

Minimizing the regularity of maximal regular antichains of 2-sets and 3-sets

THOMAS KALINOWSKI

*School of Mathematical and Physical Sciences
University of Newcastle
Callaghan, NSW 2308
Australia*

UWE LECK

*Universität Flensburg
Flensburg
Germany*

CHRISTIAN REIHER

*Universität Hamburg
Hamburg
Germany*

IAN T. ROBERTS

*School of Engineering and Information Technology
Charles Darwin University
Darwin 0909
Australia*

Abstract

Let $n \geq 3$ be a natural number. We study the problem of finding the smallest r such that there is a family \mathcal{A} of 2-subsets and 3-subsets of $[n] = \{1, 2, \dots, n\}$ with the following properties: (1) \mathcal{A} is an antichain, i.e., no member of \mathcal{A} is a subset of any other member of \mathcal{A} , (2) \mathcal{A} is maximal, i.e., for every $X \in 2^{[n]} \setminus \mathcal{A}$ there is an $A \in \mathcal{A}$ with $X \subseteq A$ or $A \subseteq X$, and (3) \mathcal{A} is r -regular, i.e., every point $x \in [n]$ is contained in exactly r members of \mathcal{A} . We prove lower bounds on r , and we describe constructions for regular maximal antichains with small regularity.

1 Introduction

Let $n \geq 3$ be an integer and $[n] := \{1, 2, \dots, n\}$. By $2^{[n]}$ we denote the family of all subsets of $[n]$ and by $\binom{[n]}{k}$ the family of all k -subsets of $[n]$. A family $\mathcal{A} \subseteq 2^{[n]}$ is an *antichain* if $A \not\subseteq B$ for all distinct $A, B \in \mathcal{A}$. An antichain \mathcal{A} is called *flat* if it is contained in two consecutive levels of $2^{[n]}$, i.e., if $\mathcal{A} \subseteq \binom{[n]}{k} \cup \binom{[n]}{k+1}$ for some k . The remarkable *Flat Antichain Theorem*, which follows from results of Lieby [14] (see also [15]) and Kisvölcsy [13], says that for every antichain \mathcal{A} there is an equivalent flat antichain \mathcal{A}' , where the equivalence relation is defined by: \mathcal{A}' is equivalent to \mathcal{A} if and only if $|\mathcal{A}'| = |\mathcal{A}|$ and $\sum_{A \in \mathcal{A}'} |A| = \sum_{A \in \mathcal{A}} |A|$. Griggs et al. [10] showed that the flat antichains minimize the *BLYM-values* $\sum_{A \in \mathcal{A}} \binom{n}{|A|}^{-1}$ within their equivalence classes, and more generally, they minimize (maximize) $\sum_{A \in \mathcal{A}} w(|A|)$ for every convex (concave) function w . Grüttmüller et al. [11] initiated the study of maximal flat antichains. In particular, they asked for the minimum size of a maximal antichain consisting only of k -sets and $(k+1)$ -sets, and settled the first nontrivial case $k=2$. Gerbner et al. [8] derived asymptotic bounds for general k and studied some related problems for r -Sperner families, i.e., families of subsets of $[n]$ which do not contain $r+1$ sets that form a chain with respect to inclusion. A different generalization was proposed in [12] where the flatness condition is replaced by the requirement that the cardinalities of the members of the set family \mathcal{A} have to be in a fixed small set K . Böhm [2, 3, 4] proposed the study of regular antichains, i.e., antichains \mathcal{A} such that every element of the ground set appears in the same number of sets in \mathcal{A} . In particular, he studied the problem for which values of r there are maximal flat antichains such that every element of the ground set occurs exactly r times. The same problem without the maximality and flatness conditions has been studied by Roberts and Böhm [17].

In this paper we focus on maximal flat antichains consisting of 2-sets and 3-sets: we consider *maximal (2,3)-antichains* \mathcal{A} on $[n] := \{1, 2, \dots, n\}$, i.e., $\mathcal{A} = \mathcal{A}_2 \cup \mathcal{A}_3$ with $\mathcal{A}_2 \subseteq \binom{[n]}{2}$ and $\mathcal{A}_3 \subseteq \binom{[n]}{3}$. The antichain \mathcal{A} is called r -regular if every element of $[n]$ is contained in exactly r members of \mathcal{A} . Clearly, the maximum possible value of r is $\binom{n-1}{2}$, obtained by choosing $\mathcal{A}_3 = \binom{[n]}{3}$, but the smaller value $r = n-1$ can be achieved by choosing $\mathcal{A}_2 = \binom{[n]}{2}$. We are interested in antichains of small regularity, so henceforth we will assume that $r < n-1$.

Following [11, 12] we associate a graph G with \mathcal{A} . The vertex set is $[n]$ and the edge set is $\binom{[n]}{2} \setminus \mathcal{A}_2$. The elements of \mathcal{A}_3 are the triangles in G , and every edge of G is contained in at least one triangle. For a vertex x let $d(x)$ denote its degree and $t(x)$ the number of triangles containing x , i.e., the number of edges in the neighbourhood of x . Similarly, for an edge xy , $t(xy)$ is the number of triangles containing xy , i.e., the number of common neighbours of x and y . The antichain \mathcal{A} is r -regular if and only if in the corresponding graph, for all vertices x ,

$$d(x) - t(x) = \bar{r}$$

where \bar{r} is defined as $\bar{r} = n-1-r > 0$. In the present paper we study the question for which values of r an r -regular maximal (2,3)-antichain exists.

In Section 2 we prove two lower bounds for r . The first one $r \geq (5n - 9)/6$ is valid for all n (Corollary 2) and the second one $r \geq (9/10 + o(1))n$ holds asymptotically for $n \rightarrow \infty$ (Corollary 5). Section 3 contains two constructions for regular maximal $(2, 3)$ -antichains of small regularity, and in the final Section 4 we propose two conjectures strengthening our results.

2 Lower bounds

In this section we start with a lower bound for the regularity of a maximal $(2, 3)$ -antichain which is valid for all values of n . The cases in which this lower bound is achieved are completely described, and we prove an asymptotic lower bound for large n .

Lemma 1. *Let G be a graph with the property that every edge is contained in at least one triangle. Then for every edge xy ,*

$$t(x) \geq \frac{d(x) + t(xy) - 1}{2}.$$

Proof. Fix an edge xy . There are $d(x) - t(xy) - 1$ neighbours of x that are not contained in a triangle with x and y . To cover these there must be at least $(d(x) - t(xy) - 1)/2$ triangles containing x , but not y . So the total number of triangles containing x is at least

$$t(xy) + \frac{d(x) - t(xy) - 1}{2} = \frac{d(x) + t(xy) - 1}{2}. \quad \square$$

Theorem 1. *If G corresponds to an r -regular maximal $(2, 3)$ -antichain on n points, then $\bar{r} \leq (n + 3)/6$.*

Proof. Fix a triangle xyz and let a_i denote the number of vertices outside xyz that have exactly i neighbours in xyz ($i = 0, 1, 2, 3$). Then

$$\begin{aligned} a_0 + a_1 + a_2 + a_3 &= n - 3, \\ a_1 + 2a_2 + 3a_3 &= d(x) + d(y) + d(z) - 6, \\ a_2 + 3a_3 &= t(xy) + t(yz) + t(xz) - 3. \end{aligned}$$

Subtracting the third from the second equation and using the first equation to bound $a_1 + a_2$, we obtain

$$[d(x) - t(xy)] + [d(y) - t(yz)] + [d(z) - t(xz)] = a_1 + a_2 + 3 \leq n.$$

At least one of the three terms in square brackets on the left hand side, say $d(x) - t(xy)$, is at most $n/3$. We use Lemma 1 to estimate \bar{r} ,

$$\bar{r} = d(x) - t(x) \leq d(x) - \frac{d(x) + t(xy) - 1}{2} = \frac{d(x) - t(xy) + 1}{2} \leq \frac{n + 3}{6}. \quad \square$$

Corollary 2. *If there exists an r -regular maximal $(2, 3)$ -antichain on n points, then $r \geq (5n - 9)/6$.*

Next we characterize the cases when equality in Corollary 2 can occur.

Corollary 3. *There exists a $(5n - 9)/6$ -regular maximal $(2, 3)$ -antichain if and only if $n \in \{3, 9, 15, 27\}$. Furthermore, the extremal antichains for these values are unique up to isomorphism.*

Proof. We use the observation from [11] that the graphs on n vertices having $n(n + 3)/6$ edges and every edge contained in exactly one triangle are the triangle for $n = 3$ and graphs corresponding to generalized quadrangles $GQ(2, (n - 3)/6)$ which exist precisely for $n \in \{9, 15, 27\}$ (see [16]). So we just have to prove that in order to achieve $d(x) - t(x) = (n + 3)/6$ we must have $n(n + 3)/6$ edges and every edge has to be contained in exactly one triangle. In order to have equality in the proof of Theorem 1 we need $d(x) - t(xy) = n/3$ for every edge xy . In particular, n is divisible by 3 and $t(xy) = d(x) - n/3$ is the same for all neighbours y of x . If $d(x) \geq n/3 + 2$ then

$$t(x) = \frac{1}{2} \sum_{y:xy \in E} t(xy) \geq d(x).$$

Hence we can conclude that $d(x) = n/3 + 1$ for all vertices x and $t(xy) = 1$ for all edges xy . □

Now we prove an asymptotic upper bound on \bar{r} . Let E and T be the sets of edges and triangles in G , respectively. Intuitively, it seems reasonable to look for a graph with $d(x) = 2\bar{r}$ and $t(x) = \bar{r}$ for every vertex x . The deviation from these values is measured by $\delta(x) = d(x) - 2\bar{r} = t(x) - \bar{r}$. Summing over all vertices gives

$$2|E| = \sum_{x \in V} d(x) = 2\bar{r}n + \sum_{x \in V} \delta(x), \quad 3|T| = \sum_{x \in V} t(x) = \bar{r}n + \sum_{x \in V} \delta(x).$$

For the calculations it is convenient to define

$$\gamma = \frac{\bar{r}}{n} \quad \beta = \frac{1}{2n^2} \sum_{x \in V} \delta(x).$$

With this notation,

$$\begin{aligned} d(x) &= 2\gamma n + \delta(x), & t(x) &= \gamma n + \delta(x), \\ |E| &= (\gamma + \beta)n^2, & |T| &= \frac{\gamma + 2\beta}{3}n^2. \end{aligned}$$

Lemma 2. *If G corresponds to an r -regular maximal $(2, 3)$ -antichain then $\gamma \leq \beta + o(1)$.*

Proof. We may assume that $|T| = o(n^3)$, since otherwise \bar{r} is negative for large n . The triangle removal lemma [7, 18], implies that $|T| \geq |E|/2 + o(n^2)$, hence $(\gamma + 2\beta)/3 \geq (\gamma + \beta)/2 + o(1)$, and rearranging terms gives the claimed inequality. □

Lemma 3. $\sum_{xy \in E} t(xy)^2 \leq \frac{1}{2} \sum_{x \in V} \delta(x)^2 + o(n^3)$.

Proof. Fix a vertex x and let $N(x)$ denote its neighbourhood. Then $|N(x)| = d(x) = 2\gamma n + \delta(x)$ and

$$\sum_{y \in N(x)} t(xy) = 2t(x) = 2(\gamma n + \delta(x)).$$

Note that $X = (X_y)_{y \in N(x)}$ is an optimal solution for the optimization problem

$$\begin{aligned} &\text{maximize } f(X) = \sum_{y \in N(x)} X_y^2 \\ &\text{subject to } \sum_{y \in N(x)} X_y = 2(\gamma n + \delta(x)), \text{ and} \\ &\qquad X_y \geq 1 \text{ for all } y \in N(x), \end{aligned}$$

if and only if $d(x) - 1$ of the variables are equal to 1 and the remaining variable is equal to

$$2(\gamma n + \delta(x)) - (d(x) - 1) = 2(\gamma n + \delta(x)) - (2\gamma n + \delta(x) - 1) = \delta(x) + 1.$$

This is true because if $X_y > 1$ and $X_{y'} > 1$ for two distinct $y, y' \in N(x)$ then we obtain a better solution \tilde{X} by setting $X_y = 1$, $\tilde{X}_{y'} = X_{y'} + (X_y - 1)$ and $\tilde{X}_{y''} = X_{y''}$ for all $y'' \in N(x) \setminus \{y, y'\}$, because this yields

$$f(\tilde{X}) = f(X) + 2(X_y - 1)(X_{y'} - 1) > f(X).$$

Consequently,

$$\sum_{y \in N(x)} t(xy)^2 \leq (\delta(x) + 1)^2 + (2\gamma n + \delta(x) - 1) = \delta(x)^2 + o(n^2),$$

and the claim follows by summation over all $x \in V$. □

Theorem 4. *If G corresponds to an r -regular maximal $(2, 3)$ -antichain, then $\bar{r} \leq (1/10 + o(1))n$.*

Proof. From the proof of Theorem 1 we take the inequality

$$d(x) + d(y) + d(z) - t(xy) - t(yz) - t(xz) \leq n$$

which is valid for every triangle $xyz \in T$. Summation over all triangles gives

$$\sum_{x \in V} t(x)d(x) - \sum_{xy \in E} t(xy)^2 \leq n|T| = \frac{\gamma + 2\beta}{3}n^3.$$

Substituting

$$t(x)d(x) = (\gamma n + \delta(x))(2\gamma n + \delta(x)) = 2\gamma^2 n^2 + 3\gamma n\delta(x) + \delta(x)^2$$

we obtain

$$\sum_{x \in V} \delta(x)^2 - \sum_{xy \in E} t(xy)^2 \leq \left[\frac{\gamma + 2\beta}{3} - 2\gamma^2 - 6\beta\gamma \right] n^3,$$

and using Lemma 3 and the Cauchy-Schwarz inequality,

$$\frac{4\beta^2 n^4}{2n} \leq \frac{1}{2} \sum_{x \in V} \delta(x)^2 \leq \left[\frac{\gamma + 2\beta}{3} - 2\gamma^2 - 6\beta\gamma + o(1) \right] n^3,$$

hence

$$\frac{\gamma + 2\beta}{3} - 2\gamma^2 - 6\beta\gamma - 2\beta^2 \geq o(1). \tag{1}$$

For fixed γ we consider the left hand side of this inequality as a function $f_\gamma(\beta)$. Then $f'_\gamma(\beta) = -4\beta + (2/3 - 6\gamma)$, and f has its maximum at $\beta^* = (1 - 9\gamma)/6$. For $\gamma \geq 1/10$, we have $\beta^* < \gamma$, and consequently, for every $\beta \geq \gamma$,

$$\frac{\gamma + 2\beta}{3} - 2\gamma^2 - 6\beta\gamma - 2\beta^2 = f_\gamma(\beta) \leq f_\gamma(\gamma) = \gamma - 10\gamma^2.$$

If $\gamma \geq 1/10 + \varepsilon$ for some positive ε , then this implies $f_\gamma(\beta) < -\varepsilon$ for all $\beta \geq \gamma$. Therefore, (1) together with Lemma 2 implies $\gamma \leq 1/10 + o(1)$, and this concludes the proof. \square

Corollary 5. *If there exists an r -regular maximal $(2, 3)$ -antichain on n points, then $r \geq (9/10 + o(1))n$.*

3 Constructions

Here we present two different constructions for regular maximal $(2, 3)$ -antichains with small regularity. Section 3.1 contains an algebraic construction for $n \equiv 0 \pmod{3}$. The regularity of these antichains is still $(1 - o(1))n$, while in Section 3.2 we describe how certain two-coloured digraphs can be used to achieve regularity cn with c strictly less than 1.

3.1 An algebraic construction

Let $n = 3n'$, and let Γ be an abelian group of order $|\Gamma| = n'$. Suppose that there are two sequences

$$A = (a_1, a_2, \dots, a_p) \subseteq \Gamma \quad \text{and} \quad B = (b_1, \dots, b_p) \subseteq \Gamma$$

satisfying the following conditions.

- The entries in both sequences are pairwise distinct, i.e., $a_i \neq a_j$ and $b_i \neq b_j$ for $i \neq j$.
- The sequence $C = (b_1 - a_1, b_2 - a_2, \dots, b_p - a_p)$ has pairwise distinct elements, i.e., $b_i - a_i \neq b_j - a_j$ for $i \neq j$.

- $b_j - a_i \notin C$ for all $i \neq j$.

Let G be the graph with vertex set $V = \Gamma \times [3]$ and edge set

$$E = \{ \{(g, 1), (h, 2)\} : h - g \in A \} \cup \{ \{(g, 2), (h, 3)\} : h - g \in C \} \\ \cup \{ \{(g, 1), (h, 3)\} : h - g \in B \}.$$

The triangles in this graph are the sets of the form

$$\{(g, 1), (g + a_s, 2), (g + b_s, 3)\}$$

for $g \in \Gamma$ and $s \in [p]$. Every vertex has degree $d(x) = 2p$ and is contained in $t(x) = p$ triangles, so the corresponding maximal $(2, 3)$ -antichain is $(n - 1 - p)$ -regular.

Let $p_{\max}(n')$ denote the maximal p such that a group Γ and sequences A and B satisfying the above conditions exist. Then our construction implies the following existence result.

Theorem 6. *If $n = 3n'$, there exists a maximal, r -regular $(2, 3)$ -antichain for every $r \in \{n - 1 - p_{\max}(n'), \dots, n - 1\}$.*

For $r \in \{n - 1 - p_{\max}(n'), \dots, n - 1\}$ the constructed antichain has $\binom{n}{2} - n(n - 1 - r)$ pairs and $n(n - 1 - r)/3$ triples. So the size is $\binom{n}{2} - \frac{2}{3}n(n - 1 - r)$ and this is the largest possible size for a r -regular maximal $(2, 3)$ -antichain which follows from the next lemma.

Lemma 4. *Let G be a graph on n vertices in which every edge is contained in at least one triangle and $d(x) - t(x) = n - 1 - r$ for all vertices x . Then $|E| - |T| \geq \frac{2}{3}n(n - 1 - r)$.*

Proof. We have

$$2|E| - 3|T| = \sum_{x \in V} d(x) - t(x) = n(n - 1 - r)$$

and, using $t(xy) \geq 1$ for every edge xy ,

$$\frac{3}{2}|T| = \frac{1}{2} \sum_{xy \in E} t(xy) \geq \frac{|E|}{2}$$

Adding these two inequalities we get $\frac{3}{2}(|E| - |T|) \geq n(n - 1 - r)$ which is the claim. □

As observed in [18], sets without 3-term arithmetic progressions can be used to construct graphs with every edge contained in exactly one triangle. This is a special case of the above construction with $A = C$ and $B = 2A$. Let A be a subset of Γ without nontrivial 3-term arithmetic progressions, i.e., the only solutions to $x + y = 2z$ in A are of the form $x = y = z = a$ for some $a \in A$. Then $B = 2A = \{a + a : a \in A\}$ and $C = A$ satisfy our conditions, and the triangles in the graph are precisely the sets $\{(g, 1), (g + a, 2), (g + 2a, 3)\}$ for $g \in \Gamma$. So if $r_3(\Gamma)$ denotes the maximal size of a subset of Γ without 3-APs we get $p_{\max}(n') \geq r_3(\Gamma)$.

Example 1. For an integer N , let $r_3(N)$ denote the maximal size of a subset of $[N]$ without 3-term APs (in \mathbb{Z}). The best known lower bound for $r_3(N)$, based on a modification of a very old construction [1], can be found in [6] and [9]:

$$r_3(N) \geq \frac{cN \log^{1/4}(N)}{2^2 \sqrt{2 \log_2 N}}$$

for some positive constant c . Clearly, $r_3(\mathbb{Z}_{n'}) \geq r_3(\lfloor n'/2 \rfloor)$, and so we get that for integers $n \equiv 0 \pmod{3}$ there exists an r -regular maximal $(2, 3)$ -antichain \mathcal{A} on n points of size $|\mathcal{A}| = \binom{n}{2} - \frac{2}{3}n(n-1-r)$ whenever $n > r \geq n-1-r_3(\lfloor n/6 \rfloor)$. In particular, there is a positive constant c such that

$$r \geq n \left(1 - \frac{c \log^{1/4}(n)}{2^2 \sqrt{2 \log_2 n}} \right)$$

is sufficient for the existence of an r -regular maximal $(2, 3)$ -antichain on n points.

Example 2. Let A be the set of all integers t , $0 \leq t < n'/2$ such that the ternary representation of t contains only the digits 0 and 1, and interpret A as a subset of $\mathbb{Z}_{n'}$. Then A does not contain 3-term APs and this yields a regular maximal $(2, 3)$ -antichain with regularity

$$r \approx n - 1 - \lfloor n/6 \rfloor^{\log_3 2} \leq n - cn^{0.63}.$$

Example 3. Let Γ be the additive group of the vector space \mathbb{F}_3^m , i.e., $n = 3^{m+1}$. The 3-AP free sets in this group are called *cap sets* and there are constructions of size greater than 2.217^m for arbitrary large m (see [5]). This gives r -regular maximal antichains for

$$r \approx n - 1 - n^{\frac{m \log 2.217}{(m+1) \log 3}} < n - 1 - n^{\frac{0.724m}{m+1}}.$$

3.2 A blowup construction

Note that all of our constructions did not get a regularity below $(1 - o(1))n$. We now describe a way to achieve regularity less than cn for some positive c . We start with a special class of digraphs with a 2-colouring of the arcs. Let $D = (N, A_0 \cup A_1)$ be a 2-coloured digraph with node set N and arc set $A = A_0 \cup A_1$ where the sets A_0 and A_1 are the colour classes. All our digraphs contain no loops and no parallel arcs, and we assume that every node has at least one outgoing edge. For such a digraph D and any positive integer t we can define a graph $G = G(D, t)$ on $2t|N|$ vertices as follows. To every node $v \in N$ correspond $2t$ vertices labeled by (v, i, j) for $i \in \{1, 2, \dots, t\}$ and $j \in \{0, 1\}$, and these $2t$ vertices induce a matching containing the edges $\{(v, i, 0), (v, i, 1)\}$ for $i \in \{1, 2, \dots, t\}$. Let E_2 denote the set of the $t|N|$ matching edges. Furthermore, let

$$E_0 = \{ \{(u, i, j), (v, i', 0)\} : (u, v) \in A_0, i, i' \in \{1, \dots, t\}, j \in \{0, 1\} \}, \text{ and}$$

$$E_1 = \{ \{(u, i, j), (v, i', 1)\} : (u, v) \in A_1, i, i' \in \{1, \dots, t\}, j \in \{0, 1\} \}.$$

The edge set of the graph G is then $E_0 \cup E_1 \cup E_2$. By construction, every edge of G is contained in at least one triangle (for the edges in E_2 this is ensured by the assumption that every node in D has out-degree at least 1). Also every triangle contains at most one edge from the matching E_2 . The next lemma provides a condition on the (undirected) triangles in the coloured digraph D which ensures that every edge in $E_0 \cup E_1$ is contained in precisely one triangle.

Lemma 5. *If all triangles in $D = (N, A_0 \cup A_1)$ have the form $\{u, v, w\}$ with $(u, v), (v, w) \in A_0$ and $(u, w) \in A_1$ or $(u, v), (v, w) \in A_1$ and $(u, w) \in A_0$, then in $G = G(D, t)$ every edge $\{x, y\} \in E_0 \cup E_1$ is contained in a unique triangle.*

Proof. An edge $e = \{(u, i, j), (v, i', j')\}$ with $u \neq v$ is contained in precisely one triangle corresponding to the arc $(u, v) \in A_{j'}$ or to the arc $(v, u) \in A_j$. Any other triangle containing the edge e comes from a triangle $\{u, v, w\}$ in the digraph D . But if all triangles in D have the form required in the hypothesis of the lemma then the structure of the corresponding subgraphs in G is shown in Figure 1 and it does not contain any unwanted triangles. \square

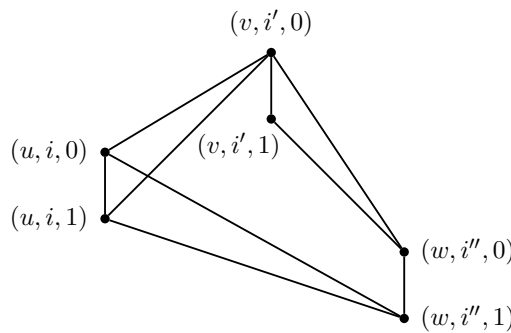


Figure 1: Subgraph from a triangle $\{u, v, w\}$ with $(u, v), (v, w) \in A_0$ and $(u, w) \in A_1$.

Under the conditions of Lemma 5 it is easy to determine $d(x)$ and $t(x)$ for the vertices of the graph G . Let $d_i^-(v)$ and $d_i^+(v)$ denote the number of arcs in A_i entering and leaving node v , respectively. Then we have

$$d(x) = \begin{cases} 1 + t(d_0^+(u) + d_1^+(u) + 2d_0^-(u)) & \text{if } x = (u, i, 0) \text{ for some } i, \\ 1 + t(d_0^+(u) + d_1^+(u) + 2d_1^-(u)) & \text{if } x = (u, i, 1) \text{ for some } i, \end{cases}$$

$$t(x) = \begin{cases} t(d_0^+(u) + d_1^+(u) + d_0^-(u)) & \text{if } x = (u, i, 0) \text{ for some } i, \\ t(d_0^+(u) + d_1^+(u) + d_1^-(u)) & \text{if } x = (u, i, 1) \text{ for some } i. \end{cases}$$

In order to make $d(x) - t(x)$ constant we have to require that the in-degrees in both colours are the same constant over the node set N , i.e., $d_0^-(u) = d_1^-(u) = d$ for some d and all $u \in N$. In the graph $G = G(D, t)$ with $n = 2t|N|$ vertices we have $d(x) - t(x) = 1 + td = 1 + dn/2|N|$, and the regularity of the corresponding maximal $(2, 3)$ -antichain is

$$r = \left(1 - \frac{d}{2|N|}\right) n - 2.$$

Figure 2 shows two digraphs on 7 and 8 nodes, respectively, satisfying the condition in Lemma 5, and having constant in-degree 1 in both colours. For the corresponding

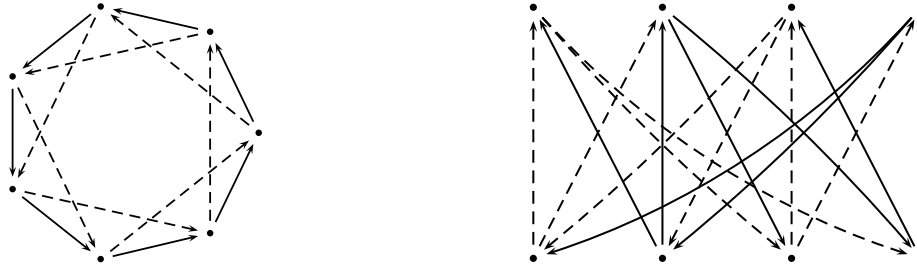


Figure 2: Two arc-coloured digraphs (colours indicated by solid and dashed lines).

graphs $G = G(D, t)$ we obtain $d(x) = 1 + 4t$ and $t(x) = 3t$ for all vertices x , and this implies the following theorem.

Theorem 7. *For every $n \equiv 0 \pmod{14}$ there is a $(13n/14 - 2)$ -regular maximal $(2, 3)$ -antichain on n points, and for every $n \equiv 0 \pmod{16}$ there is a $(15n/16 - 2)$ -regular maximal $(2, 3)$ -antichain.*

4 Open problems

We conjecture that the first part of Theorem 7 gives asymptotically the smallest possible regularity.

Conjecture 1. *The smallest possible regularity of a maximal $(2, 3)$ -antichain on n points is $(13/14 + o(1))n$.*

In the proof of the $9/10$ -bound in Corollary 5 we assumed the worst case that for every triangle xyz every vertex outside this triangle is adjacent to either one or two of the triangle vertices. It is not difficult to see that the case where an outside vertex is adjacent to all three vertices of a triangle can be neglected for the asymptotics, because the number of 4-cliques is $o(n^3)$. But the factor $9/10$ can be improved using any cubic lower bound on the number of copies of the graph H on four vertices consisting of a triangle and an isolated vertex. Here are some observations that might be used to force copies of H .

- The condition that $d(x) - t(x)$ is large, say cn for some $c \geq 1/14$, implies that the neighbourhood of every vertex has many connected components: at least cn (and exactly cn if the neighbourhood induces a forest).
- So every neighbourhood contains an induced matching of size cn .
- The union of n induced matchings is small, only $o(n^2)$ edges [18].

As a first step towards Conjecture 1 one might try to prove that $13n/14$ is best possible for the type of construction described in Section 3.2.

Conjecture 2. *Let $D = (N, A_0 \cup A_1)$ be an arc-coloured digraph with the following properties.*

1. *All triangles have the form $\{u, v, w\}$ with $(u, v), (v, w) \in A_0$ and $(u, w) \in A_1$ or $(u, v), (v, w) \in A_1$ and $(u, w) \in A_0$.*
2. *Every node has exactly d incoming arcs in each of the sets A_0 and A_1 .*

Then $d \leq |N|/7$.

References

- [1] F. Behrend, On sequences of integers containing no arithmetic progression, *Čas. Mat. Fys.* **67**(4) (1938), 235–239.
- [2] M. Böhm, *Regular antichains*, Dissertation, Universität Rostock, 2011.
- [3] M. Böhm, k -regular antichains on $[m]$ with $k \leq m - 2$, *Australas. J. Combin.* **53** (2012), 41–52.
- [4] M. Böhm, Maximal flat regular antichains, *J. Combin. Math. Combin. Comput.* **88** (2014), 237–261.
- [5] Y. Edel, Extensions of generalized product caps, *Des. Codes Cryptogr.* **31**(1) (2004), 5–14.
- [6] M. Elkin, An improved construction of progression-free sets, *Isr. J. Math.* **184** (2011), 93–128.
- [7] J. Fox, A new proof for the graph removal lemma, *Ann. Math.* **174** (2011), 561–579.
- [8] D. Gerbner, B. Keszegh, N. Lemons, C. Palmer, D. Pálvölgyi and B. Patkós. Saturating sperner families. *Graphs Combin.* **29**(5) (2013), 1355–1364.
- [9] B. Green and J. Wolf, A note on Elkins improvement of Behrend’s construction, In *Additive Number Theory. Festschrift in honor of the sixtieth birthday of Melvyn B. Nathanson* (2010), 141–144.
- [10] J.R. Griggs, S. Hartmann, U. Leck and I.T. Roberts, Squashed full flat antichains of minimum weight, (to appear).
- [11] M. Grützmüller, S. Hartmann, T. Kalinowski, U. Leck and I. Roberts, Maximum flat antichains of minimum weight, *Electr. J. Comb.* **16** (2009), R69.
- [12] T. Kalinowski, U. Leck and I.T. Roberts, Maximal antichains of minimum size, *Electr. J. Comb.* **20**(2) (2013), #P3.
- [13] Á. Kisvölcsy, Flattening antichains, *Combinatorica* **26**(1) (2006), 65–82.

- [14] P. Lieby, *Extremal problems in finite sets*, PhD thesis, Northern Territory University, Darwin (1999).
- [15] P. Lieby, Antichains on three levels, *Electr. J. Comb.* **11** (2004), R50.
- [16] S.E. Payne and J.A. Thas, *Finite generalized quadrangles*, vol. 9 of *EMS Series of Lectures in Mathematics* (2009).
- [17] I.T. Roberts and M. Böhm, On the existence of regular antichains, *Discrete Math.* **313**(4) (2013), 348–356.
- [18] I.Z. Ruzsa and E. Szemerédi, Triple systems with no six points carrying three triangles, *Combinatorics, Keszthely 1976*, Colloq. Math. Soc. János Bolyai **18** (1978), 939–945.

(Received 4 July 2014; revised 20 Oct 2015)