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On a pursuit-evasion model without instantaneous movement

Jeong-Ok Choi

Gwangju Institute of Science and Technology
Gwangju
South Korea
jchoi351@gist.ac.kr

John P. Georges   David W. Mauro

Department of Mathematics
Trinity College
Hartford, CT 06106
U.S.A.
john.georges@trincoll.edu   david.mauro@trincoll.edu

Abstract

There is a large body of literature devoted to the graph-theoretic modeling of searching in networks. Using such terms as cops, robbers, watchmen, searchers, and intruders, these works posit the movements of pursuers and evaders from one vertex to another in a graph G, not necessarily along edges, with the respective purposes of capturing the evaders and escaping the pursuers. Various assumptions on the mobility and knowledge of the pursuers and evaders, as well as various definitions of capture, determine the graph-theoretic parameter of primary interest: the minimum number of pursuers needed to guarantee the capture of all evaders regardless of the routes elected by the evaders for escape. In this paper, we analyze a model under which pursuers and evaders move simultaneously from one vertex to another along the edges of G, none of whom moves with infinite speed. Capture of an evader occurs if and only if the evader and a pursuer are within distance one. Denoting the indicated minimum number of pursuers under our model by $w(G)$, we determine conditions under which $w(G)$ is at least $w(H)$ for H a subgraph of G. We consider the relationship between $w(G)$ and various invariants of G such as girth and diameter. And we determine $w(G)$ for G in various classes.

1 Introduction

In graph theory, a pursuit-evasion model is a collection of axioms by which pursuers and evaders move throughout a graph with the respective purposes of capturing each evader and evading the pursuers indefinitely. For a given graph and a fixed model that specifies such things as the means by which a capture is made and the nature of the mobility of the pursuers and evaders, the task is to determine the minimum number of pursuers required to guarantee the capture of each evader, no matter what movements the evader may take. This number is generally called the *search number* of the graph under the given model.

There are many different pursuit-evasion models in the literature. The earliest, which involves not so much evasion as pursuit, is credited to Parsons [19] whose work was inspired by searchers (pursuers) who must locate a missing explorer (evader) wandering aimlessly in a network of caverns and connecting tunnels. In Parsons' work, a graph $G = (V, E)$ is considered to be an embedding in 3-space, and the locations of the evader and the pursuers are given by continuous functions of time into $V \cup E$. A rescue (or capture) of the evader thus occurs if and only if there exists a finite time at which the location of a pursuer is identical to the location of the evader. Accordingly, the minimum number $s(G)$ of pursuers required to guarantee the capture of the evader is the smallest integer k with the following property: that there exist k pursuer location functions $l_1, l_2, \ldots, l_k$ such that for every evader location function f, there exists a finite time t_f where $f(t_f) = l_i(t_f)$ for some i. Note that the usage of the quantifiers codifies the assumption that the pursuers needn't have any knowledge of the whereabouts or the intentions of the evader, wheras the evader, who may elect a route that is most inimical to capture or rescue, may have complete knowledge of the whereabouts or the intentions of the pursuers. A discrete characterization of this model, presented by Golovach [15], is given in terms of *edge clearing*. In this characterization, the movement of a pursuer along edge e will clear e of evaders (or, in another setting, will clear e of a noxious gas). However, e will return to uncleared status if at some step there is a path of uncleared edges through e that does not contain any pursuers. Under Parsons' formulation (and hence under Golovach's as well), $s(K_n) = n$ for $n \geq 4$, where K_n is the complete graph on n vertices.

A number of pursuit-evasion models are summarized in surveys by Alspach [3] and Fomin and Thilikos [12]. Alspach distinguishes between models that *sweep* a graph and models that *search* a graph. In the former type (which includes Parson's model), the evader may be located at a vertex or an edge, while in the latter type the evader may be located only at a vertex. Among the five sweep models mentioned in [3], four of them (all but Parsons') admit loops or multiple edges. Of those four, three require the pursuers to occupy only vertices at each time t, and furthermore constrain the pursuers' moves so that only one pursuer may move per unit interval of time. The first of those three models (called the *wormhole* model) allows the pursuer in motion to move to a neighboring vertex either by a continuous slide or an instantaneous jump. A capture is made if the evader and some pursuer occupy the same point at the same time. The second of the three models (called the *laser* or *node search* model) provides for capture if the intruder is on an edge whose incident vertices are occupied by pursuers (see [17]). And finally, the third of the three models represents a strategy that combines the wormhole with the laser. Turning to the search models, Alspach focuses on the *basic pursuit-evasion* model (the *BPE* model) with *complete information*. In this model, pursuers and evaders move alternately, and have complete knowledge of the locations of all parties. Capture occurs if the evader and a pursuer are located at the same vertex at the same time. Here, it can be shown that any finite tree T has search number 1.

Fomin and Thilikos likewise survey various models of the pursuit-evasion problem, using vocabulary that differs somewhat from Alspach's. Graph sweeping models such as Parsons' are called *edge search models* as suggested by Golovach's equivalent characterization indicated above. Other edge search models mentioned in [12] include that of [5] in which an edge is cleared either by its traversal by a pursuer or by the simultaneous occupation of its incident vertices by two pursuers. Fomin and Thilikos refer to this model as a *mixed strategy* model. Still other variations of edge-clearing models set particular limits on the degrees of knowledge held by the involved parties. For example, in [22], a node-searching model is accompanied by an assumption that the pursuers can see the evader. In a *radius of capture* model (as given in [11] and [20]), a graph is assumed to be embedded in $\mathcal{R}^3$, and edges are assumed to be made up of polygonal lines. The distance between two adjacent vertices is thus the Euclidean length of the edge, and a capture is made whenever the evader is within some specified distance ϵ of a pursuer. For example, it is known from [11] that the tetrahedron (in which each edge is assumed to have length 1) has search number 2 if $0.5 \leq \epsilon < 1.5$ and 1 if $\epsilon \geq 1.5$. And finally, the cops and robbers model introduced by Quilliot [21] and the duo Nowakowski and Winkler in [18], is the BPE model reviewed by Alspach. In [2], Aigner and Fromme show that under this model the search number of any planar graph is at most 3. Bonato et al. [6] extend this model by specifying that a capture takes place when the evader is within a specified distance of the pursuer, and in [13], Frieze et al. consider a model in which the pursuers and evader may move more than one edge at a time. Major results of this model are surveyed by Bonato and Nowakowski in [7].

Recently, Dyer and Milley [10] have extended the work of Hartnell et al. [16] by considering the minimum number of guards required to ensure that no vertex remains unobserved for more than a fixed time t_0 as the guards execute closed walks through a dominating set. Beaton et al. [4] continued the work in [10] by proving a conjectured bound on the minimum number of guards when the underlying graph is a tree and t_0 is even.

See Abramovskaya and Petrov [1] for a recent survey of models and applications.

Using the terms *watchman* and *intruder*, this paper presents a model that is inspired largely by the works of Brugger [8] and Brugger et al. [9] on the Paranoid Watchman Problem. In [8], watchmen and an intruder move alternately such that at their respective turns to move, (1) each watchman moves instantly from one vertex to an adjacent vertex, and (2) the intruder moves instantly from one vertex to another along a path. A capture occurs if some non-initial vertex of the intruder's path is adjacent to a watchman, or if the intruder and a watchman occupy the same vertex.

In this paper, we consider a model under which watchmen and intruders initiate their movements simultaneously, and edge traversals by watchmen and intruders require positive time. Since watchmen may thus be regarded as occupying edges during time intervals of positive length, we further specify that any watchman who is located in an edge may inspect only that edge and its incident vertices. On the other hand, any watchman at some vertex q may inspect the edges incident with q and the vertices adjacent to q. A capture occurs if an intruder occupies an inspected location.

Though these conditions will be formalized in the next section, we give an example from [8] that will serve to distinguish the model of this paper from the Brugger model.

Graph G is given in Figure 1.1.

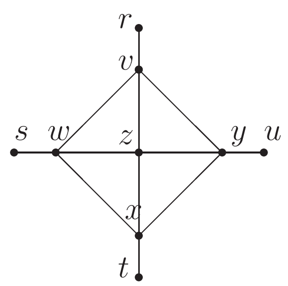

Figure 1.1
The graph G

Under the Brugger model, suppose a lone watchman occupies vertex v while the intruder is at t. If the watchman subsequently moves to z, then w, he prevents the intruder from fleeing to r without capture. This is due to the fact that any walk to r would take the intruder through v, which is visible from z and w. On the other hand, under the model of this paper, the intruder could successfully flee along the path t, x, y, v, r as the watchman is in transit from z to w. We note that this is a real difference in models, since under the model of [8], only one watchman suffices to clear G of intruders, while two will prove necessary under the model proposed herein.

For the remainder of this section, we give an informal elaboration of the proposed model and a summary of the paper.

We will assume that we have a network of finitely many rooms and connecting, non-intersecting corridors with the properties that each corridor leads to precisely two distinct rooms and any two distinct rooms are connected by at most one corridor. An occupant of any given room may fully inspect the corridors that emanate from that room as well as all rooms to which those corridors are connected. Likewise, by the symmetry of visibility, an occupant of any given corridor may fully inspect that corridor and each of the two rooms that are connected by the corridor.

At step 0, a finite number of watchmen and a number of intruders (perhaps infinitely many) are deployed throughout the rooms. Between integral steps i and $i+1$, each watchman may remain in his current room or move through a connecting corridor to an adjacent room. Each intruder, however, may remain in his current room or move to any room that is reachable via the network of rooms and corridors. (Thus, both rooms and corridors may be occupied strictly between consecutive integral steps, but at each integral step, only rooms may be occupied.) The capture of a particular intruder occurs at the earliest integral step i such that strictly between steps $i-1$ and i, or at step i, the location of the intruder is inspectable from the location of some watchman.

Clearly, if each room contains one watchman at step i, then each intruder is captured at step i if not prior. Thus the capture of all intruders in finitely many steps can be accomplished if the number of watchman is at least the number of rooms. However, it is possible that fewer watchmen are needed for the task. (Imagine, for instance, that each two distinct rooms are connected by a corridor. Then only 1 watchman is needed, and the capture of each intruder occurs at step 0.) For a given configuration of rooms and corridors, we therefore require the fewest number of watchmen needed to guarantee that each intruder can be captured in finitely many steps, regardless of the number of intruders present. We will refer to this number as the *watchman number*.

Of course, in a graph-theoretic context, rooms are represented by vertices and corridors are represented by edges. Thus the above conditions require that G is finite, simple, and loopless. From one integral step to the next, intruders move along walks in G of arbitrary finite length and watchmen move along paths in G of length at most 1 such that at each step, each party is located on some vertex. From any vertex v at which a watchman may be located, he can inspect the edges incident with v and the vertices adjacent to v. From any edge e on which a watchman may be travelling, he can inspect e and the two vertices incident with e. The capture of an intruder occurs as described two paragraphs ago.

In Section 2, we establish notation and definitions, thereby giving rigor to the above description. We also present a number of results that allow the analysis of watchman numbers to be conducted under various simplifying conditions. In Section 3, we establish relationships between the watchman numbers of graphs and their subgraphs, thereby facilitating the determination of watchman numbers of graphs in certain structural classes. In Section 4, we consider the relationship between the watchman number and other graph invariants such as diameter, girth, and minimum degree. We then study the watchman numbers of paths, cycles, complete multipartite graphs, interval graphs, hypercubes, and various Cartesian products. (Trees are considered in a separate paper.) Finally, Section 5 is given to closing remarks.

2 Notation, definitions, and preliminary results

Throughout this paper, all graphs will be finite, simple (no multiple edges), and loopless. Unless otherwise indicated, G shall denote an arbitrary graph and H shall denote an arbitrary subgraph of G. For $v \in V(G)$, the *closed neighborhood of* v, $N_G[v]$, shall denote the set of vertices that are equal or adjacent to v in G.

For any integer $n \geq 1$, a *walk of length* n *in* G is an $(n+1)$-tuple $W = (w_0, w_1, w_2, \ldots, w_n)$ of vertices in $V(G)$ such that for $1 \leq i \leq n$, w_{i-1} is adjacent to w_i. The vertices w_0 and w_n are respectively the *initial* and *terminal* vertices of W, and for $1 \leq i \leq n-1$, vertex w_i is an *interior* vertex of W. (Note that since the coordinate entries of W need not be pairwise distinct, a vertex w_i may belong to more than one category.) A *walk of length* 0 *in* G is a 1-tuple (w) for $w \in V(G)$. Such a walk, which is also a path, is said to have initial and terminal vertex w, and no interior vertices.

We now formalize the model in a way that not only captures the description given in Section 1, but generalizes it. Particularly, we will develop the problem allowing for the possibility that the locations of the intruders are constrained to some fixed subgraph of the graph over which the watchmen may move. This generalization will facilitate arguments to be found later in the paper. Additionally, we alert the reader to the fact that the formalization will codify the existence of only one intruder. This simplifies the notation and the argumentation of this section, yet results in no loss of generality since we conclude in Theorem 2.8 that watchman numbers are invariant to the number of intruders.

For positive integer k, a *k-watchman pursuit in* G shall denote any infinite sequence $S_0, S_1, S_2, S_3, \ldots$ such that

(1) each S_i is a k-tuple $\left(v_{i,1}, v_{i,2}, v_{i,3}, \ldots, v_{i,k}\right)$ of (not necessarily distinct) vertices in V, and

(2) $v_{i,j}$ is equal to or adjacent to $v_{i-1,j}$ for each $i \geq 1$.

The vertex $v_{i,j}$ shall denote the *location of watchman j at step i*, and the vector S_i shall be called the *watchman location vector at step i*. In the case $k = 1$, we may dispense with the formality of referring to S_i as the 1-tuple $(v_{i,1})$, and instead merely identify S_i with the vertex v_i or the 1-tuple (v_i) as clarity requires. We may make occasional reference to a *pursuit in* G; this shall refer to a k-watchman pursuit in G for some k. Note that (2) codifies the informal condition that from one step to the next, each watchman will either not change rooms or will move to an adjacent room.

Similarly, an *intruder evasion in* H (where H is a subgraph of G) shall denote an infinite sequence of walks $W_0, W_1, W_2, W_3, \ldots$ in H such that

(3) no W_i of positive length has equal terminal and initial vertices, and

(4) the terminal vertex of W_{i-1} is the initial vertex of W_i.

The walk W_i shall represent the route taken by the intruder between step i and step $i+1$. The initial vertex of W_i shall denote the *location of the intruder at step i*

(implying that the terminal vertex of W_i, which equals the initial vertex of W_{i+1}, is the location of the intruder at step $i+1$). Note that (4) codifies the informal condition that from one step to the next, the intruder may flee to a different room in H that is reachable by a walk in H or remain at his current location (in the event that the corresponding walk is of length 0).

Now let $S = S_0, S_1, S_2, \ldots$ denote a k-watchman pursuit in G, and let $T = W_0, W_1, W_2, \ldots$ denote an intruder evasion in H. Then the ordered pair (S, T) shall be called a *k-search of H from G.* (In the event $H = G$, the phrase "from G" may be omitted.) We say that *the intruder is captured at step i under* (S, T) if and only if i is the smallest integer such that for some h, $1 \leq h \leq k$,

(5) the initial vertex of W_i is adjacent or equal to $v_{i,h}$, or

(6) $v_{i-1,h}$ and $v_{i,h}$ are distinct and some interior vertex of W_{i-1} is equal to either $v_{i-1,h}$ or $v_{i,h}$, or

(7) $v_{i-1,h} = v_{i,h}$ and some interior vertex of W_{i-1} is adjacent to $v_{i,h}$.

Moreover, we say that *watchman h_0 captures the intruder at step i under* (S, T) if and only if the intruder is captured at step i under (S, T) and condition (5), (6) or (7) holds with $h = h_0$.

It may be helpful to think of the movements of watchmen and the intruder as occuring between steps along edges, during which time the as-yet-uncaptured intruder will be captured if (by 6) his walk takes him through a vertex that is incident with an edge along which a watchman is moving, or (by 7) his walk takes him through a vertex that is adjacent to a vertex occupied by a watchman who has elected not to move. A capture will also occur (by 5) if at the ends of the moves of the watchmen and the as-yet-uncaptured intruder, the intruder is located on a vertex that is equal or adjacent to a vertex that locates some watchman. Note that for fixed h, conditions (6) and (7) are mutually exclusive, but that (5) and (6) may both be true as may (5) and (7). In each of the two cases, both conditions declare that the capture occurs at step i, so there is no ambiguity. In the sequel, it will be useful to refer to the captures described by (5), (6), and (7) as captures of Type 1, Type 2, and Type 3, respectively.

Let (S^*, T^*) be a fixed k-search of H from G. Then (S^*, T^*) *is successful for H relative to G* if and only if there exists a step at which the intruder is captured under (S^*, T^*). Furthermore, S^* *is successful for H relative to G* if and only if for all intruder evasions T in H, (S^*, T) is successful for H relative to G. Finally, we say that *H is k-guardable relative to G* if and only if there exists a k-watchman pursuit in G that is successful for H relative to G. (In the event that $H = G$, the phrase "relative to G" may be omitted.)

It is clear that H is $|V(G)|$-guardable relative to G. (The deployment of one watchman to each of the vertices of G suffices for the capture of the intruder at step 0.) Thus there exists a smallest integer k such that H is k-guardable relative to G, motivating the next definition.

Definition 2.1. Let G be a graph with subgraph H. Then the *watchman number of H relative to G*, denoted $w(H|G)$, is the smallest positive integer k such that H is k-guardable relative to G. If $H = G$, then $w(H|G)$ shall be denoted $w(G)$ and shall be called the *watchman number of G*.

Since H is a subgraph of G, it follows that the set of pursuits S in G such that (S,T) is successful for all intruder evasions T in G is a subset of the set of pursuits S in G such that (S,T) is successful for all intruder evasions T in H. Thus we have

Observation 2.2. *For graph G with subgraph H, $w(H|G) \leq w(G)$.*

The next theorem indicates that if an intruder can avoid capture by taking walks, then he can avoid capture by taking paths. This will prove useful in the proof of Theorem 2.6.

Theorem 2.3. *Let (S,T) be a k-search of H from G such that for fixed integer $c \geq 0$, the intruder is not captured at or prior to step c under (S,T). Then there exists an intruder evasion $T^* = W_0^*, W_1^*, W_2^* \ldots$ in H such that*

(1) for $0 \leq i \leq c$, W_i^* is a path in H, and

(2) under (S,T^*), the intruder is not captured at or prior to step c.

Proof: Let $T = W_0, W_1, W_2, \ldots$. For either $0 \leq i \leq c$ such that W_i is a path or $i > c$, define W_i^* to be W_i. For $0 \leq i \leq c$ such that W_i is not a path, it follows that W_i has positive length, and hence has distinct initial and terminal vertices α_i and β_i respectively. In this case, define W_i^* to be P_i where P_i is some fixed path from α_i to β_i such that all interior vertices (if any) of P_i are found among the interior vertices incident with W_i. Then it is clear that $T^* = W_0^*, W_1^*, W_2^*, \ldots$ is an intruder evasion in H satisfying (1). To see that (2) holds, note that under (S,T^*), the intruder cannot experience a capture of Type 1, Type 2, or Type 3 at or prior to step c since under (S,T) the intruder does not experience a capture of Type 1, Type 2, or Type 3 at or prior to step c. ∎

We now give the definition of a set $D_i(G,H,S)$ that is analogous to the complement of the set A_i of cleared vertices discussed in [8]. Intuitively, $D_i(G,H,S)$ will be the set of vertices at which the intruder may be located at step i, uncaptured under the movements of the watchmen given by S.

Definition 2.4. Let G be a graph with subgraph H and let S denote a pursuit in G. Then for non-negative integer i, $D_i(G,H,S)$ denotes the set of vertices in $V(H)$ such that $v \in D_i(G,H,S)$ if and only if there exists an intruder evasion T in H with the properties that

(1) the intruder is located on vertex v at step i under T, and
(2) the intruder is not captured at any step $i_0 \leq i$ under (S,T).

We shall denote $D_i(G,H,S)$ by D_i when there is no chance for confusion.

An example follows. In Figure 2.1 and Table 2.2, we give $G = P_4 \square P_4$ (the

Cartesian product of P_4 with P_4) and demonstrate a 2-watchman pursuit S in G and $D_i(G, G, S)$ through step 11.

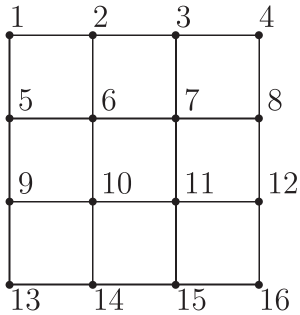

Figure 2.1
The graph $P_4 \Box P_4$

	S_i	$D_i(G, G, S)$		S_i	$D_i(G, G, S)$
$i = 0$	(1,9)	$\{3, 4, 6, 7, 8, 11, 12, 14, 15, 16\}$	$i = 6$	(6,11)	$\{3, 4, 8, 16\}$
$i = 1$	(2,9)	$\{4, 7, 8, 11, 12, 14, 15, 16\}$	$i = 7$	(2,11)	$\{4, 8, 16\}$
$i = 2$	(6,9)	$\{3, 4, 8, 11, 12, 14, 15, 16\}$	$i = 8$	(3,11)	$\{8, 16\}$
$i = 3$	(6,13)	$\{3, 4, 8, 11, 12, 15, 16\}$	$i = 9$	(4,11)	$\{16\}$
$i = 4$	(6,14)	$\{3, 4, 8, 11, 12, 16\}$	$i = 10$	(8,11)	$\{16\}$
$i = 5$	(6,15)	$\{3, 4, 8, 12\}$	$i = 11$	(12,11)	$\emptyset$

Table 2.2
A 2-watchman pursuit in $P_4 \Box P_4$

Observation 2.5. *For fixed k, let S and S^* be k-watchman pursuits in G. For fixed non-negative integers x and x^*, suppose $D_x(G, H, S) =$ (resp. $\subseteq$) $D_{x^*}(G, H, S^*)$. Suppose also that, for $i \geq 0$, $S_{x+i} = S^*_{x^*+i}$. Then for $i \geq 0$, $D_{x+i}(G, H, S) =$ (resp. $\subseteq$) $D_{x^*+i}(G, H, S^*)$*

Suppose S is a successful pursuit for H relative to G. From the next theorem, it will follow that there exists a finite upper bound on the number of steps required for the capture of the intruder under (S, T) over the set of all intruder evasions T in H. That is, there exists a finite number of steps n_S such that no matter which intruder evasion T is elected, the intruder will be captured in n_S or fewer steps under (S, T). It will follow as a corollary that $w(H|G)$ will equal the fewest number of watchmen sufficient to the task of capturing an arbitrary number of simultaneous intruders.

Theorem 2.6. *The pursuit S in G is successful for H relative to G if and only if $D_i(G, H, S) = \emptyset$ for some i.*

Proof: $\Leftarrow$. Suppose that c is a fixed integer such that $D_c(G, H, S) = \emptyset$. Select an arbitrary intruder evasion T in H, and suppose that the intruder is located on vertex v_0 at step c under T. Since $v_0 \notin D_c(G, H, S)$, it follows by Definition 2.4 that under (S, T), the intruder is captured at some step $i \leq c$. So, by the arbitrariness of T, S is successful for H relative to G.

$\Rightarrow$. Now suppose that for all $i \geq 0$, $D_i(G, H, S) \neq \emptyset$. By Theorem 2.3, it follows that for each $i \geq 0$, there exists an intruder evasion in H, say,

$$T^i = P_0^i, P_1^i, P_2^i, P_3^i, \ldots, P_i^i, W_{i+1}^i, W_{i+2}^i \ldots$$

such that the intruder is not captured at any step at or prior to step i under (S, T^i), and the first $i + 1$ terms of T^i are paths in H. Since $V(H)$ is finite and hence contains only finitely many distinct paths, there exists a path P_0 in H and an infinite subsequence SS_0 of $0, 1, 2, 3, 4, 5, 6, \ldots$ such that for every term x of SS_0, the first term of T^x is P_0. Note that any intruder evasion T in H such that $T_0 = P_0$ has the property that the intruder is not captured at step 0 under (S, T).

Now, since $V(H)$ is finite, there exists a path P_1 in H and an infinite subsequence SS_1 of SS_0 such that for every term x of SS_1, the first two terms of T^x are P_0 and P_1. Note that any intruder evasion T in H such that $T_0 = P_0$ and $T_1 = P_1$ has the property that the intruder is not captured at step 0 or step 1 under (S, T).

Continuing this process, we create an intruder evasion

$$T^* = P_0, P_1, P_2, P_3, \ldots$$

in H such that at no finite step i is the intruder captured under (S, T^*). Thus S is not successful for H relative to G. ■

Let S be a k-watchman pursuit that is successful for H relative to G. Then by the preceding theorem, we may find the smallest integer i such that $D_i(G, H, S) = \emptyset$. Denoting this integer by $m_{G,H,S}$ and observing that it is not a function of any given intruder evasion, we have the following result.

Corollary 2.7. *Let S be a pursuit that is successful for H relative to G. Then for any intruder evasion T in H, the intruder is captured at or prior to step $m_{G,H,S}$ under (S, T). Moreover, by Definition 2.4, there exists an intruder evasion T such that under (S, T) the intruder is uncaptured at step $m_{G,H,S} - 1$.*

Let $\mathcal{PE}$ denote our formalized pursuit-evasion model under which there exists precisely one intruder. Let $\mathcal{PE}'$ denote the pursuit-evasion model that is identical to $\mathcal{PE}$ except for the constraint that there exist at least two simultaneous intruders. Continuing our assumption that S is a k-watchman pursuit that is successful for H relative to G, we observe that under $\mathcal{PE}'$, each of the two or more intruders will be captured no later than $m_{G,H,S}$ as determined under $\mathcal{PE}$. Expanding (in the obvious way) our use of the phrase "watchman number of H relative to G" to the case of more than one intruder, we have that the watchman number of H relative to G under $\mathcal{PE}'$ is thus no more than the watchman number of H relative to G under $\mathcal{PE}$. This inequality is clearly reversible, resulting in the next theorem.

Theorem 2.8. *Let G be a graph with subgraph H. Then the fewest number of watchmen sufficient to the task of capturing an arbitrary number $m \leq \infty$ of simultaneous intruders is invariant to m. Therefore, to study the general model of this paper in* *which the number of intruders is at least 1, it suffices to assume the existence of just one intruder.*

We now consider additional simplifying conditions. Let S be a k-watchman pursuit in G and let $\vec{v}$ denote a k-tuple of vertices in $V(G)$. Then S *initiates at* $\vec{v}$ if and only if $S_0 = \vec{v}$. Additionally, for positive integer c, S *is aggressive to step* c (resp. *elementary to step* c) if and only if for $1 \leq i \leq c$, S_{i-1} and S_i differ in at least one coordinate (resp. differ in at most one coordinate). (Thus, in a search that is elementary and aggressive to step c, precisely one watchmen will move between step $i-1$ and step i, $1 \leq i \leq c$.) We close this section by showing that if $\vec{v}$ is an arbitrary k-tuple of vertices in $V(G)$ and H is k-guardable relative to connected G, then there exists a k-watchman pursuit S in G such that S initiates at $\vec{v}$, S is successful for H relative to G, and S is both elementary and aggressive to step $m_{G,H,S}$. Thus, in the determination of $w(H|G)$, it will suffice to consider only those pursuits that are aggressive and elementary to finitely many steps, as well as initiated at the most convenient vertices. We begin with the matter of elementariness.

Theorem 2.9. *Let H be k-guardable relative to G. Then there exists a k-watchman pursuit S that is successful for H relative to G and elementary to step $m_{G,H,S}$.*

Proof: Throughout the proof, all successful pursuits are understood to be for H relative to G.

The claim is clearly true if $k = 1$. Thus we suppose that k is a fixed integer at least 2, and we let $\mathcal{K}$ denote the set of successful k-watchman pursuits. By assumption, $\mathcal{K} \neq \emptyset$.

For each pursuit $S \in \mathcal{K}$ we define the following:

(1) for each i, $1 \leq i \leq m_{G,H,S}$, $\sigma_i(S)$ denotes the number of coordinates in which S_{i-1} and S_i differ;

(2) $\sigma(S)$ denotes $\sum \sigma_i(S)$ where the summation is taken over all i, $1 \leq i \leq m_{G,H,S}$ such that $\sigma_i(S) \geq 2$. (We take $\sigma(S)$ to be 0 if the indexing set is empty.)

Since we are claiming that there exists a pursuit S in $\mathcal{K}$ such that $\sigma_i(S) \leq 1$ for $1 \leq i \leq m_{G,H,S}$, we assume the contrary that for every pursuit S in $\mathcal{K}$, $\sigma_i(S) \geq 2$ for some i, $1 \leq i \leq m_{G,H,S}$, and hence $\sigma(S) \geq 2$. We may thus select a pursuit $S^* = S^*_0, S^*_1, S^*_2, \ldots$ in $\mathcal{K}$ such that $\sigma(S^*)$ minimizes $\sigma(S)$ over all pursuits S in $\mathcal{K}$, and we may find an integer $c \leq m_{G,H,S^*}$ such that $\sigma_c(S^*) \geq 2$. With no loss of generality, we assume that $v_{c-1,1}$ differs from $v_{c,1}$ and that $v_{c-1,2}$ differs from $v_{c,2}$.

Now let S denote the k-watchman pursuit that agrees with S^* except for the insertion of a particular watchman location vector S^*_z; particularly,

$$S = S^*_0, S^*_1, S^*_2, \ldots, S^*_{c-1}, S^*_z, S^*_c, S^*_{c+1}, \ldots,$$

where S^*_z is the watchman location vector that has first coordinate entry $v_{c-1,1}$ and agrees with S^*_c otherwise. We aim to show that S is in $\mathcal{K}$ with $\sigma(S) < \sigma(S^*)$, thereby contradicting the minimality of $\sigma(S^*)$.

Let $T = W_0, W_1, W_2, \ldots$ be an arbitrary intruder evasion in H. Let the walks W_{c-1} and W_c be, respectively, ${}^{\alpha}W^{\beta}$ and ${}^{\beta}W^{\gamma}$, where the presuperscripts and postsuperscripts respectively indicate initial and terminal vertices of the walks. Let ${}^{\alpha}W^{\gamma}$ denote the walk formed by catenating ${}^{\alpha}W^{\beta}$ and ${}^{\beta}W^{\gamma}$ where, if $\alpha = \gamma$, ${}^{\alpha}W^{\gamma}$ is understood to be the path of 0 length. Then the following sequence T^* is easily verified to be an intruder evasion in H:

$$T^* = W_0, W_1, W_2, \ldots, W_{c-2},\ {}^{\alpha}W^{\gamma},\ W_{c+1}, W_{c+2}, \ldots$$

Moreover, (S^*, T^*) is successful for H relative to G since S^* is successful for H relative to G by assumption. This implies that under (S^*, T^*), the intruder is captured prior to step c, at step c, or after step c.

Since S^* and S (resp. T^* and T) are termwise identical prior to step c (resp. prior to step $c-1$), the intruder is captured at step $s < c$ under (S^*, T^*) if and only if he is captured at step $s < c$ under (S, T). Likewise, for $x \geq c$, the x^{th} step of S^* (resp. T^*) is identical to the $(x+1)^{st}$ step of S (resp. T). So, if the intruder is captured at step $s > c$ under (S^*, T^*), he will be captured by step $s+1$ under (S, T). So let us assume that the intruder is captured at step c under (S^*, T^*). Since the intruder then cannot be captured prior to step c under (S, T) (for otherwise he would be captured prior to step c under (S^*, T^*)), we will show that he is captured at step c or step $c+1$ under (S, T).

Since the intruder is captured at step c under (S^*, T^*), then the intruder experiences at least one of the following:

(1) a Type 1 capture: some coordinate entry $v_{c,h}$ of S^*_c is equal or adjacent to the initial vertex γ of the walk W^*_c, or

(2) a Type 2 capture: some interior vertex of the walk $W^*_{c-1} = {}^{\alpha}W^{\gamma}$ is equal to one of the distinct coordinate entries $v_{c-1,h}$ and $v_{c,h}$ of S^*_{c-1} and S^*_c, respectively, or

(3) a Type 3 capture: some interior vertex of the walk $W^*_{c-1} = {}^{\alpha}W^{\gamma}$ is adjacent to the indistinct coordinate entries $v_{c-1,h}$ and $v_{c,h}$ of S^*_{c-1} and S^*_c, respectively.

Suppose (1) holds. Since $S_{c+1} = S^*_c$ and $W_{c+1} = W^*_c$, then the coordinate entry $v_{c,h}$ of S_{c+1} is equal or adjacent to the initial vertex γ of the walk in W_{c+1}, implying that the intruder is captured by step $c+1$ under (S, T).

Suppose (3) holds. Then $\alpha \neq \gamma$ and the walk ${}^{\alpha}W^{\gamma}$ contains some interior vertex v that is adjacent to $v_{c-1,h} = v_{c,h}$ where necessarily $h \neq 1, 2$. We observe that v is in the interior of ${}^{\alpha}W^{\beta} = W_{c-1}$ or in the interior of ${}^{\beta}W^{\gamma} = W_c$ or is the initial vertex β of W_c. We also observe that $v_{c-1,h}$ is in the h^{th} coordinate of S_{c-1}, S_c, and S_{c+1}. Thus, the intruder is captured by step $c+1$ under (S, T).

Finally, suppose that (2) holds. Then $\alpha \neq \gamma$, $v_{c-1,h}$ is adjacent to $v_{c,h}$, and the walk ${}^{\alpha}W^{\gamma}$ contains some interior vertex v that is equal to $v_{c-1,h}$ or $v_{c,h}$. As in the case above, we observe that v is in the interior of ${}^{\alpha}W^{\beta} = W_{c-1}$ or in the interior of ${}^{\beta}W^{\gamma} = W_c$ or is the initial vertex β of W_c. Consider two cases.

Suppose $h = 1$. Then $v_{c-1,h}$ is the first coordinate entry of S_{c-1} and S_c, and $v_{c,h}$ is the first coordinate entry of S_{c+1}. Moreover, as noted above, $W_{c-1} = {}^{\alpha}W^{\beta}$ and $W_c = {}^{\beta}W^{\gamma}$. Therefore, the following hold: if v is an interior vertex of ${}^{\alpha}W^{\beta}$, then the intruder will experience a capture of Type 3 at step c under (S,T); if v is in the interior of ${}^{\beta}W^{\gamma}$, then the intruder will experience a capture of Type 2 by step $c+1$ under (S,T); if v is β, then the intruder will experience a capture of Type 1 at step c under (S,T).

Suppose $h \neq 1$. Then $v_{c-1,h}$ is the h^{th} coordinate entry of S_{c-1} and $v_{c,h}$ is the h^{th} coordinate entry of both S_c and S_{c+1}. Moreover, as noted above, $W_{c-1} = {}^{\alpha}W^{\beta}$ and $W_c = {}^{\beta}W^{\gamma}$. Therefore, the following hold: if v is an interior vertex of ${}^{\alpha}W^{\beta}$, then the intruder will experience a capture of Type 2 at step c under (S,T); if v is in the interior of ${}^{\beta}W^{\gamma}$, then the intruder will experience a capture of Type 3 at step $c+1$ under (S,T) if he hasn't been captured at step c; if v is β, then the intruder will experience a capture of Type 1 at step c under (S,T).

Thus S is successful since T was arbitrary. Moreover, we have seen that for any intruder evasion T, the intruder is captured under (S,T) at step no later than $s+1$ where s is the step at which the intruder would be caught under (S^*,T^*). Thus by Corollary 2.7, $m_{G,H,S} \leq m_{G,H,S^*}+1$. We also observe that through step $m_{G,H,S^*}+1$, the terms of S are

$$S_0^*, S_1^*, S_2^*, \ldots, S_{c-1}^*, S_z^*, S_c^*, S_{c+1}^*, \ldots, S_{m_{G,H,S^*}}^*,$$

while through step m_{G,H,S^*}, the terms of S^* are

$$S_0^*, S_1^*, S_2^*, \ldots, S_{c-1}^*, S_c^*, S_{c+1}^*, \ldots, S_{m_{G,H,S^*}}^*.$$

Since S_z^* and S_c^* differ in exactly one coordinate, and S_z^* and S_{c-1}^* differ in one fewer coordinates than S_{c-1}^* and S_c^*, it follows that $\sigma(S) < \sigma(S^*)$, contradicting the minimality of $\sigma(S^*)$. ■

We next turn to the matter of aggressive pursuits.

Theorem 2.10. *Let H be k-guardable relative to G. Then there exists a k-watchman pursuit S that is successful for H relative to G and both elementary and aggressive to step $m_{G,H,S}$.*

Proof: By Theorem 2.9, we may select a k-watchman pursuit

$$S^* = S_0^*, S_1^*, S_2^*, \ldots$$

that is successful for H relative to G and elementary to step m_{G,H,S^*}. If S^* is aggressive to step m_{G,H,S^*}, we are done. So, suppose that c is a fixed integer such that $c \leq m_{G,H,S^*}$ and $S_{c-1}^* = S_c^*$. Let S^{**} be the elementary k-watchman pursuit in G that results by deleting the term S_c^* from the sequence S^*:

$$S^{**} = S_0^*, S_1^*, S_2^*, \ldots, S_{c-1}^*, S_{c+1}^*, S_{c+2}^*, \ldots$$

Since $S_i^* = S_i^{**}$ for $0 \leq i \leq c-1$, then $D_i(G,H,S^{**}) = D_i(G,H,S^*)$ for $0 \leq i \leq c-1$, giving in particular $D_{c-1}(G,H,S^{**}) = D_{c-1}(G,H,S^*)$. But $D_{c-1}(G,H,S^*) =$

$D_c(G,H,S^*)$ since $S^*_{c-1} = S^*_c$. So $D_{c-1}(G,H,S^{**}) = D_c(G,H,S^*)$. Moreover, for $i \geq 0$, $S^{**}_{c-1+i} = S^*_{c+i}$. Thus by Observation 2.5, $D_{c-1+i}(G,H,S^{**}) = D_{c+i}(G,H,S^*)$ for $i \geq 0$. Letting $i = m_{G,H,S^*} - c$, it therefore follows that $D_{m_{G,H,S^*}-1}(G,H,S^{**}) = D_{m_{G,H,S^*}}(G,H,S^*) = \emptyset$, which implies by Theorem 2.6 that S^{**} is successful with $m_{G,H,S^{**}} < m_{G,H,S^*}$. (Particularly, $m_{G,H,S^{**}} = m_{G,H,S^*} - 1$.) Moreover, it is clear that S^{**} is elementary to step $m_{G,H,S^{**}}$. Thus, by the iterative deletion of consecutive repeated terms of S^*, we construct the desired k-watchman pursuit. ■

Finally, we consider initiation.

Theorem 2.11. *Let H be k-guardable relative to connected G, and let $\vec{v}$ denote a k-tuple of vertices in $V(G)$. Then there exists a k-watchman pursuit S such that S is successful for H relative to G, S initiates at $\vec{v}$, and S is both aggressive and elementary to step $m_{G,H,S}$.*

Proof: By Theorem 2.10, we may select a k-watchman pursuit

$$S^* = S^*_0, S^*_1, S^*_2, \ldots$$

that is successful for H relative to G and both elementary and aggressive to step m_{G,H,S^*}. Because G is connected, it follows that for some integer h, we may construct a k-watchman pursuit in G

$$S = S_0, S_1, \ldots, S_h, S^*_0, S^*_1, S^*_2, S^*_3, \ldots$$

such that S initiates at $\vec{v}$ and is elementary and aggressive to step $h + 1 + m_{G,H,S^*}$. To conclude the proof, it therefore suffices to show that S is successful for H relative to G with $m_{G,H,S} \leq h + m_{G,H,S^*} + 1$. But by Observation 2.5 with $x^* = 0$ and $x = h+1$, $D_{h+1+m_{G,H,S^*}}(G,H,S) \subseteq D_{m_{G,H,S^*}}(S^*) = \emptyset$. The result now follows from Theorem 2.6. ■

3 Watchman numbers of subgraphs relative to graphs

It is a property of watchman numbers that if H is a subgraph of G, then $w(H)$ is not necessarily less than or equal to $w(G)$. Take, for instance, K_1 join C_4. This graph clearly has watchman number 1, yet its subgraph C_4 will prove to have watchman number 2. Under what conditions is it the case that $w(H) \leq w(G)$?

Let H be a subgraph of G. Then the *closure of* H, denoted $\bar{H}$, is the subgraph of G induced by $\bigcup_{v \in V(H)} N_G[v]$. Additionally, the *vertex boundary of* H is $B(H) = V(\bar{H}) - V(H)$. We note that $v \in V(\bar{H})$ if and only if either $v \in V(H)$ or v is adjacent to some vertex in $V(H)$. Since $\bar{H}$ is a subgraph of G, then any k-watchman pursuit in $\bar{H}$ is a k-watchman pursuit in G. Thus we have

Observation 3.1. *Let H be a subgraph of G. Then $w(H|G) \leq w(H|\bar{H})$.*

In the following discussion, it will be occasionally convenient to denote the infinite sequence of watchman location vectors $S_r, S_{r+1}, S_{r+2}, \ldots$ by $S_r, S_{r+1}, S_{r+2}, \ldots, S_\infty$, where S_∞ is merely a symbolic reminder of the infinitude of the sequence.

Let H be a subgraph of G and let $S = S_0, S_1, S_2, \ldots$ be a k-watchman pursuit in G that is successful for H relative to G. Suppose also that $V(\bar{H})$ is a proper subset of $V(G)$. Then a *detour from $\bar{H}$ by watchman j under S* is a finite or infinite sequence

$$S_r, S_{r+1}, S_{r+2}, \ldots, S_t$$

of at least 3 consecutive terms of S such that

(1) $r < m_{G,H,S}$,

(2) the location of watchman j at step r is in $B(H)$, and

(3) t is the maximum extended integer ($r+2 \leq t \leq \infty$) such that for all i, $r < i < t$, watchman j is located in $V(G) - V(\bar{H})$ at step i. (Note that if t is finite, the location of watchman j at step t must be in $B(H)$.)

We observe that since $r < m_{G,H,S}$, it follows that for each j, there are only finitely many distinct detours from $\bar{H}$ by watchman j under fixed S. We also note that since detours have at least 3 terms, the set of integers strictly between r and t is not empty.

Theorem 3.2. *Let H be a connected subgraph of G that is k-guardable relative to G. Then H is k-guardable relative to $\bar{H}$ and $w(H|\bar{H}) \leq w(H|G)$.*

Proof: Throughout the proof, all k-watchman pursuits are understood to be in G and all successes are understood to be for H relative to G. We may assume that $V(\bar{H})$ is a proper subset of $V(G)$, for otherwise $\bar{H} = G$, rendering the theorem trivial. Additionally, we note that if H is k-guardable relative to $\bar{H}$ when H is k-guardable relative to G, then it easily follows that $w(H|\bar{H}) \leq w(H|G)$. We thus devote our attention to proving that H is k-guardable relative to $\bar{H}$.

Since H is k-guardable relative to G, then by Theorem 2.11 we may select a successful k-watchman pursuit $S' = S'_0, S'_1, S'_2, \ldots$ such that the watchman location vector S'_0 is a vector of vertices in $V(H)$. If the number of detours from $\bar{H}$ of watchman j under S' is 0 for each j, then we are done. Thus, suppose without loss of generality that the number of detours from $\bar{H}$ of watchman 1 under S' is $d \geq 1$. It will suffice to construct a successful k-watchman pursuit S^* such that (1) S^*_0 contains only vertices in $V(H)$, and (2) under S^*, the number of detours from $\bar{H}$ of watchman 1 is $d - 1$ while the number of detours from $\bar{H}$ of every other watchman is the same as under S'.

Let $S'_r, S'_{r+1}, \ldots, S'_t$ be a detour from $\bar{H}$ of watchman 1 under S', where $t \leq \infty$. Then $r \geq 1$ since S'_0 contains only vertices in $V(H)$.

We assume that the location of watchman 1 at step r is $v_r \in B(H)$ and, if t is finite, the location of watchman 1 at step t is $v_t \in B(H)$.

If $t < \infty$, then due to the connectedness of H, it follows that there exists a finite sequence of vertices $h_r, h_{r+1}, h_{r+2}, \ldots, h_{y-1}, h_y$ in $\bar{H}$ such that

(1) $y \geq t$,
(2) $h_r = v_r$, $h_y = v_t$ and all other terms are in $V(H)$, and

(3) consecutive terms of the sequence are either equal or adjacent.

In this case, we define S to be the k-watchman pursuit in G that results by inserting $y-t$ copies of S'_{r+1} between S'_r and S'_{r+1} in S'. On the other hand, if $t=\infty$, we set $y=\infty$, and we define S to equal S'. In each case, by an application of Observation 2.5, S is successful. Moreover, watchman 1 has d detours from $\bar{H}$. If t is finite, one of those detours is given by

$$S'_r, \overbrace{S'_{r+1}, S'_{r+1}, \dots, S'_{r+1}}^{y-t+1 \ copies}, S'_{r+2}, S'_{r+3}, \dots, S'_t \quad = \quad S_r, S_{r+1}, S_{r+2}, \dots S_y.$$

If t is infinite, one of those detours is

$$S'_r, S'_{r+1}, S'_{r+2}, \dots S'_\infty \quad = \quad S_r, S_{r+1}, S_{r+2}, \dots S_y.$$

We now show the existence of S^*. For $0 \le i \le r$, define S^*_i to be S_i. If $t=\infty$, then for $i > r$, define S^*_i to be the k-tuple of vertices that equals S_i in the last $k-1$ coordinates and v_r in the first coordinate. If $t<\infty$, then for $r+1 \le i \le y-1$, define S^*_i to be the k-tuple of vertices that equals S_i in the last $k-1$ coordinates and h_i in the first coordinate; and for $i \ge y$, define S^*_i to be S_i. Then it is readily seen that $S^* = S^*_0, S^*_1, S^*_2, \dots$ is a k-watchman pursuit in G under which watchman 1 has $d-1$ detours from $\bar{H}$ and each watchman $j \neq 1$ has the same number of detours from $\bar{H}$ under S^* as under S. We show that for every i, $D_i(G,H,S^*) \subseteq D_i(G,H,S)$, thereby proving that S^* is successful by Theorem 2.6.

Fix an arbitrary integer $c \ge 0$ and $v \in D_c(G,H,S^*)$. Then there exists an intruder evasion T in H such that at step c, the intruder is located on v and is uncaptured at or prior to step c under (S^*,T). But for all $i \ge 0$ and j, $2 \le j \le k$, the j^{th} coordinate entry of S^*_i equals the j^{th} coordinate entry of S_i. Thus for such j, watchman j does not capture the intruder at or prior to step c under (S,T). We therefore see that if the intruder is captured at or prior to step c under (S,T), he is captured by watchman 1. But for $0 \le i \le r$, watchman 1 has the same location at step i under S^* as under S. Therefore, if $0 \le c \le r$, the intruder is not captured by watchman 1 at or prior to step c under (S,T). Moreover, for steps $r < i < y$, the intruder is confined to vertices (in $V(H)$) that are not equal or adjacent to the locations of watchman 1 under S, implying that the intruder is not captured by watchman 1 at or prior to step c under (S,T) for $r < c < y$. And finally, we observe that for $i \ge y$ (which is vacuous if $y=\infty$), watchman 1 has the same location at step i under S as under S^*. Therefore the intruder is not captured by watchman 1 at any step at or prior to step c under (S,T) if $c \ge y$. Hence $v \in D_c(G,H,S)$, implying that $D_c(G,H,S^*) \subseteq D_c(G,H,S)$ and therefore that S^* is successful for H relative to G. Moreover, since $S^*_0 = S_0$, then S^*_0 contains only vertices in $V(H)$. The proof is thus concluded since watchman 1 has $d-1$ detours from $\bar{H}$ under S^*. ■

Corollary 3.3. *Let H be a connected subgraph of G. Then by Observation 3.1 and Theorem 3.2, $w(H|G) = w(H|\bar{H})$.*

Let H be a subgraph of G such that $B(H)$ is a non-empty set $\{x_1, x_2, \dots, x_m\}$,

$m \geq 1$. Then $\bar{H}$ is said to be *basic* if and only if (1) each vertex x_i in $B(H)$ is adjacent to precisely one vertex y_i in $V(H)$, and (2) if x_{i_1} and x_{i_2} are adjacent elements of $B(H)$, then y_{i_1} and y_{i_2} are indistinct or adjacent elements of $V(H)$.

Theorem 3.4. *Let H be an induced connected subgraph of G and suppose that $\bar{H}$ is basic. Then $w(H) \leq w(G)$.*

Proof: We have seen that $w(H|\bar{H}) = w(H|G) \leq w(G)$. It thus suffices to show that $w(H) = w(H|\bar{H})$. Since it is clear that $w(H) \geq w(H|\bar{H})$, we show $w(H) \leq w(H|\bar{H})$.

We recall that $\bar{H}$ is the graph induced by the vertices of H and their neighbors. Thus, since H is induced, it follows that two vertices in $V(H)$ are adjacent in H if and only if they are adjacent in $\bar{H}$.

Let $S = S_0, S_1, S_2, \ldots$ be a k-watchman pursuit in $\bar{H}$, where $S_i = (v_{i,1}, v_{i,2}, \ldots, v_{i,k})$. For each $i \geq 0$, let S_i^* denote the k-tuple of vertices $(v_{i,1}^*, v_{i,2}^*, \ldots, v_{i,k}^*)$ in H such that (1) if $v_{i,j}$ is in $V(H)$, then $v_{i,j}^* = v_{i,j}$, and (2) if $v_{i,j}$ is in $B(H)$, then $v_{i,j}^*$ is the unique vertex in $V(H)$ to which $v_{i,j}$ is adjacent. Since $\bar{H}$ is basic, it is readily seen that $S^* = S_0^*, S_1^*, S_2^*, \ldots$ is a k-watchman pursuit in H. It thus suffices to show that $D_i(H, H, S^*) \subseteq D_i(\bar{H}, H, S)$ for each i. We proceed by induction on i.

To establish the base case, select $v \in D_0(H, H, S^*)$. Then $v \in V(H)$ and $v \notin \bigcup_{j=1}^{k} N_H[v_{0,j}^*]$. Since $\bar{H}$ is basic, it follows that $v \notin \bigcup_{j=1}^{k} N_{\bar{H}}[v_{0,j}]$, giving $v \in D_0(\bar{H}, H, S)$. Thus $D_0(H, H, S^*) \subseteq D_0(\bar{H}, H, S)$.

Now suppose that c is a positive integer such that for $0 \leq i \leq c-1$, $D_i(H, H, S^*) \subseteq D_i(\bar{H}, H, S)$. Select $v \in D_c(H, H, S^*)$. Then there exists an intruder evasion T^* in H

$$T^* = \ {}^{\alpha_0}W^{\alpha_1}, \ {}^{\alpha_1}W^{\alpha_2}, \ \ldots \ {}^{\alpha_{c-2}}W^{\alpha_{c-1}} \ {}^{\alpha_{c-1}}W^{v}, \ {}^{v}W^{\alpha_{c+1}}, \ldots$$

such that under (S^*, T^*) the intruder is on vertex v at step c and is uncaptured at step c or prior. Thus the intruder is uncaptured at step $c-1$ under (S^*, T^*), implying that α_{c-1} is in $D_{c-1}(H, H, S^*)$, and hence (by the inductive assumption) also in $D_{c-1}(\bar{H}, H, S)$. Therefore, there exists an intruder evasion T

$$T = \ {}^{\beta_0}W^{\beta_1}, \ {}^{\beta_1}W^{\beta_2}, \ \ldots \ {}^{\beta_{c-2}}W^{\alpha_{c-1}}, \ {}^{\alpha_{c-1}}W^{v}, \ {}^{v}W^{\beta_{c+1}} \ldots$$

in H such that under (S, T), the intruder is on α_{c-1} at step $c-1$ and is uncaptured at or prior to step $c-1$. Noting the equality of the walks W_{c-1} and W_{c-1}^* of T and T^* respectively (each is ${}^{\alpha_{c-1}}W^{v}$), we show that the intruder is uncaptured at step c under (S, T), concluding the proof.

To show that a capture of Type 1 is not made at step c under (S, T), we observe that since the intruder is uncaptured at step c under (S^*, T^*), then $v \notin \bigcup_{j=1}^{k} N_H[v_{c,j}^*]$. Since $\bar{H}$ is basic, it follows that $v \notin \bigcup_{j=1}^{k} N_{\bar{H}}[v_{c,j}]$, implying no Type 1 capture under (S, T) at step c.

To show that a capture of Type 2 is not made at step c under (S, T), assume otherwise so that, with no loss of generality, some interior vertex u of ${}^{\alpha_{c-1}}W^{v}$ equals

$v_{c,1}$ or $v_{c-1,1}$, where $v_{c-1,1} \neq v_{c,1}$. Assume (with no loss of generality) that $u = v_{c,1}$. Then $v_{c,1}$ must be in $V(H)$ since $u \in V(H)$, which implies that $v_{c,1} = v^*_{c,1}$. Therefore, the interior vertex u of ${}^{\alpha_{c-1}}W^v$ equals a vertex of S^*_c, a contradiction of the assumption that the intruder is uncaptured at or prior to step c under (S^*, T^*).

To show that a capture of Type 3 is not made at step c under (S, T), assume otherwise so that, with no loss of generality, some interior vertex u of ${}^{\alpha_{c-1}}W^v$ is adjacent to $v_{c-1,1} = v_{c,1}$ in $\bar{H}$. If $v_{c,1}$ is in $B(H)$, then $u = v^*_{c,1}$, a contradiction of the assumption that the intruder is uncaptured at or prior to step c under (S^*, T^*). On the other hand, if $v_{c,1}$ is not in $B(H)$, then $v_{c,1} = v^*_{c,1} = v_{c-1,1} = v^*_{c-1,1}$. Thus, at steps $c - 1$ and c, watchman 1 is at the vertex $v_{c,1}$ in $V(H)$, which is adjacent to u in H since H is induced. Therefore we have the contradiction that the intruder is captured by step c under (S^*, T^*). ■

4 Watchman numbers for several classes of graphs

We begin this section with the derivation of relationships between $w(G)$ and certain graph invariants.

For graph G, let the minimum degree of G be denoted $\delta(G)$ and let the domination number of G be denoted $\gamma(G)$. It is clear that $w(G) \leq \gamma(G)$. Moreover, if G has diameter 2, then for any vertex v in G with degree $\delta(G)$, the neighbors of v form a dominating set, and hence $\gamma(G) \leq \delta(G)$. We state these results as

Observation 4.1. *For graph G, $w(G) \leq \gamma(G)$. If G has diameter 2, then $w(G) \leq \delta(G)$.*

We state a more general bound on $w(G)$ with the next observation.

Observation 4.2. *Let G be a connected graph and let $X \subseteq V(G)$. If X is a dominating set, set $y = 0$. Otherwise, let H be the graph induced by $V(G) - \bigcup_{x \in X} N[x]$, let $C^1, C^2, \ldots C^h$ be the components of H, and let y denote $max\{w(C^i) | 1 \leq i \leq h\}$. Then for every walk W in G, either W is incident with only vertices in $V(C^i)$ for some i or W is incident with some vertex in $N[x]$ where $x \in X$. Thus, letting k denote $|X| + y$, we may form a successful k-watchman pursuit S such that for every vertex x in X, there exists j where watchman j is located on x at every step. Hence $w(G) \leq k$.*

Now let G be a connected graph with finite girth $L \geq 4$ and let $C = c_0, c_1, c_2, \ldots, c_{L-1}$ be an L-cycle in G. By Theorem 2.11, there is no 1-watchman pursuit in G that is successful for G if there is no 1-watchman pursuit in G that is successful for G and initiates at c_0. So, let $S = (v_0), (v_1), (v_2), \ldots$ be an arbitrary 1-watchman pursuit in G where $v_0 = c_0$. We show that $w(G) \geq 2$ by showing that S is not successful for G.

For each $i \geq 0$, let c_{x_i} denote a fixed vertex from among those vertices of C that minimize the distance d_i between c_{x_i} and v_i. We construct an intruder evasion T in C (and therefore in G) such that (S, T) is not successful, thus:

Let the location of the intruder at step 0 be c_2. Then for all $i \geq 1$,

(1) if $d_i = 0$, let the location of the intruder at step i be $c_{(2+x_i)\ mod\ L}$. (Note that if movement on the part of the intruder is necessary to attain the prescribed position, there is an appropriate direction about the cycle in which the intruder may move without being observed by the watchman.)

(2) if $d_i = 1$, let the location of the intruder at step i be $c_{(1+x_i)\ mod\ L}$. (Note the same comment.)

(3) if d_i is at least 2, let the location of the intruder at step i be the same as at step $i-1$.

Since the intruder is captured at no step i, we have

Theorem 4.3. *For a connected graph G with finite girth at least 4, $w(G) \geq 2$.*

Now suppose that $\delta(G) \geq 2$ and S is a successful $\big(\delta(G)-1\big)$-watchman pursuit in G. With no loss of generality, we may assume that for fixed vertex v_0 with degree $\delta(G)$, every watchman has location v_0 at step 0. Let m denote $m_{G,G,S}$. If $m = 0$, then the component entries of S_0 form a dominating set, implying that G is isomorphic to $K_{\delta(G)+1}$, and hence has girth 3. If $m > 0$, then $D_{m-1}(G,G,S)$ contains some vertex v; hence there exists an intruder evasion $T = W_0, W_1, W_2, \ldots$ in G such that at step $m-1$, the intruder is located on v and is yet uncaptured under (S,T). Let the distinct neighbors of v be $y_1, y_2, y_3, \ldots, y_h$, where $h \geq \delta(G)$. Since $D_m(G,G,S) = \emptyset$, then for any i, $1 \leq i \leq h$, the intruder is captured at step m if W_{m-1} is the path of length 1 from v to y_i. Thus, at step m, each y_i is either the location of some watchman or is adjacent to the location of some watchman. Since the number of watchmen is less than h, some watchman (at step m) has a location that is either adjacent to at least 2 distinct neighbors of v, or is both a neighbor of v and adjacent to at least one neighbor of v. These two conditions respectively imply the existence of a 4-cycle in G and a 3-cycle in G, giving the next theorem.

Theorem 4.4. *For a connected graph G with finite girth at least 5, $w(G) \geq \delta(G)$.*

We now turn to the watchman numbers of graphs in the following classes: paths, cycles, complete r-partite graphs, interval graphs, r-paths, Moore graphs, the Heawood graph, and various Cartesian products. Trees will considered in another paper.

The watchman numbers of paths and cycles are clear.

Observation 4.5. *For path P_n and cycle C_m where $n \geq 1$ and $m = 3$, $w\big(P_n\big) = w\big(C_m\big) = 1$. For $m \geq 4$, $w\big(C_m\big) = 2$.*

We note that the difference between $w(G)$ and $\gamma(G)$ can be arbitrarily large, as illustrated by the case $G = P_n$. We also confirm that for graph G with subgraph H, each of the two relationships $w(H) \leq w(G)$ and $w(G) \leq w(H)$ is a possibility. Particularly, for $n \geq 4$, $1 = w(P_n) \leq w(C_n) = 2$, and $1 = w(K_1 + C_n) < w(C_n) = 2$, where $+$ is the join operator.

Theorem 4.6. *Let $G = K_{n_1,n_2,n_3,\ldots,n_r}$ be the complete r-partite graph on $\sum n_i$ vertices, where $n_1 \leq n_2 \cdots \leq n_r$. Then $w(G) = 2$.*

Proof: If $n_1 = 1$, then the domination number of G is 1, and hence $w(G) = 1$. If $n_1 \geq 2$, then the domination number of G is 2, and therefore $1 \leq w(G) \leq 2$. But for any 1-watchman pursuit S in G, it is easy to find an intruder evasion T such that for each i, $D_i(G, G, S)$ contains at least one vertex in the same part as the location of the watchman at step i. Thus S is not successful for G, giving $w(G) = 2$. ■

Theorem 4.7. *Let G be a connected interval graph with order n. Then $w(G) \leq 2$.*

Proof: The vertices of G can be identified $v_1, v_2, \ldots, v_n$ such that for $1 \leq i < j < k \leq n$, v_j is adjacent to v_k if v_i is adjacent to v_k. We will say that v_x *is to the left* (resp. *right*) *of* v_y if and only if $x < y$ (resp. $x > y$).

By the connectedness of G, for arbitrary y, $1 \leq y < n$, some vertex v_x in $\{v_1, v_2, \ldots, v_y\}$ is adjacent to some vertex v_z in $\{v_{y+1}, v_{y+2}, \ldots, v_n\}$. Thus, v_y is also adjacent to v_z, establishing that for any vertex v_y, $1 \leq y < n$, v_y is adjacent to some vertex to its right. Setting $k_0 = 1$, we may therefore let $k_0, k_1, k_2, \ldots, k_m$ be the strictly increasing sequence of integers such that $k_m = n$ and for $1 \leq i \leq m$, k_i is the largest integer such that $v_{k_{i-1}}$ is adjacent to v_{k_i}. We observe the following.

(1) for $1 \leq i \leq m$, each vertex that is to the left of v_{k_i} and equal or to the right of $v_{k_{i-1}}$ is adjacent to v_{k_i};

(2) for $2 \leq i \leq m-1$, any walk from some vertex to the right of v_{k_i} to some vertex to the left of $v_{k_{i-1}}$ is incident with some vertex v_z such that v_z is to the right of $v_{k_{i-1}}$ and either equal to or left of v_{k_i}. Thus, by (1), v_z either equals or is adjacent to v_{k_i}.

We now show $w(G) \leq 2$ by demonstrating a succesful 2-watchman pursuit.

Let S be the 2-watchman pursuit in which each watchman is located on v_1 at step 0 and ultimately located on $v_n = v_{k_m}$:

$$S = (v_{k_0}, v_{k_0}), (v_{k_1}, v_{k_0}), (v_{k_1}, v_{k_1}), (v_{k_2}, v_{k_1}), (v_{k_2}, v_{k_2}), (v_{k_3}, v_{k_2}), (v_{k_3}, v_{k_3}), \ldots,$$
$$\ldots, (v_{k_{m-1}}, v_{k_m}), (v_{k_m}, v_{k_m}), (v_{k_m}, v_{k_m}), (v_{k_m}, v_{k_m}), \ldots,$$

We show by induction that for each $i \geq 0$, $D_i(G, G, S)$ contains no vertices to the left of each watchman at step i. It will then follow that D_i is empty if $S_i = (v_{k_m}, v_{k_m}) = (v_n, v_n)$.

It is clear that D_i contains no vertices to the left of the watchmen for $i = 0$. Let $h \geq 1$ be an integer such that D_{h-1} contains no vertices to the left of each watchman at step $h-1$, and select arbitrary vertex β in D_h. Then there exists an intruder evasion T with $W_{h-1} = {}^{\alpha}W^{\beta}$ such that the intruder is uncaptured at or prior to step h under (S, T). We consider two cases.

Case 1. $S_{h-1} = (v_{k_j}, v_{k_{j-1}})$ for some j. Since $\alpha \in D_{h-1}$, the inductive assumption implies that α is not to the left of $v_{k_{j-1}}$. Nor is α in the union of the closed neighborhoods of $v_{k_{j-1}}$ and v_{k_j}, for otherwise a Type 1 capture would occur at step $h-1$. Therefore by (1) α is to the right of v_{k_j}. By (2), if β is to the left of $v_{k_{j-1}}$, then there is a vertex v_z along ${}^{\alpha}W^{\beta}$ that is adjacent or equal to v_{k_j}. Since watchman 1 is at v_{k_j}

at steps $h-1$ and h, a capture occurs at step h, contradicting the assumption that the intruder is uncaptured at or prior to step h. Thus β is not to the left of each watchman.

Case 2. $S_{h-1} = (v_{k_j}, v_{k_j})$ for some j. Since $\alpha \in D_{h-1}$, the inductive assumption implies that α is not to the left of v_{k_j}. Nor is α in the closed neighborhood of v_{k_j}, for otherwise a Type 1 capture would occur at step $h-1$. Therefore α is to the right of v_{k_j}. Suppose that β is to the left of v_{k_j}. If β is equal or to the right of $v_{k_{j-1}}$, then the capture of the intruder at step h is guaranteed by (1) and the presence of watchman 2 on v_{k_j} at step h. And if β is to the left of $v_{k_{j-1}}$, then the capture of the intruder at step h is guaranteed by (2). Thus β is not to the left of each watchman. ■

For an integer $r \geq 2$, the infinite r-path $P_\infty(r)$ is the interval graph on vertices $\ldots v_{-3}, v_{-2}, v_{-1}, v_0, v_1, v_2, v_3 \ldots$ such that v_s is adjacent to v_t if and only if $|s-t| \leq r-1$. The r-path on n vertices is the subgraph $P_n(r)$ of $P_\infty(r)$ induced by vertices $v_1, v_2, \ldots, v_n$.

If $r = 2$, then $w\big(P_n(r)\big) = 1$ since $P_n(r)$ is the path P_n. If $r = 3$, then the following is easily verified to be a successful 1-watchman pursuit in $P_n(r)$:

$$v_1, v_2, \ldots, v_{n-2}, v_{n-1}, v_n, v_n, v_n, \ldots.$$

Now suppose $r \geq 4$. If $n \leq 2r-1$, then the domination number of $P_n(r)$ is 1, implying $w\big(P_n(r)\big) = 1$. Thus, suppose that $2r \leq n < \infty$. Since $P_n(r)$ is an interval graph, then $w\big(P_n(r)\big) \leq 2$. Let S be a 1-watchman pursuit such that the location of the watchman at step 0 is v_1. It is easy to see that for any i, $D_i(P_n(r), P_n(r), S)$ contains v_1 if the watchman is located on v_z for some $z \geq r+1$, and $D_i(P_n(r), P_n(r), S)$ contains v_n if the watchman is located on v_z for some $z \leq r$. Thus $w\big(P_n(r)\big) = 2$. We summarize these results as follows.

Corollary 4.8. *Let $P_n(r)$ denote the r-path on $n < \infty$ vertices. Then $w\big(P_n(r)\big) \leq 2$. Moreover, $w\big(P_n(r)\big) = 1$ if and only if $r = 2$ or $r = 3$ or $r \geq 4$ with $n \leq 2r-1$.*

A Moore graph is a graph with diameter d and girth $2d+1$. It is known that all Moore graphs are regular. Additionally, in the case $d = 2$, each Moore graph has degree 2, 3, 7, or 57, the first three of which are uniqely represented by the 5-cycle, the Petersen graph, and the Hoffman-Singleton graph, respectively. (It is not known whether there exists a 57-regular Moore graph with diameter 2.) By Observation 4.1 and Theorem 4.4, the Petersen graph and the Hoffman-Singleton graph have watchman numbers 3 and 7, respectively. If a Moore graph of diameter 2 and degree 57 exists, then its watchman number is 57. This, along with Theorem 4.4, gives the following.

Theorem 4.9. *Among Moore graphs with diameter 2, the Petersen and Hoffman-Singleton graphs have respective watchmen numbers 3 and 7. Any Moore graph with diameter 2 and degree 57 has watchman number 57. Any Moore graph with degree $d > 2$ and finite girth at least 7 has watchman number at least d.*

The Heawood graph HW is a 3-regular bipartite graph with 7 vertices in each part and girth 6. Properties of HW include (1) any two distinct vertices in the same part have a union of open neighborhoods that contains exactly 5 vertices, and (2) for any three distinct vertices u, v, w in the same part of HW, vertex u has a neighbor that is not among the neighbors of v or w.

Theorem 4.10. *Let HW denote the Heawood graph. Then $w(HW) = 3$.*

Proof: By Theorem 4.4, $w(HW) \geq 3$. To show that $w(HW)$ is exactly 3, we construct a successful 3-watchman search.

Denote the two parts of HW by $\{1, 3, 5, 7, 9, 11, 13\}$ and $\{2, 4, 6, 8, 10, 12, 14\}$. Let

1 be adjacent to $2, 6, 14$;	3 be adjacent to 2, 4, 8;	5 be adjacent to $4, 6, 10$;
7 be adjacent to $6, 8, 12$;	9 be adjacent to $8, 10, 14$;	11 be adjacent to $2, 10, 12$;
	13 be adjacent to $4, 12, 14$.	

Let S be a 3-watchman search of HW such that $S_0 = (1, 3, 11)$, $S_1 = (1, 4, 11)$, $S_2 = (1, 3, 11)$ and $S_4 = (1, 8, 11)$. It is easy to verify that $D_4(HW, HW, S)$ is empty. ■

Let G_1 and G_2 be connected graphs where $V(G_1)$ and $V(G_2)$ are disjoint vertex sets $\{v_1, v_2, v_3, \ldots, v_{n_1}\}$ and $\{u_1, u_2, u_3, \ldots, u_{n_2}\}$, respectively. Let $G = G_1 \Box G_2$. For fixed j, let H_j be the subgraph of G induced by the set of vertices $\{(v_i, u_j) | 1 \leq i \leq n_1\}$. Then H_j is isomorphic to G_1 and has a basic closure. By Theorem 3.4, it follows that $w(G_1 \Box G_2) \geq w(G_i)$ for each $i = 1, 2$. Moreover, it readily follows from the commutivity and associativity of Cartesian product that if $G_1, G_2, \ldots, G_n$ are connected graphs and X is a non-empty subset of $\{1, 2, \ldots n\}$, then by the preceding inequality, we have

Theorem 4.11. $w\big(\Pi_{i \in X} G_i\big) \leq w\big(\Pi_{i=1}^{n} G_i\big)$.

We now turn to the products of particular graphs with emphasis on complete graphs and paths.

Theorem 4.12. *For positive integers $m \leq n$ and $r \leq s$, $w(K_m^r) \leq w(K_n^s)$.*

Proof: We denote vertices of K_p^t in the usual way: as t-tuples with coordinate entries in $\{1, 2, \ldots, p\}$. Accordingly, the t-tuple representation of any vertex of K_m^r shall also represent a vertex of K_n^r.

By Theorem 4.11, we have $w(K_n^r) \leq w(K_n^s)$. It thus suffices to show that $w(K_m^r) \leq w(K_n^r)$.

For each vertex v of K_n^r, we define v^* to be the r-tuple that results by replacing each coordinate entry of v that exceeds m with m. Then v^* represents a vertex of both K_n^r and K_m^r. We observe the following for r-tuple x that represents a vertex of K_m^r and r-tuple y that represents a vertex of K_n^r:

(1) if $x = y$ in K_n^r, then $x = y^*$ in K_m^r, and

(2) if x is adjacent to y in K_n^r, then x is adjacent or equal to y^* in K_m^r.

Now let S denote a k-watchman pursuit that is successful for K_n^r. Denoting S_i by the k-tuple $(v_{i,1}, v_{i,2}, \ldots, v_{i,k})$ of vertices of K_n^r, we define S_i^* to be the k-

tuple $(v^*_{i,1}, v^*_{i,2}, \ldots, v^*_{i,k})$ of vertices in K^r_m. It is clear that $S^* = S^*_0, S^*_1, S^*_2, \ldots$ is a k-watchman pursuit in K^r_m, Thus, it suffices to show that S^* is successful for K^r_m. To that end, we will show that for all $i \geq 0$, if $D_i(K^r_n, K^r_n, S) = \emptyset$, then $D_i(K^r_m, K^r_m, S^*) = \emptyset$.

Proceeding contrapositively, fix arbitrary i and assume $D_i(K^r_m, K^r_m, S^*) \neq \emptyset$. We may thus select vertex $z \in D_i(K^r_m, K^r_m, S^*)$, which implies that there exists an intruder evasion $T^* = W^*_0, W^*_1, W^*_2, \ldots, W^*_{i-1}, W^*_i, \ldots$ in K^r_m such that under (S^*, T^*), the intruder is on vertex z at step i and is uncaptured at or prior to step i. Noting that T^* is also an intruder evasion in K^r_n, we assume that under (S, T^*) in K^r_n, the intruder is captured at or prior to step i, from which a contradiction will be shown.

With no loss of generality, suppose that the intruder is captured by watchman 1 at step j, $0 \leq j \leq i$, under (S, T^*) in K^r_n. If the capture is of Type 1, then watchman 1 and the intruder are located on equal or adjacent vertices in K^r_n at step j under (S, T^*). But the location of the intruder at step j is a vertex of K^r_m. Hence, by (1) and (2), this implies that the watchman and the intruder are located on equal or adjacent vertices in K^r_m under (S^*, T^*) at step j, a contradiction of the assumption that the intruder is uncaptured through step i under (S^*, T^*). If the capture is of Type 2, then some interior vertex x of W_{j-1} is equal to $v_{j-1,1}$ or $v_{j,1}$ in K^r_n, where $v_{j-1,1} \neq v_{j,1}$. But again, x is a vertex of K^r_m, implying by (1) that x is equal to $v^*_{j-1,1}$ or $v^*_{j,1}$ in K^r_m, giving the same contradiction. If the capture is of Type 3, then some interior vertex x of W_{j-1} is adjacent to $v_{j-1,1} = v_{j,1}$ in K^r_n. But since x is a vertex of K^r_m, then by (2), x is adjacent or equal to $v^*_{j-1,1} = v^*_{j,1}$ in K^r_m, implying that the intruder is captured under (S^*, T^*) at or prior to step j.

We thus have that for all $i \geq 0$, $D_i(K^r_m, K^r_m, S^*) = \emptyset$ if $D_i(K^r_n, K^r_n, S) = \emptyset$. But since S is successful for K^r_n, $D_i(K^r_n, K^r_n, S) = \emptyset$ for $i = m_{K^r_n, K^r_n, S}$ by Theorem 2.6. Thus $D_i(K^r_m, K^r_m, S^*) = \emptyset$ for $i = m_{K^r_n, K^r_n, S}$, which implies that S^* is successful for K^r_m. Since S^* and S have equal lengths, it follows that $w(K^r_m) \leq w(K^r_n)$. ■

Now consider the graph $G = K_{a_1} \square K_{a_2}$, vertices of which we represent in the usual way by $v_{x,y}$, $1 \leq x \leq a_1$, $1 \leq y \leq a_2$. We show

Theorem 4.13. *For positive integers $a_1 \leq a_2 \cdots \leq a_n$ and $n \geq 2$, $w\big(K_{a_1} \square K_{a_2}\big) = a_1$ and $w\big(\Pi^n_{i=1} K_{a_i}\big) \geq a_{n-1}$.*

Proof: Consider $G = K_{a_1} \square K_{a_2}$. Since $\gamma(G) = a_1$, $w(G) \leq a_1$. To show that $w(G) = a_1$, let S be an arbitrary elementary and aggressive $(a_1 - 1)$-watchman pursuit in G. Since the vertices of S_0 are too few to dominate G, $D_0(G, G, S)$ is non-empty, containing those vertices not equal or adjacent to any coordinate entry of S_0. Now, proceeding by induction to show that no $D_i(G, G, S)$ is empty, let $i^* - 1$ be a positive integer such that $D_{i^*-1}(G, G, S)$ contains some vertex $v_{r,c}$. Considering two cases, first suppose that S_{i^*-1} and S_{i^*} differ in the j^{th} coordinate, where those coordinate entries are respectively v_{r_1,c_1} and v_{r_1,c_2}, $r_1 \neq r$ and $c_1 \neq c$. Since the number of watchmen is less than the number of columns, there must be a column $c_0 \neq c_2$ of the array that contains no vertex of S_{i^*}, implying that between step $i^* - 1$

and i^*, the intruder may flee (or stay put) without inducing capture at step i^*. Hence $D_{i^*}(G, G, S)$ is not empty. On the other hand, suppose that S_{i^*-1} and S_{i^*} differ in the j^{th} coordinate, where those coordinate entries are respectively v_{r_1,c_1} and v_{r_2,c_1}. Since the number of watchmen is less than the number of rows, we may apply a similar argument.

To show $w\big(\Pi_{i=1}^{n} K_{a_i}\big) \geq a_{n-1}$, let $G_1 = \Pi_{i=1}^{n} K_{a_i}$ and let $G_2 = K_{a_{n-1}} \Box K_{a_n}$. By Theorem 4.11 and the first part of Theorem 4.13, $w(G_1) \geq w(G_2) = a_{n-1}$. ■

In [14], it is proved that $\gamma(K_n^3) = \lceil \frac{n^2}{2} \rceil$. Therefore the following result holds.

Theorem 4.14. $w(K_n^3) \leq \lceil \frac{n^2}{2} \rceil$.

We note without proof that we have found $w(K_3^3) = 4$. More generally, through constructive methods, we have found that for a positive integer j,
$w(K_{2j+1}^3) \leq 2j^2 + j + 1$ and $w(K_{2j}^3) \leq 2j^2 - 1$. These too imply that $w(K_n^3) < \lceil \frac{n^2}{2} \rceil$.

Since the n-cube Q_n is the graph K_2^n, we have by Theorem 4.12 that if $m \leq n$, then $w(Q_m) \leq w(Q_n)$. Additionally, $w(Q_2) = 2$ (since Q_2 is the 4-cycle) and $w(Q_3) = 2$ since $w(Q_3) \geq 2$ and $\gamma(Q_3) = 2$. For the consideration of Q_4, we appeal to the characterization of Q_4 in which $V(Q_4)$ is the power set of the 4-set $\{a, b, c, d\}$ and vertex x is adjacent to vertex y if and only if their cardinalities differ by precisely 1 and one is a subset of the other. We see that $w(Q_4) \leq 3$, since we form a successful 3-watchman pursuit S in Q_4 as follows. Permanently station one watchman on each of $\emptyset$ and $\{a, b, c, d\}$. Then an uncaptured intruder must be located among the 2-subsets of $\{a, b, c, d\}$ at each step, between any two of which no edge exists. The third watchman may then visit each 2-subset, forcing the capture of the intruder. To see that $w(Q_4) = 3$, we argue that each 2-watchman pursuit in Q_4 is not successful. Assuming the contrary, suppose that S is an elementary, aggressive 2-watchman pursuit that is successful for Q_4. Since $\gamma(Q_4) > 2$, then the capture occurs at step $m = m_{Q_4,Q_4,S} > 0$. Hence $D_{m-1} \neq \emptyset$ and $D_m = \emptyset$. Owing to the symmetry of Q_4, we may assume that from step $m-1$ to step m, watchman 1 (with no loss of generality) moves from $\{a\}$ to $\emptyset$. If D_{m-1} contains a vertex y of order 2, 3 or 4, then y is in D_m due to the stationarity of watchman 2 from step $m-1$ to step m, contradicting that D_m is empty. Thus D_{m-1} contains either $\{b\}, \{c\}$, or $\{d\}$. If D_{m-1} contains $\{b\}$, then D_m contains either $\{a, b\}, \{b, c\}$ or $\{b, d\}$ since the location of watchman 2 cannot be adjacent or equal to all three. Analogous conclusions hold if D_{m-1} contains $\{c\}$ or $\{d\}$. Thus we again have our contradiction, concluding the argument that $w(Q_4) = 3$.

We note that we have found a successful 5-watchman pursuit in Q_5, a successful 8-watchman pursuit in Q_6, and a successful 13-watchman pursuit in Q_7, establishing upper bounds on their respective watchman numbers of 5, 8 and 13.

Theorem 4.15. *Let G be connected. Then*

(1) for $n \geq 2$, $w(G \Box P_n) \leq \gamma(G) + 1$;

(2) for $n \geq 3$, $w(G \Box C_n) \leq \gamma(G) + w(G \Box P_{n-1}) \leq 2\gamma(G) + 1$;

(3) for $n \geq 2$, $w(G\square K_n) \leq \gamma(G) + w(G\square K_{n-1}) \leq (n-1)\gamma(G) + 1$.

Proof: Let $\gamma(G)$ be denoted by γ. We give an informal description of a $(\gamma+1)$-watchman pursuit S that is successful for $G\square P_n$ and elementary to arbitrarily large step. Parts (2) and (3) are consequences of (1); details are omitted.

Let $V(G\square P_n) = \{v_{i,j} \mid 1 \leq i \leq |V(G)| \text{ and } 1 \leq j \leq n\}$. For fixed j, the set $\{v_{i,j} \mid 1 \leq i \leq |V(G)|\}$ induces a copy G_i of G, and for fixed i, the set $\{v_{i,j} \mid 1 \leq j \leq n\}$ induces a copy of P_n.

We first consider the case $\gamma \geq 2$. For fixed j, $1 \leq j \leq n$, let $X(j) = \{x_1(j), x_2(j), x_3(j), \ldots, x_\gamma(j)\}$ be a dominating set of G_j, where the graph induced by $x_i(1), x_i(2), \ldots, x_i(n)$ is isomorphic to P_n. We define S by describing the locations and movements of watchmen as follows:

At step 0, for $1 \leq i \leq \gamma$, watchman i is located on $x_i(1)$ and watchman $\gamma+1$ is located on $x_1(1)$. Clearly D_0 contains no vertices of G_1;

in the next step, watchman 1 moves from $x_1(1)$ to $x_1(2)$;

in the next one or more steps, watchman $\gamma+1$ moves (in G_1) from $x_1(1)$ to $x_2(1)$ (possible since G_1 is connected);

in the next step, watchman 2 moves from $x_2(1)$ to $x_2(2)$;

in the next one or more steps, watchman $\gamma+1$ moves (in G_1) from $x_2(1)$ to $x_3(1)$;

in the next step, watchman 3 moves from $x_3(1)$ to $x_3(2)$;

⋮

in the next step, watchman $\gamma-1$ moves from $x_{\gamma-1}(1)$ to $x_{\gamma-1}(2)$;

in the next one or more steps, watchman $\gamma+1$ moves (in G_1) from $x_{\gamma-1}(1)$ to $x_\gamma(1)$. At all steps k at or prior to the step k_1 at which watchman $\gamma+1$ reaches $x_\gamma(1)$, D_k contains no vertices of G_1;

in the next step, watchman γ moves from $x_\gamma(1)$ to $x_\gamma(2)$. At this step (which is step k_1+1), D_{k_1+1} contains no vertices of G_1 or G_2;

in the next steps, watchman $\gamma+1$ moves from $x_\gamma(1)$ to $x_1(2)$. At the step k_2 of his arrival at $x_1(2)$, for $1 \leq i \leq \gamma$, watchman i is located on $x_i(2)$ and watchman $\gamma+1$ is located on $x_1(2)$. No vertices of G_1 or G_2 are contained in D_{k_2}.

If $n = 2$, then $D_{k_2} = \emptyset$ and we are done. Otherwise, we repeat the pattern of movements by the watchmen until for $1 \leq i \leq \gamma$, watchman i is located on $x_i(n)$, at which step k_3, $D_{k_3} = \emptyset$.

In the event that $\gamma = 1$, let $X(j) = \{x_1(j)\}$ be a dominating set of G_j such that graph induced by $x_1(1), x_1(2), \ldots, x_1(n)$ is isomorphic to P_n. The following is readily checked to be a successful elementary and aggressive 2-watchman pursuit in $G\square P_n$: At step i for $0 \leq i \leq 2n-3$, let the first watchman be located at $x_1\left(\lceil \frac{i+1}{2} \rceil\right)$ and let the second watchman be located at $x_1\left(\lfloor \frac{i+3}{2} \rfloor\right)$. ■

We note that since Q_5 is isomorphic to $Q_4\square P_2$, then $w(Q_5) \leq 5$ by Theorem 4.15(1). But as well, Q_5 is isomorphic to $Q_3\square C_4$. Hence, by Theorem 4.15(2), we again have $w(Q_5) \leq 5$. But not always are the bounds the same. By Theorem 4.15(1) and 4.15(2), respectively, $w(C_{13}\Box P_{17}) \leq 6$ and $w(C_{13}\Box P_{17}) \leq 13$. On the other hand, by Theorem 4.15(1) and 4.15(2), respectively, $w(C_{13}\Box P_6) \leq 6$ and $w(C_{13}\Box P_6) \leq 5$. It can be easily verified that $w(C_m\Box P_n)$ is given the smaller bound by Theorem 4.15(1) if and only if $\lceil \frac{m}{3} \rceil < 2\lceil \frac{n}{3} \rceil$.

Theorem 4.16. *If* $2 \leq m, n$, *then* $w(K_m\Box P_n) = 2$.

Proof: By Theorem 4.15(1), $w(K_m\Box P_n) \leq 2$. Now, $K_m\Box P_n$ has induced connected subgraph $K_m\Box P_2$, which is $K_m\Box K_2$. But by Theorem 4.13, $w(K_m\Box K_2) \geq 2$. Moreover, $K_m\Box K_2$ has a basic closure in $K_m\Box P_n$, implying by Theorem 3.4 that $w(K_m\Box P_n) \geq w(K_m\Box K_2)$. Hence, $w(K_m\Box P_n) = 2$. ■

We turn to the products of paths. For positive $a_i \leq b_i$, $1 \leq i \leq n$, with at least one strict inequality, $\Pi_{i=1}^n P_{a_i}$ is an induced connected subgraph of $\Pi_{i=1}^n P_{b_i}$ with a basic closure. Thus by Theorem 3.4, $w(\Pi_{i=1}^n P_{a_i}) \leq w(\Pi_{i=1}^n P_{b_i})$.

Referring to Figure 2.1, we have demonstrated a successful 2-watchman pursuit in P_4^2. Since C_4 is a subgraph of P_4^2 with basic closure, then $w(P_4^2) \geq w(C_4) = 2$, giving $w(P_4^2) = 2$. An extension of the strategy illustrated in Table 2.2, yields the following.

Theorem 4.17. *For integers* $a_1 \leq a_2$, $w(P_{a_1}\Box P_{a_2}) \leq \lfloor \frac{a_1+4}{3} \rfloor$.

Therefore we have the following.

If $a_1 = 1$, then $w(P_{a_1}\Box P_{a_2}) = w(P_{a_2}) = 1$; and

if $a_1 = 2, 3$ or 4, then $w(P_{a_1}\Box P_{a_2}) = 2$.

We state without proof that $w(P_5^2) > 2$. Hence, $w(P_5^2) = 3$ by Theorem 4.17. Additionally, for $a_1 = 6$ or 7, $P_{a_1}\Box P_{a_2}$ has an induced connected subgraph isomorphic to P_5^2 with a basic closure. Thus, $w(P_{a_1}\Box P_{a_2}) = 3$ by Theorem 4.17.

5 Closing remarks

As noted in the introduction, the literature presents many pursuit-evasion models. Some of these models have the property of *monotonicity*; that is, if the search number of a graph is s, then there exists a pursuit executed by the pursuers that not only clears the graph of all evaders, but does so in a way that once a vertex is first cleared of an evader, that vertex can never again be occupied by an evader without capture. We observe that the model presented in this paper does not have the property of monotonicity. The graph that results by subdividing each edge of $K_{1,3}$ has watchman number 1, but every successful 1-watchman pursuit has the property that $D_0, D_1, D_2, \ldots$ is not monotonically decreasing.

A number of open questions are inspired by our exploration of this topic.

(1) Are there more general conditions than those of Theorem 3.4 that guarantee $w(H) \leq w(G)$ for H a subgraph of G?

(1) We have seen $w(G) = 1$ or 2 for interval graph G. Is there a simple set of necessary and sufficient conditions for $w(G) = 1$?

(3) We have seen that $w(Q_n) \leq f_n$ for $n = 3, 4, 5, 6, 7$, where f_n is the n^{th} Fibonacci number. More generally, is $w(Q_n) \leq f_n$ for $n \geq 3$? For which n does $w(Q_n)$ equal f_n?

(4) We have bounded $w(K_{2j}^3)$ with $2j^2 - 1$ for $j \geq 3$ and $w(K_{2j+1}^3)$ with $2j^2 + j + 1$ for $j \geq 2$. Can this bound be improved? What is $w(K_n^d)$ for $d \geq 3$?

(5) In Theorem 4.17, we bound $w(P_{a_1} \Box P_{a_2})$ from above, with equality for $a_1 \leq 7$. Does equality hold for for $a_1 \geq 8$? What is $w(P_n^d)$ for $d \geq 2$?

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