_3-to-y_3 subpath of $C[x_2, y_1]$ with the edge x_3y_3. By the inductive hypothesis, at least one of the crossing chords x_1y_1 and x_2y_2 of C' would be double-crossed in C' (contradicting that neither x_1y_1 nor x_2y_2 is double-crossed in C). □

The converse to Theorem 3 will hold for distance-hereditary graphs, since being distance-hereditary is equivalent to every ≥5-cycle having crossing chords [3]. Therefore, in a distance-hereditary graph, every ≥5-cycle will have a double-crossed chord if and only if, for every ≥5-cycle C, at least one chord of every pair of crossing chords of C is double-crossed in C.

3 When all chords of ≥5-cycles are double-crossed

Theorem 3 motivates strengthening the condition studied in section 2 to require now that *every* chord of every ≥5-cycle is double-crossed. Theorems 4 and 5 will characterize this stronger condition with intriguing parallels to Theorems 1 and 2. Theorems 4 and 5 are restricted to *hole-free graphs* [2], meaning graphs in which every ≥5-cycle has a chord. This prevents conditions (4a) and (5a) from being vacuously true; note that conditions (4b), (4c), (5b), and (5c) are false in the non-hole-free cycles C_n for $n \geq 5$. Every distance-hereditary graph is, of course, hole-free.

In Theorem 4, requiring every block that is not an edge to contain a ≥5-cycle avoids the (hole-free) complete tripartite graphs $K_{1,1,c}$ with $c \geq 2$, for which (4a) would hold vacuously while (4b) and (4c) would be false. In the proof, let $v \sim w$ and $v \not\sim w$ denote, respectively, that vertices v and w are or are not adjacent.

Theorem 4 *The following are equivalent for all hole-free graphs in which every block that is not an edge contains a ≥5-cycle:*

(4a) *Every chord of every ≥5-cycle is double-crossed.*

(4b) *No block contains an induced paw, P_4, or $K_{1,1,2}$ subgraph.*

(4c) *Every block is either complete or complete bipartite.*

Proof. Suppose G is a hole-free graph in which in which every block that is not an edge contains a ≥5-cycle.

First, suppose condition (4a) holds and H is a block (toward showing that (4c) holds). Thus every ≥5-cycle has a double-crossed chord, and so H is complete multipartite by Theorem 1. If $H \cong K_2$, then H would be complete (and also complete bipartite); hence, assume $H \ncong K_2$, and so H contains a ≥5-cycle. If H is complete k-partite with $2 < k < n$, then $H \ncong K_{1,1,n-2}$ (since H contains a ≥5-cycle), and so H contains an induced $K_{1,2,2}$ or $K_{1,1,1,2}$, and so would contain a 5-cycle with a chord that is not double-crossed, contradicting (4a). Therefore, $k = 2$ or $k = n$, so the complete multipartite graph H is complete bipartite or complete, and so (4c) holds.

Next, condition (4c) implies (4b) since complete graphs and complete bipartite blocks contain no induced paw, P_4, or $K_{1,1,2}$.

Finally, suppose condition (4b) holds and xy is a chord of a ≥5-cycle C (toward showing that (4b) holds). Suppose x_1, x, x_2 and y_2, y, y_1 are subpaths of C where x_1, x, x_2, y_2, y, y_1 come in that order around C (possibly with $x_1 = y_1$ or $x_2 = y_2$, but not both). Let H' be the subgraph of G induced by $\{x, x_1, y, y_2\}$.

Suppose for the moment that $x_1 = y_1$ (and so $x_2 \neq y_2$). Thus either $x \sim y_2$ or $x_1 \sim y_2$ (to avoid H' being a paw), so both $x \sim y_2$ and $x_1 \sim y_2$ (to avoid $H' \cong K_{1,1,2}$), and so xy is crossed by the chord x_1y_2. Using the subgraph of G induced by $\{x, x_1, x_2, y\}$ similarly shows that xy is crossed by the chord y_1x_2, which makes xy double-crossed.

Now suppose (instead) that $x_1 \neq y_1$ and (similarly) $x_2 \neq y_2$. If $x_1 \not\sim y_2$, then either $x_1 \sim y$ or $x \sim y_2$ (to avoid $H' \cong P_4$), so both $x_1 \sim y$ and $x \sim y_2$ (to avoid H' being a paw), and so $H' \cong K_{1,1,2}$, contradicting (4b). Thus $x_1 \sim y_2$. Similarly, $x_2 \sim y_1$, which makes xy double-crossed.

Therefore, (4b) implies (4a). □

Theorem 5 *The following are equivalent for all hole-free graphs*:

(5a) *Every chord of every* ≥5-*cycle is double-crossed.*

(5b) *For every* ≥5-*cycle* C, $\langle V(C)\rangle$ *contains a* K_5 *or* $K_{3,3}$ *subgraph.*

(5c) *For every* ≥5-*cycle* C, $\langle V(C)\rangle$ *is nonplanar.*

Proof. Suppose G is a hole-free graph.

First, suppose condition (5a) holds, and so every ≥5-cycle has a double-crossed chord. Thus, for every ≥5-cycle C, Theorem 2 ensures that $\langle V(C)\rangle$ contains a (not necessarily induced) $K_{1,2,2}$ or $K_{3,3}$ subgraph H. In the case of H being a $K_{1,2,2}$ subgraph, (5a) forces $\langle V(H)\rangle \cong K_5$ and so $\langle V(C)\rangle$ contains a K_5 subgraph. Thus, in every case, $\langle V(C)\rangle$ contains a K_5 or $K_{3,3}$ subgraph and so (5b) holds.

Next, suppose condition (5b) holds. Thus every ≥5-cycle has a double-crossed chord, and so, for every ≥5-cycle C, $\langle V(C)\rangle$ is complete k-partite for some $2 \leq k \leq |V(C)|$ by Theorem 1. If some ≥5-cycle C has $2 < k < |V(C)|$, then $\langle V(C)\rangle$ contains an induced $K_{1,2,2}$ or $K_{1,1,1,2}$, and so $\langle V(C)\rangle$ would contain a 5-cycle C' such that $\langle V(C')\rangle$ contains neither a K_5 nor a $K_{3,3}$ subgraph, contradicting (5b). Therefore, for every ≥5-cycle C, either $k = 2$ and $\langle V(C)\rangle$ is complete bipartite or $k = |V(C)|$ and $\langle V(C)\rangle$ is complete, so every chord of C is double-crossed, and so (5a) holds.

Therefore, conditions (5a) and (5b) are equivalent.

Note that condition (5b) implies (5c), since K_5 and $K_{3,3}$ are nonplanar.

Finally, suppose condition (5c) holds (toward showing that (5b) holds). Therefore, for every ≥5-cycle C, $\langle V(C)\rangle$ contains a subdivision H_C of a K_5 or $K_{3,3}$ subgraph by Kuratowski's Theorem. Argue by induction on the length of C that, for every ≥5-cycle C, $\langle V(C)\rangle$ contains a K_5 or $K_{3,3}$ subgraph that is not subdivided. If C is a 5-cycle, then $H_C = \langle V(C)\rangle \cong K_5$. If C is a 6-cycle, then either $H_C = \langle V(C)\rangle \cong K_{3,3}$ or C contains vertices a_1, a_2, a_3, a_4, a_5 that induce a K_5 and a sixth vertex b that has (say) neighbors a_1 and a_2 along C, in which case $\{a_1, a_2, a_3, a_4, b\}$ induces a 5-cycle

C' with $\langle V(C')\rangle \cong K_5$ by (5c). Finally, if C is a ${}^{\geq}7$-cycle, then C has a chord xy that combines with a x-to-y subpath of C to form a ${}^{\geq}5$-cycle C' such that, by the induction hypothesis, $\langle V(C')\rangle$ (and so also $\langle V(C)\rangle$) contains a K_5 or $K_{3,3}$ subgraph. Thus, (5b) holds. □

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