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The 3-color Ramsey number for a 3-uniform loose path of length 3

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Abstract

The values of hypergraph 2-color Ramsey numbers for loose cycles and paths have already been determined. The only known value for more than 2 colors is $R(C^3_3;3) = 8$, where C^3_3 is a 3-uniform loose cycle of length 3. Here we determine that $R(P^3_3;3) = 9$, where P^3_3 is a 3-uniform loose path of length 3. Our proof relies on the determination of the Turán number $\mathrm{ex}_3(9;P^3_3)$. We also find the Turán number $\mathrm{ex}_3(12;P^3_3)$ and use it to estimate $R(P^3_3;4)$.

1 Introduction

In this note we consider the problem of finding the 3-color Ramsey number for the 3-uniform loose path of length 3 and estimate the corresponding Ramsey number for 4 colors. A *hypergraph* H is a pair $H = (V, E)$, where V is a finite nonempty set of vertices and E is a collection of distinct nonempty subsets of V. A vertex v is of *degree* i when it belongs to i edges in a hypergraph H. We consider only k*-uniform hypergraphs* in which all edges have size k, and call them k*-graphs*, for short.

The *clique* K^k_n is a k-graph on n vertices and with $\binom{n}{k}$ edges. For a given k-graph H, the *Ramsey number* $R(H;r)$ is the least integer n such that in every r-coloring of the edges of K^k_n there is a monochromatic copy of H. If H itself is a clique, we are dealing with classical Ramsey numbers, which are so hard to calculate that the only known value for $k \geqslant 3$ is $R(K^3_4;2) = 13$ ([7]). Instead of cliques, sometimes sparser structures like *cycles* and *paths* have been studied.

There are several natural definitions of a cycle and a path in a uniform hypergraph. Here we focus only on loose cycles and loose paths. A k-uniform *loose cycle* C^k_n of length n is a k-graph whose edges form a cyclic list $(f_1, \ldots, f_n)$ such that consecutive edges intersect in exactly one element and nonconsecutive ones are disjoint.

By removing one edge from a loose cycle of length $n+1$, we obtain a k-uniform *loose path* P_n^k of length n. Note that $|V(C_n^k)| = n(k-1)$ and $|V(P_n^k)| = n(k-1)+1$.

Further, a *k-star* with n arms is a k-graph with edges $f_1, \dots, f_n$, $n \geqslant 2$, such that $\bigcap_{i=1}^n f_i \neq \emptyset$. A star S is called *full* if $|E(S)| = \binom{|V(S)|-1}{k-1}$, that is, a vertex v forms edges with all $(k-1)$-element subsets of $V(S)\backslash\{v\}$. For $k = 2$ we get the usual graph definitions of the cycle C_n, the path P_n with n edges, and the star $K_{1,n}$. Given a k-graph H and a k-element set e, we denote by $H + e$ the k-graph $(V(H) \cup e, E(H) \cup \{e\})$.

There are many results in graph Ramsey theory related to cycles and paths (see [9]). For hypergraphs though, much less is known. First, it was proved in [5] that $R(P_n^3; 2)$ and $R(C_n^3; 2)$ are asymptotically equal to $\frac{5n}{2}$. Subsequently, Omidi and Shahsiah in [8] proved that

$$R(P_n^3; 2) = R(C_n^3; 2) + 1 = \left\lfloor \frac{5n+1}{2} \right\rfloor.$$

Gyárfás and Raeisi [4] found the values for $R(P_n^k; 2)$ and $R(C_n^k; 2)$ for $n \leqslant 4$ and $k \geqslant 3$. They also determined the 3-color Ramsey number for C_3^3,

$$R(C_3^3; 3) = 8.$$

In this note we prove two theorems about multicolored Ramsey numbers for P_3^3.

Theorem 1.1. $R(P_3^3; 3) = 9$

Theorem 1.2. $10 \leqslant R(P_3^3; 4) \leqslant 12$

Turán numbers may sometimes provide upper bounds on Ramsey numbers (see, e.g. Prop. 13 in [4] and Proposition 3.2 below). Indeed, the proofs of Theorems 1.1 and 1.2 are based on the corresponding Turán numbers. In Section 2, we will first determine the Turán numbers $\mathrm{ex}_3(9; P_3^3)$ and $\mathrm{ex}_3(12; P_3^3)$, and then, in Section 3, deduce Theorems 1.1 and 1.2.

2 Turán numbers

Given a k-graph H and a positive integer n, the *k-graph Turán number* $ex_k(n; H)$ is the maximum number of edges in a k-graph F on n vertices that does not contain H as a subhypergraph.

The numbers $ex_k(n; P_l^k)$, for all fixed k and l, where $k \geqslant 4$ or $l \geqslant 4$, and sufficiently large n, are determined in [3] and [6]. There are, however, no corresponding results for $k = l = 3$. The method of the proof used in [3] does not quite work for the case $k = 3$. In turn, Kostochka, Mubayi and Verstraëte skipped this case, assuming that it was determined in [3].

In order to determine $ex_3(9; P_3^3)$ and $ex_3(12; P_3^3)$, we will use the following result for 3-cycles of length 3, proved by Csákány and Kahn (see also [2]).

Theorem 2.1. *[1] For $n \geqslant 6$, $ex_3(n; C_3^3) = \binom{n-1}{2}$. Moreover, for $n \geqslant 8$, the only extremal 3-graph is the full star.*

We begin with a determination of $ex_3(9; P_3^3)$.

Lemma 2.2. *We have $ex_3(9; P_3^3) = 28$. Moreover, the only extremal 3-graph is the full star.*

Before proving Lemma 2.2, we will show some useful facts. In these facts, e always stands for a 3-element subset of a vertex set V. Let us consider a copy C of C_3^3 with $V(C) \subset V$. We partition $V(C) = V_1 \cup V_2$ where, for $i = 1, 2$, V_i stands for the set of vertices of degree i in C, that is the vertices which belong to exactly i edges of C.

We define two families of triples:

$E_1 = \left\{e \in \binom{V}{3} : |e \cap V_1| = |e \cap V_2| = 1, \text{ and } \forall f \in E(C) : e \cap f \neq \emptyset\right\}$,

$E_2 = \left\{e \in \binom{V}{3} : V_1 = \emptyset, |e \cap V_2| = 2\right\}$,

and $E' = E_1 \cup E_2$.

The edges in E_1 are formed by taking a vertex of degree 1 in C, then another one of degree 2 in C but which does not belong to the same edge as the first one, and the third vertex belongs to the set $V \backslash V(C)$. Similarly the edges in E_2 are formed by taking two vertices of degree 2 in C and one vertex from the set $V \backslash V(C)$ (see Figures 1 and 2).

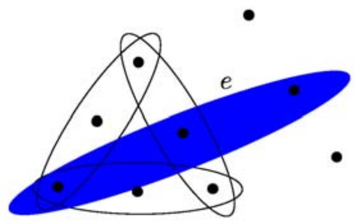

Figure 1: An edge from the family E_1 is shaded.

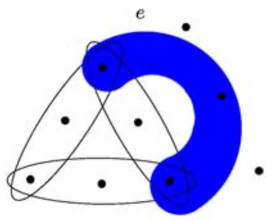

Figure 2: An edge from the family E_2 is shaded.

Fact 2.3. *For every $e \in \binom{V}{3}$ such that either $|e \cap V(C)| = 1$, or $|e \cap V(C)| = 2$ but $e \notin E'$, we have $C + e \supset P_3^3$.*

Fact 2.3 says that the existence of edges listed therein implies the presence of P_3^3. In particular, the family E' consists of all triples e, with $1 \leq |e \cap V(C)| \leq 2$, whose addition to C does not create a copy of P_3^3. However, if we consider these edges more carefully, we will notice that some of them, if occur together, do lead to a formation of P_3^3. This is formalized in Fact 2.4 below, for which, as well as for the two subsequent facts, we introduce some further notation and assumptions.

For $s \geqslant 2$, let $V = V(C) \cup W$ where $V(C) \cap W = \emptyset$ and $|W| = s$.

Fact 2.4. *Let H be a P_3^3-free 3-graph with $V(H) = V$ and $C \subseteq H$. Then $|E' \cap E(H)| \leqslant 3s$.*

Proof. If $e \in E_1$, $f \in E_2$ and $e \cap f = \emptyset$, then $C+e+f \supset P_3^3$. We have $|E_1| = |E_2| = 3s$. Construct an auxiliary bipartite graph $B = (E_1, E_2; \mathcal{E})$, where $\{e, f\} \in \mathcal{E}$ if $e \cap f = \emptyset$. It follows that if $\{e, f\} \in \mathcal{E}$, then $|\{e, f\} \cap E(H)| \leqslant 1$. Observe also that the graph B is $(s-1)$-regular, thus by Hall's theorem it has a perfect matching M. At most one edge of each pair $\{e, f\} \in M$ is in $E(H)$, which implies that $|E' \cap E(H)| \leqslant 3s$. □

As a further preparation toward the proof of Lemma 2.2, let us consider the set of three edges $E_3 = \{V(C) \backslash e : e \in C\}$. One edge of E_3 is presented in Figure 3.

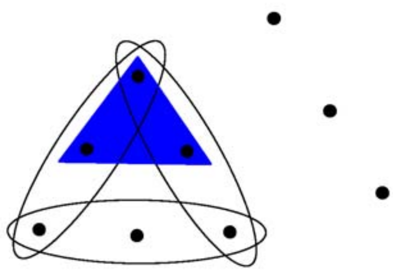

Figure 3: An edge from the family E_3 is shaded.

Fact 2.5. *Let H be a P_3^3-free 3-graph with $V(H) = V$ and $C \subseteq H$. If $e \in E' \cap E(H)$, then $|E_3 \cap E(H)| \leqslant 1$.*

Proof. Let $f \in E_3$. If $e \in E_1$, then $C + e + f \supset P_3^3$, and in view of the assumption that H is P_3^3-free, we conclude that $E_3 \cap E(H) = \emptyset$. If $e \in E_2$ and $e \cap f \neq \emptyset$, then $C + e + f \supset P_3^3$, and, as two of the three edges in E_3 intersect e, we conclude that $|E_3 \cap E(H)| \leqslant 1$. □

It turns out that we can ban some more edges from being present in H. Let us set $E_4 = \binom{W}{3}$.

Fact 2.6. *Let H be a P_3^3-free 3-graph with $V(H) = V$ and $C \subseteq H$. If $e \in E'$, $f \in E_4$, and $f \cap e \neq \emptyset$, then $C + e + f \supset P_3^3$. Consequently, only one of e and f may belong to H.* □

We are now going to use Facts 2.3–2.6 to prove Lemma 2.2.

Proof of Lemma 2.2. Notice that the full star on 9 vertices has $\binom{8}{2} = 28$ edges and contains no P_3^3.

Consider a 3-graph H with 9 vertices and at least 28 edges which is not a star. Based on Theorem 2.1, H contains a copy C of C_3^3. Suppose $P_3^3 \not\subseteq H$. Then, by Fact 2.3,

$$|E(H)| \leqslant \left|\binom{V(C)}{3} \setminus E_3\right| + |E_3 \cap E(H)| + |E_4 \cap E(H)| + |E' \cap E(H)|.$$

Note that $\left|\binom{V(C)}{3} \setminus E_3\right| = \binom{6}{3} - 3 = 17$, $|E_3 \cap E(H)| \leqslant |E_3| = 3$, and $|E_4 \cap E(H)| \leqslant |E_4| = 1$. Hence, if $E' \cap E(H) = \emptyset$ then $|E(H)| \leqslant 17 + 3 + 1 + 0 = 21 < 28$, a contradiction. Otherwise, if $|E' \cap E(H)| \geqslant 1$ then, by Fact 2.4 with $s = 3$, $|E' \cap E(H)| \leqslant 9$. Moreover, by Fact 2.5, $|E_3 \cap E(H)| \leqslant 1$, and by Fact 2.6, $E_4 \cap E(H) = \emptyset$. Consequently, $|E(H)| \leqslant 17 + 1 + 0 + 9 = 27 < 28$, a contradiction again. □

Based on Lemma 2.2, we can determine $ex_3(12; P_3^3)$.

Lemma 2.7. *We have $ex_3(12; P_3^3) = 55$. Moreover, the only extremal 3-graph is the full star.*

Proof. Notice that the full star on 12 vertices has $\binom{11}{2} = 55$ edges and contains no P_3^3. Consider a 3-graph H with 12 vertices and at least 55 edges, which is not a star. It follows from Theorem 2.1 that $C_3^3 \subseteq H$. Let C be a copy of C_3^3 in H, set $W = V(H) \backslash V(C)$, and notice that $|W| = 6$. Assume that there is no copy of P_3^3 in H and consider two cases.

Case 1. $\binom{W}{3} \cap E(H) \neq \emptyset$.

Let $f \in H[W]$. By Facts 2.3 and 2.6, there is no edge e in H such that $f \cap e \neq \emptyset$ and $e \cap V(C) \neq \emptyset$. By Lemma 2.2, $|H[V \backslash f]| \leq 27$. Also $|E(H) \cap \binom{W}{3}| \leqslant 20$. Thus, $|E(H)| \leqslant 27 + 20 = 47 < 55$, a contradiction.

Case 2. $\binom{W}{3} \cap E(H) = \emptyset$.

Partition the set W in two triples f_1 and f_2 and define two induced subhypergraphs $H_1 = H[V \backslash f_1]$ and $H_2 = H[V \backslash f_2]$. By Lemma 2.2, $|E(H_1)| \leqslant 28$ and $|E(H_2)| \leqslant 28$. Moreover, $|E(H_1) \cap E(H_2)| \geqslant |E(C)| = 3$. Consequently $|E(H)| \leqslant |E(H_1)| + |E(H_2)| - |E(C)| = 28 + 28 - 3 = 53 < 55$, a contradiction again. □

3 Proofs of Theorem 1.1 and Theorem 1.2

The derivation of the lower bounds in Theorems 1.1 and 1.2 is based on a construction used already by Gyárfás and Raeisi in [4] to determine $R(C_3^3; 3)$. For future references we state this result in a general form.

Proposition 3.1. *Let $r \geqslant 2$. If a k-graph F is not a star, then*

$$R(F; r) \geqslant r + |V(F)| - 1.$$

Proof. Let us consider the following r-coloring of the edges of the clique K_n^k with vertex set $\{1, 2, \ldots, n\}$, where $n = r + |V(F)| - 2$. We color an edge e by color i, for $i \in \{1, 2, \ldots, r-1\}$, if the minimum vertex in e equals i, that is $\min(e) = i$, and by color r otherwise. Hence, there is no monochromatic copy of F in colors $1, 2, \ldots, r-1$, because F is not a star. We do not obtain a copy of F in color r either, because the edges of color r form a clique K_{n-r+1}^k, while $|V(F)| = n - r + 2$. □

A relation between the Turán and Ramsey numbers is captured by the following simple observation.

Proposition 3.2. *Let $r \geqslant 2$, $k \geqslant 2$, and $n \geqslant r + k$. If $ex_k(n; F) = \frac{1}{r}\binom{n}{k}$, but the unique F-free k-graph with n vertices and $\frac{1}{r}\binom{n}{k}$ edges is a star, then $R(F; r) \leqslant n$.*

Proof. Let us consider an r-coloring of the complete k-graph K_n^k. If there are more than $\frac{1}{r}\binom{n}{k}$ edges in one color, then, by the definition of $ex_k(n; F)$, there is a copy of F in that color. Otherwise, there are exactly $\frac{1}{r}\binom{n}{k}$ edges in each color, but not all the colors may form stars. Indeed, since $n \geqslant r + k$, there would be at least k vertices which are not centers of any monochromatic star. But then an edge of K_n^k would have no color assigned, a contradiction. Thus, for some i, the edges colored by i do not form a star, which, by our assumption on $ex_k(n; F)$, implies that there is a copy of F in that color. □

Propositions 3.1 and 3.2, together with Lemma 2.2 quickly imply Theorem 1.1.

Proof of Theorem 1.1. From Proposition 3.1 we obtain the lower bound $R(P_3^3; 3) \geqslant 3 + 7 - 1 = 9$. For the upper bound we use Proposition 3.2 with $k = 3$, $r = 3$, and $n = 9$. Indeed, the assumptions of Proposition 3.2 follow by Lemma 2.2, and thus $R(P_3^3; 3) \leqslant 9$. □

Similarly, Theorem 1.2 follows from Proposition 3.1, Proposition 3.2, and Lemma 2.7.

4 Concluding remarks

It would be interesting to determine the Turán numbers $ex_3(n; P_3^3)$ for all n. As far as the next Ramsey numbers are concerned, we conjecture that $R(P_3^3; 4) = 10$. We would also like to determine or estimate the Ramsey numbers $R(P_n^k; r)$ for at least some cases where $\max\{n, k, r\} \geq 4$.

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