AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **64(2)** (2016), Pages 327–333

Dimension 4 and dimension 5 graphs with minimum edge set

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Abstract

The dimension of a graph G is defined to be the minimum n such that G has a representation as a unit-distance graph in $\mathbb{R}^n$. A problem posed by Paul Erdős asks for the minimum number of edges in a graph of dimension 4. In a recent article, R. F. House showed that the answer to Erdős' question is 9. In this article, we give a shorter (and we feel more straightforward) proof of House's result, and then extend our methods to answer the question for dimension 5 as well. It is ultimately shown that a dimension 5 graph has at least 15 edges, and that this lower bound is realized only by two graphs: K_6 and $K_{1,3,3}$.

1 Introduction

The concept of graph dimension originated in a 1965 article [1] by three giants of twentieth century mathematics, Erdős, Harary, and Tutte. For a finite simple graph G, define G to be of dimension n and write $\dim(G) = n$ if n is the smallest integer such that G can be represented with vertices as points of $\mathbb{R}^n$ with vertices being adjacent only if they are a Euclidean distance 1 apart. Note that in the definition, we are not forced to include an edge if two vertices are distance 1 apart, so G is not necessarily induced. In the parlance associated with the subject, if $\dim(G) = n$ for a given graph G, then for all $m \geq n$, it is said that G is embeddable in $\mathbb{R}^m$ or that G has a unit-distance representation in $\mathbb{R}^m$. Alexander Soifer devotes a chapter of *The Mathematical Coloring Book* [3] to the discussion of graph dimension and related topics, and in his discourse relates the following problem posed by Erdős in private communication.

"What is the smallest number of edges in a graph G such that $\dim(G) = 4$?"

In a 2013 article [2], R. F. House showed that the answer to the above question is 9 and furthermore, that the complete bipartite graph $K_{3,3}$ is the unique graph that achieves this bound. In our current work, we provide an alternate proof of House's result, in particular, one that has the added benefit of sidestepping much of the case analysis required in his original paper. We then extend our methods to answer Erdős' question in dimension 5, ultimately showing that the minimum number of edges in a graph G with $\dim(G) = 5$ is 15 and that this minimum is realized only in the cases of K_6 and the complete tripartite graph $K_{1,3,3}$.

2 Preliminary Results

In the previously mentioned [1], Erdős, Harary, and Tutte determine the dimension of a number of common graphs. Several of their results will be used repeatedly in the arguments that follow, and for easy reference we give them as lemmas below.

Lemma 1. $\dim(K_n) = n - 1$.

Lemma 2. $\dim(K_n - e) = n - 2$ *where* e *is any edge of* K_n.

Lemma 3. $\dim(K_{n,m}) = 4$ *for* $m, n \geq 3$.

Lemma 4. *Let* H *be any subgraph of* G. *Then* $\dim(H) \leq \dim(G)$.

Lemma 4 can be rephrased in terms of graph complements. For a graph G, define $\overline{G}$ to be the complement of G where $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv : uv \notin E(G)\}$. It is easily seen that for graphs G and H with $|V(G)| = |V(H)|$, if $\overline{G}$ is a subgraph of $\overline{H}$, then H is a subgraph of G. This gives the following corollary.

Corollary 5. *Let* $\overline{G}$ *be a subgraph of* $\overline{H}$ *where* $|V(G)| = |V(H)|$. *Then* $\dim(H) \leq \dim(G)$.

3 Erdős' Question in Dimension 4

The following two theorems were first proven by House in [2]. We reprove his results below in a new and more concise manner.

Theorem 6. *The minimum number of edges of a graph* G *with* $\dim(G) = 4$ *is nine.*

Proof. Let G be a graph which cannot be embedded in $\mathbb{R}^3$ where $|E(G)|$ is minimum. Assume $|E(G)| \leq 8$. Since $|E(G)|$ is minimum, obviously G cannot contain a vertex of degree 1.

We claim that G also cannot contain a vertex of degree 2. To see this, let $u \in V(G)$ with $\deg u = 2$ and let uv_1 and uv_2 be distinct edges of G. Let G' be defined by $V(G') = V(G) \setminus \{u\}$ and $E(G') = (E(G) \setminus \{uv_1, uv_2\}) \cup \{v_1v_2\}$. Since $|E(G')| < |E(G)|$, G' can be embedded in $\mathbb{R}^3$. Since v_1 and v_2 are distance 1 apart in this embedding, the spheres of radius 1 centered at v_1 and v_2 intersect in a circle. In particular, there are an infinite number of points distance 1 from both v_1 and v_2. We may choose one of these to be u. This gives us a supergraph of G—that is, G along with the edge v_1v_2—which is embeddable in $\mathbb{R}^3$ (note here that we are following the nomenclature convention of a graph being a supergraph of itself as v_1v_2 may already be an edge of G). So by Lemma 4, G has dimension at most 3, a contradiction which completes the proof of the claim.

We may now begin with G and insert edges between vertices of G (if $|E(G)| < 8$) to create a graph H where $\dim(H) > 3$, $|E(H)| = 8$, and $\deg v \geq 3$ for each $v \in V(H)$. As $|E(H)| = 8$, the sum of the degree measures of all vertices of H is equal to 16. It is then easy to see that the only possible degree sequence of H is $(4, 3, 3, 3, 3)$. This, however, implies that H is a subgraph of $K_5 - e$ and by Lemmas 2 and 4, we have that $\dim(H) \leq 3$. This contradiction, along with the facts that $\dim(K_{3,3}) = 4$ and $|E(K_{3,3})| = 9$, completes the proof of the theorem. □

Theorem 7. *The only dimension 4 graph with nine edges is $K_{3,3}$.*

Proof. Let G be a graph with $\dim(G) = 4$ and $|E(G)| = 9$. By the arguments in the preceding theorem, it must be the case that for any $v \in V(G)$, $\deg v \geq 3$. K_4 has only six edges so G must have at least 5 vertices. If G has 5 vertices, then each vertex must have degree at least 3 as argued above and at most 4 since G has 5 vertices. It follows that G must have degree sequence $(4, 4, 4, 3, 3)$. If G is on 6 vertices, the fact that each vertex must have degree at least 3 immediately implies that the only degree sequence is $(3, 3, 3, 3, 3, 3)$. The sequence $(4, 4, 4, 3, 3)$ corresponds solely to the graph $K_5 - e$. By Lemma 2, $\dim(K_5 - e) = 3$, so this degree sequence cannot represent G. Note now that the complement of a graph with degree sequence $(3, 3, 3, 3, 3, 3)$ is a 2-regular graph on six vertices; in other words, a cycle C_6 or a graph whose two components are each copies of K_3. If $\overline{G} = C_6$, then G has an embedding in $\mathbb{R}^2$ with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$, $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, and $(\frac{1}{2}, 1 + \frac{\sqrt{3}}{2})$. If $\overline{G}$ consists of two copies of K_3, then $G = K_{3,3}$. □

4 Erdős' Question in Dimension 5

In this section, we show that there are exactly two dimension 5 graphs with minimum edge set: K_6, whose dimension is given in Lemma 1, and $K_{1,3,3}$, which is addressed in the following theorem.

Theorem 8. $\dim(K_{1,3,3}) = 5$.

Proof. As $K_{1,3,3}$ is a subgraph of $K_7 - e$, we have that $\dim(K_{1,3,3}) \leq 5$. We will now suppose there exists an embedding of $K_{1,3,3}$ in $\mathbb{R}^4$ and establish a contradiction. Label $K_{1,3,3}$ as having partite sets A, B, and C where $|A| = |B| = 3$ and $|C| = 1$. We may freely assume that one of the vertices of A is the origin $(0, 0, 0, 0)$. Call the other two vertices of that part u and v. Rotate all of $\mathbb{R}^4$ about the origin (thus preserving distance) so that u and v both land in the xy-plane. We may now write $u = (u_1, u_2, 0, 0)$ and $v = (v_1, v_2, 0, 0)$. As any of the points of B or C are equidistant from $(0, 0, 0, 0)$, u, and v, it is impossible for $(0, 0, 0, 0)$, u, and v to be collinear. So let (h, k) be the circumcenter of the triangle with vertices $(0, 0)$, (u_1, u_2), and (v_1, v_2) and note that (h, k) is the unique point equidistant from $(0, 0)$, (u_1, u_2), and (v_1, v_2).

Now consider an arbitrary point (x, y, z, w) that is distance 1 from $(0, 0, 0, 0)$, u, and v. We have that $1 = x^2 + y^2 + z^2 + w^2 = (x - u_1)^2 + (y - u_2)^2 + z^2 + w^2 = (x-v_1)^2+(y-v_2)^2+z^2+w^2$. Thus $x^2+y^2 = (u_1-x)^2+(u_2-y)^2 = (v_1-x)^2+(v_2-y)^2$ which implies that (x, y) is equidistant to $(0, 0)$, (u_1, u_2), and (v_1, v_2). It follows that $x = h$, $y = k$. Furthermore, since $x^2 + y^2 + z^2 + w^2 = 1$, $x = h$, and $y = k$, it follows that $z^2 + w^2 = 1 - h^2 - k^2$. This means that the three vertices of B and the single vertex of C (call it c) must lie on a circle of radius $\sqrt{1 - h^2 - k^2}$ in the plane given by $x = h$, $y = k$. However, if a point is equidistant from three points on a circle and also in the plane containing the circle, then that point must lie at the center of the circle. As c is on the circle containing each vertex of B, c cannot be equidistant (and thus adjacent) to each of the vertices of B. This contradiction shows that $K_{1,3,3}$ cannot be embedded in $\mathbb{R}^4$ and completes the proof of the theorem. ☐

The graphs $K_7 - \{e_1, e_2, e_3\}$ where e_1, e_2, e_3 are any three independent edges and $K_8 - \{e_1, e_2, e_3, e_4\}$ where e_1, e_2, e_3, e_4 are any four independent edges will play important roles in the arguments that follow. The former graph happens to be $K_{1,2,2,2}$ and the latter graph $K_{2,2,2,2}$ and although this will be the notation used when referencing them, we feel that in regards to Corollary 5 it is more beneficial to think of the graphs as K_7 minus three independent edges and K_8 minus four independent edges respectively. We determine their dimension in the following lemma.

Lemma 9. $\dim(K_{1,2,2,2}) = \dim(K_{2,2,2,2}) = 4$.

Proof. Letting $a = \frac{\sqrt{2}}{2}$, note that $K_{1,2,2,2}$ has an embedding in $\mathbb{R}^4$ with vertices $(a, 0, 0, 0), (-a, 0, 0, 0), (0, a, 0, 0), (0, -a, 0, 0), (0, 0, a, 0), (0, 0, -a, 0)$, and $(0, 0, 0, a)$. $K_{2,2,2,2}$ has an embedding in $\mathbb{R}^4$ with those same seven vertices along with the additional vertex $(0, 0, 0, -a)$. It is easily seen that $K_{3,4}$ is a subgraph of $K_{1,2,2,2}$ and that $K_{4,4}$ is a subgraph of $K_{2,2,2,2}$ which in light of Lemmas 3 and 4 implies that $K_{1,2,2,2}$ and $K_{2,2,2,2}$ each have dimension greater than or equal to 4. ☐

Theorem 10. *The minimum number of edges of a graph G with* $\dim(G) = 5$ *is fifteen.*

Proof. Let $\mathcal{G}$ be the set of all graphs which cannot be embedded in $\mathbb{R}^4$ which have a minimum number of edges. Let $G \in \mathcal{G}$ where $|V(G)|$ is minimum. For a contradiction, assume $|E(G)| \leq 14$. By the arguments in Theorem 6, G cannot contain a vertex of degree 1 or degree 2.

We claim that G cannot contain a vertex of degree 3. To see this, let $u \in V(G)$ with $\deg u = 3$. Let uv_1, uv_2, and uv_3 be distinct edges of G. Let G' be defined by $V(G') = V(G) \setminus \{u\}$ and $E(G') = (E(G) \setminus \{uv_1, uv_2, uv_3\}) \cup \{v_1v_2, v_1v_3, v_2v_3\}$. As either $|E(G') < |E(G)|$ or $|E(G')| = |E(G)|$ and $|V(G')| < |V(G)|$, we have that

G' is embeddable in $\mathbb{R}^4$. The spheres in $\mathbb{R}^4$ of radius 1 centered at v_1, v_2, and v_3 intersect in infinitely many points. Pick one not in $V(G')$ to be u. This gives an embedding in $\mathbb{R}^4$ of a supergraph of G (again, by convention we are allowing the fact that a graph is a supergraph of itself). Hence by Lemma 4, G is embeddable in $\mathbb{R}^4$, a contradiction which proves this claim.

We may now begin with G and insert edges between vertices (if $|E(G)| < 14$) to create a graph H where $\dim(H) > 4$, $|E(H)| = 14$, and $\deg v \geq 4$ for each $v \in V(H)$. As $|E(H)| = 14$, the sum of the degree measures of all vertices of H is equal to 28. Arguing in the same manner as Theorem 7, we see that there are two possible degree sequences for H. They are $(5,5,5,5,4,4)$ and $(4,4,4,4,4,4,4)$. The degree sequence $(5,5,5,5,4,4)$ corresponds to the graph $K_6 - e$, and as $\dim(K_6 - e) = 4$ by Lemma 2, it cannot represent H. So assume H has degree sequence $(4,4,4,4,4,4,4)$. The complement of a graph with degree sequence $(4,4,4,4,4,4,4)$ is a 2-regular graph on seven vertices. It follows then that $\overline{H}$ is equal to C_7 or equal to a graph whose two components are K_3 and C_4. In either case $\overline{H}$ contains three independent edges and we have that $\overline{K_{1,2,2,2}}$ is a subgraph of $\overline{H}$. Thus by Corollary 5 and Lemma 9, $\dim(H) \leq 4$, a contradiction that completes the proof of the theorem. □

Theorem 11. *The only dimension* 5 *graphs with fifteen edges are* K_6 *and* $K_{1,3,3}$.

Proof. Let $\mathcal{G}$ be the set of all dimension 5 graphs with fifteen edges. By the arguments presented in Theorem 6 and Theorem 10, no graph of $\mathcal{G}$ can contain a vertex of degree 1 or 2. Let $\mathcal{G}_0 \subset \mathcal{G}$ where each graph of $\mathcal{G}_0$ has at least one vertex of degree 3. We claim that $\mathcal{G}_0$ is empty. To see this, suppose $\mathcal{G}_0$ is non-empty and among all graphs of $\mathcal{G}_0$, let G be one with a minimum number of vertices. Let $u \in V(G)$ with $\deg u = 3$ and let uv_1, uv_2, uv_3 be distinct edges of G. We now observe two important properties of G. First, it must be the case that $\{v_1, v_2, v_3\}$ is an independent set. Otherwise, the fact that at least one of v_1v_2, v_1v_3, v_2v_3 is an edge of G would imply that the graph G' formed by $V(G') = V(G) \setminus \{u\}$ and $E(G') = (E(G) \setminus \{uv_1, uv_2, uv_3\}) \bigcup \{v_1v_2, v_1v_3, v_2v_3\}$ would have fewer edges than G and thus be embeddable in $\mathbb{R}^4$. We would then be able to obtain a contradiction by using the the technique presented in the proof of Theorem 10 to show that G is also embeddable in $\mathbb{R}^4$. Secondly, any pair of vertices of degree 3 must be adjacent. To see this, let $v \in V(G)$ with $\deg v = 3$ and consider the graph G' defined above. If uv is not an edge of G, then G' must be embeddable in $\mathbb{R}^4$ as it would have the same number of edges as G with fewer vertices and a vertex of degree 3, namely v. Again we would be able to establish a contradiction by using the technique in the proof of Theorem 10 to show that G must also be embeddable in $\mathbb{R}^4$. Combining these two observations, we conclude that G can have at most two vertices of degree 3. We now consider the possible size of $V(G)$, in each case showing that G cannot exist of that order.

Case 1.1 $|V(G)| \leq 6$

Since G has a vertex of degree 3, G is a subgraph of $K_6 - e$. By Lemmas 2 and 4, G has dimension at most 4.

Case 1.2 $|V(G)| = 7$

The possible degree sequences for a graph of order seven with one or two vertices of degree 3 are $(6, 6, 4, 4, 4, 3, 3)$, $(6, 5, 5, 4, 4, 3, 3)$, $(6, 5, 4, 4, 4, 4, 3)$, $(5, 5, 5, 5, 4, 3, 3)$ and $(5, 5, 5, 4, 4, 4, 3)$. As any vertex of degree 6 must be adjacent to every other vertex of the graph, it will be adjacent to a vertex of degree 3 and the other two neighbors of that vertex. This cannot happen as it violates the previously given observation that the vertices adjacent to a vertex of degree 3 must form an independent set. For the degree sequence $(5, 5, 5, 5, 4, 3, 3)$, note that each vertex of degree 5 must be adjacent to a vertex of degree 3. Again with the previous observation in mind, it follows that a vertex of degree 5 cannot be adjacent to the two neighbors of a vertex of degree 3. This is a contradiction since any vertex of degree 5 must be adjacent to all but one other vertex. For the same reason, when we look at the degree sequence $(5, 5, 5, 4, 4, 4, 3)$, we see that the vertex of degree 3 cannot be adjacent to any vertex of degree 5. This uniquely identifies the graph with the vertex of degree 3 being adjacent to each vertex of degree 4 and the degree 5 vertices being adjacent to every vertex except the degree 3 vertex. But this graph can be embedded in $\mathbb{R}^4$ as follows with u_1, u_2, and u_3 having degree 5, v_1, v_2, v_3 having degree 4 and w having degree 3: $u_1 = (0, 0, 0, 0), u_2 = (1, 0, 0, 0), u_3 = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, 0), v_1 = (\frac{1}{2}, \frac{1}{2\sqrt{3}}, 0, \sqrt{\frac{2}{3}}), v_2 = (\frac{1}{2}, \frac{1}{2\sqrt{3}}, 0, -\sqrt{\frac{2}{3}}), v_3 = (\frac{1}{2}, \frac{1}{2\sqrt{3}}, \sqrt{\frac{2}{3}}, 0), w = (\frac{1}{2} + \sqrt{\frac{1}{3}}, \frac{1}{2\sqrt{3}}, 0, 0)$.

Case 1.3 $|V(G)| = 8$

The only possible degree sequence (with one or two vertices of degree 3) is $(4, 4, 4, 4, 4, 4, 3, 3)$. Regarding Corollary 5 and Lemma 9, to show a graph with this degree sequence is embeddable in $\mathbb{R}^4$ it is sufficient to show that the complement of such a graph contains four independent edges. To see that this is indeed the case, let u_1 and u_2 be the vertices of degree 3 and label the other vertices $v_1, \ldots, v_6$. Recall that u_1 and u_2 are adjacent. Since the open neighborhoods of u_1 and u_2 must each consist of a set of independent vertices, it cannot be the case that u_1 and u_2 are adjacent to the same vertex. So say u_1 is adjacent to v_1 and v_2, and say that u_2 is adjacent to v_3 and v_4. It must then be that v_1 and v_2 are not adjacent and that v_3 and v_4 are not adjacent. Additionally, u_1 and v_5 are not adjacent and u_2 and v_6 are not adjacent, giving four independent edges in the complement of the graph.

Finally, we note that any graph with nine vertices and at most two vertices of degree 3 (and none of degree less than 3) will have more than 15 edges, completing the proof of the claim that $\mathcal{G}_0$ is empty.

Now let G be a graph with $\dim(G) = 5$ and $|E(G)| = 15$. Since the minimum degree is 4, the only possible degree sequences of G are $(5, 5, 5, 5, 5, 5)$, $(6, 4, 4, 4, 4, 4, 4)$, or $(5, 5, 4, 4, 4, 4, 4)$. We analyze these cases below.

Case 2.1 G has degree sequence $(5, 5, 5, 5, 5, 5)$.

It follows that $G = K_6$.

Case 2.2 G has degree sequence $(6, 4, 4, 4, 4, 4, 4)$.

Let $u \in V(G)$ with $\deg u = 6$ and note that u is adjacent to each other vertex of G. $G - \{u\}$ is a 3-regular graph on six vertices and as such, its complement is either C_6 or a graph whose two components are each copies of K_3. If $\overline{G - \{u\}} = C_6$, we note that C_6 has three independent edges which implies that $\overline{K_{1,2,2,2}}$ is a subgraph of $\overline{G}$ and thus by Corollary 5 and Lemma 9, $\dim(G) \leq 4$. If $\overline{G - \{u\}} = K_3 \bigcup K_3$, then $G = K_{1,3,3}$.

Case 2.3 G has degree sequence $(5, 5, 4, 4, 4, 4, 4)$.

Let $u, v \in V(G)$ with $\deg u = \deg v = 5$. If $uv \notin E(G)$, then u, v are each adjacent to the other five vertices of G. $G - \{u, v\}$ is then a 2-regular graph on five vertices. In other words, $G - \{u, v\}$ equals the cycle C_5. It follows that $\overline{G}$ contains three independent edges – uv and two independent edges from $\overline{G - \{u, v\}}$. Thus $\overline{K_{1,2,2,2}}$ is a subgraph of $\overline{G}$ and by Corollary 5 and Lemma 9, $\dim(G) \leq 4$.

Now assume $uv \in E(G)$. Label the other five vertices of G as w_1, w_2, w_3, w_4, w_5. If the open neighborhoods of u and v have four vertices in common – say w_1, w_2, w_3, w_4 – then w_5 is adjacent to w_1, w_2, w_3, w_4 as well. It follows that $\overline{G - \{u, v, w_5\}}$ is a 1-regular graph on 4 vertices. We then have that $\overline{G}$ contains three independent edges – uw_5 and two independent edges from $\overline{G - \{u, v, w_5\}}$. Again $\overline{K_{1,2,2,2}}$ is a subgraph of $\overline{G}$ and by Corollary 5 and Lemma 9, $\dim(G) \leq 4$.

As well as uv being an edge of G, assume also that u is adjacent to w_1, w_2, w_3, w_4 and v is adjacent to w_2, w_3, w_4, w_5. Then uw_5 and vw_1 are both independent edges of $\overline{G}$. We claim there must be an additional independent edge in $\overline{G - \{u, v, w_1, w_5\}}$. To see this, note that if w_2, w_3, w_4 constituted the vertices of a copy of K_3, we would then have that w_1 cannot be adjacent to any of w_2, w_3, w_4 and thus $\deg w_1 < 4$. So again $\overline{G}$ must contain three independent edges and by Corollary 5 and Lemma 9, $\dim(G) \leq 4$. □

Acknowledgments

This research began at the GSCC 2014, an NSF sponsored conference held at Auburn University.

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(Received 14 Jan 2015; revised 19 July 2015, 27 Oct 2015)