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Total coloring of generalized Sierpiński graphs

J. Geetha K. Somasundaram

Department of Mathematics
Amrita Vishwa Vidyapeetham
Coimbatore-641 112
India
j_geetha@cb.amrita.edu s_sundaram@cb.amrita.edu

Abstract

A total coloring of a graph is an assignment of colors to all the elements of the graph in such a way that no two adjacent or incident elements receive the same color. In this paper, we prove the tight bound of the Behzad and Vizing conjecture on total coloring for the generalized Sierpiński graphs of cycle graphs and hypercube graphs. We give a total coloring for the WK-recursive topology, which also gives the tight bound.

1 Introduction

All graphs considered here are finite, simple and undirected. Let $G = (V(G), E(G))$ be a graph with the sets of vertices and edges $V(G)$ and $E(G)$, respectively. A *total coloring* of G is a mapping $f : V(G) \cup E(G) \to C$, where C is a set of colors, satisfying the following three conditions (a)–(c):

(a) $f(u) \neq f(v)$ for any two adjacent vertices $u, v \in V(G)$;

(b) $f(e) \neq f(e')$ for any two adjacent edges $e, e' \in E(G)$; and

(c) $f(v) \neq f(e)$ for any vertex $v \in V(G)$ and any edge $e \in E(G)$ incident to v.

The *total chromatic number* of a graph G, denoted by $\chi''(G)$, is the minimum number of colors that suffice in a total coloring. It is clear that $\chi''(G) \geq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G. Behzad [1] and Vizing [21] conjectured (Total Coloring Conjecture or TCC) that for every graph G, $\chi''(G) \leq \Delta(G)+2$. This conjecture was verified by Rosenfeld [17] and Vijayaditya [20] for $\Delta(G) = 3$ and by Kostochka [14, 15, 16] for $\Delta(G) \leq 5$. For planar graphs, the conjecture was verified by Borodin [2] for $\Delta(G) \geq 9$. In 1992, Yap and Chew [22] proved that any graph G has a total coloring with at most $\Delta(G) + 2$ colors if $\Delta(G) \geq |V(G)| - 5$, where $|V(G)|$ is the number of vertices in G. In 1993, Hilton and Hind [7] proved that any graph G has a total coloring with at most $\Delta(G) + 2$ colors if $\Delta(G) \geq \frac{3}{4}|V(G)|$. It is known that the total coloring problem, which asks to find a total coloring of a given graph G with the minimum number of colors, is NP-hard [19]. In particular, McDiarmid and Arroyo [4] proved that the problem of determining the total coloring of a μ-regular bipartite graph is NP-hard, $\mu \geq 3$.

Graphs of "Sierpiński" type appear naturally in many different areas of mathematics as well as in several other scientific fields. One of the most important families of such graphs is formed by the Sierpiński gasket graphs S_n. These graphs were introduced in 1944 by Scorer, Grundy and Smith [18]. Klavžar and Milutinović [12] proved that the Sierpiński graphs $S(n, K_3)$ are isomorphic to the Tower of Hanoi graphs on 3 pegs. The generalization of $S(n, K_3)$ to $S(n, K_k)$ is done via a certain labeling technique. The motivation for this generalization came from topological studies of Lipscomb's space [9]. The graphs $S(n, K_k)$ possess many appealing properties such as coding and metric properties. Sierpiński gasket graphs play an important role in dynamic systems, probability and psychology [13]. Fu [5] studied a class of WK-recursive networks. WK-recursive networks are very similar to Sierpiński graphs. They can be obtained from Sierpiński graphs by adding a link (an open edge) to each of its extreme vertices.

In this paper, we give a total coloring for generalized Sierpiński graphs of cycle graphs and hypercube graphs. Also, we give a total coloring of WK-recursive topology of some graphs. These colorings will give the tight bound of the Behzad and Vizing conjecture.

In Section 2, we determine the total chromatic number of generalized Sierpiński graphs of cycle graphs, hypercube graphs and house graphs. In Section 3, we give a total coloring for 3D WK-recursive topology, taking the basic module as complete graphs and cycle graphs.

2 Generalized Sierpiński Graphs

The Sierpiński graphs $S(n, K_k)$, $k,\ n \geq 1,\ k,\ n \in \mathbb{N}$ are defined on the vertex set $\{1, 2, \ldots, k\}^n$, where K_k is the complete graph on k vertices. Two different vertices $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ are adjacent if and only if there exists an $h \in \{1, 2, \ldots, n\}$ such that:

a) $u_t = v_t$ for $t = 1, 2, \ldots, h-1$;
b) $u_h \neq v_h$; and
c) $u_t = v_h$ and $v_t = u_h$ for $t = h+1, \ldots, n$.

In the rest of this paper, we will use the abbreviation $v_1 v_2 \ldots v_n$ for $(v_1, v_2, \ldots, v_n)$.

Sierpiński gasket graphs (introduced by Jakovac [11]) are just a step from the Sierpiński graphs $S(n, K_3)$. The graph S_n is obtained from $S(n, K_3)$ by contracting every edge of $S(n, K_3)$ that lies in no triangle. In [11] there is also a generalization of the graph $S_n := S(n, K_3)$. These are the Sierpiński gasket graphs $S[n, k],\ k \geq 3$, obtained from the Sierpiński graphs $S(n, K_k)$ by contracting edges that lie in no complete subgraph K_k. Gravier, Kovše and Parreau [6] generalized this construction for any graph, by defining generalized Sierpiński graphs, $S(n, G)$ as follows:

$S(1,G)$ is isomorphic to the graph G and we can construct $S(n+1,G)$ by copying $|V(G)|$ times $S(n,G)$ and adding an edge between the i^{th} vertex of the j^{th} copy and the j^{th} vertex of the i^{th} copy of $S(n,G)$ (called the linking edge) whenever (i,j) is an edge in G.

Jakovac and Klavžar [10] showed that $\chi''(S(n,K_k)) = k+1$, for any $n \geq 2$ and odd $k \geq 3$. Also, they proved $\chi''(S(n,K_4)) = 5$. For even k, they proposed a conjecture, which states that "for any even $k \geq 6$, $\chi''(S(n,K_k)) = \Delta(S(n,K_k))+2$".

Hinz and Parisse [8] gave a counter example for disproving the above conjecture. Also, they proved that $\chi''(S(n,K_k)) = \Delta(S(n,K_k)) + 1$ for any k and n, $k,n \geq 2$.

The *cartesian product* of G and H is a graph, denoted by $G \,\square\, H$, whose vertex set is $V(G) \times V(H)$. Two vertices (g,h) and (g',h') are adjacent precisely if $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$ and $h = h'$. In other words, $V(G \,\square\, H) = \{(g,h)|\ g \in V(G) \text{ and } h \in V(H)\}$ and $E(G \,\square\, H) = \{((g,h),(g',h'))|\ g = g',\ hh' \in E(H),\ \text{or } gg' \in E(G),\ h = h'\}$.

The G- and H-layers are the induced subgraphs in $G \,\square\, H$ on the vertex sets $G_u = \{(x,u)|\ x \in V(G)\}$ and $H_v = \{(v,x)|\ x \in V(H)\}$, respectively.

We use the notation $[q]_0 = \{0,1,2,\ldots,q-1\}$ for the initial segment of length q. A *canonical vertex coloring* is a coloring $c_k(i)$, $c_k(i) = i$, for all $i \in [k]_0$.

Theorem 2.1. *If G is a graph with $\chi''(G) = \Delta(G)+1$ then*

$$\chi''(S(n,G)) = \Delta(S(n,G)) + 1, \quad n \geq 2, \quad n \in \mathbb{N}.$$

Proof. According to the construction of generalized Sierpiński graphs, $S(n,G)$ contains $|V(G)|$ copies of G. We color all $|V(G)|$ copies of G with the same $\Delta(G)+1$ colors. Since the adjacent vertices receive different color in G, the adjacent vertices $v_iv_j\ldots v_j$ and $v_jv_i\ldots v_i$ will also receive different colors in $S(n,G)$. Therefore we can assign a new color to the linking edges. Hence, the total chromatic number of $S(n,G)$ is $\Delta(S(n,G))+1$. □

A *house graph* is a complement of the path graph P_5. We prove for $n \geq 2$, $\chi''(S(n,G)) = \Delta(S(n,G))+1$, if G is a house graph. Figure 1. shows the Sierpiński graph of the house graph G, $S(2,G)$.

Corollary 2.1. If G is a house graph, then for $n \geq 2$, $n \in \mathbb{N}$, $\chi''(S(n,G)) = \Delta(S(n,G))+1$.

Proof. By Theorem 2.1 it suffices to prove that the total chromatic number of a house graph G is $\Delta(G)+1$. We can give the coloring in the following way: Consider the graph G. The vertices and edges of the triangle are colored with 1, 2 and 3. The remaining vertices are colored with 2 and 3. The horizontal edge is colored with 1 and the vertical edges are colored with 4. □

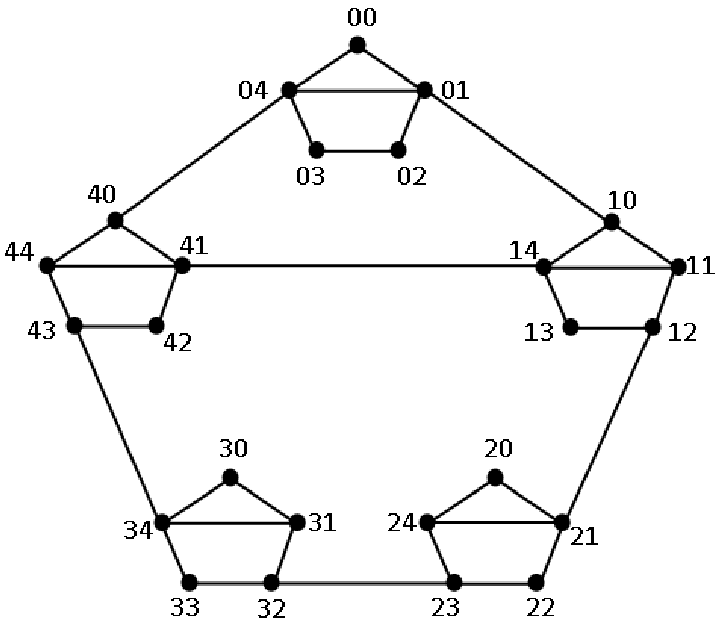

Figure 1: $S(2, G)$, G-House graph.

A *wheel graph* W_{k+1} is a graph with $k+1$ vertices ($k \geq 4$), formed by connecting a single vertex to all vertices of an C_k-cycle.

Corollary 2.2. $\chi''(S(n, W_{k+1})) = \Delta(S(n, W_{k+1})) + 1$, $n \geq 2$, $k \geq 4$, n, $k \in \mathbb{N}$.

Proof. We know that $\chi''(W_{k+1}) = \Delta(W_{k+1}) + 1$. The assertion follows from Theorem 2.1. □

The *equitable total chromatic number* of a graph G is the smallest integer μ for which G has a μ-total coloring such that the number of vertices and edges colored with each color differs by at most one. Chunling et al [3] proved that the equitable total chromatic number of the cartesian product of cycles $C_s \,\square\, C_t$ is $\Delta(C_s \,\square\, C_t) + 1$. It is easy to prove that the total chromatic number of Sierpiński graph of $C_s \,\square\, C_t$ is $\Delta(S(n, C_s \,\square\, C_t)) + 1$, by adding one color to all the linking edges.

Corollary 2.3. For $n \geq 2$, $s, t \geq 3$, and $n, s, t \in \mathbb{N}$, we have

$$\chi''(S(n, C_s \,\square\, C_t)) = \Delta(S(n, C_s \,\square\, C_t)) + 1.$$

Proof. From [3], it is easy to see that $\chi''(C_s \,\square\, C_t) = \Delta(C_s \,\square\, C_t) + 1$. Therefore by Theorem 2.1, $\chi''(S(n, C_s \,\square\, C_t)) = \Delta(S(n, C_s \,\square\, C_t)) + 1$. □

Remark:

For some Sierpiński graphs of G with $\chi''(G) = \Delta(G) + 2$, we have $\chi''(S(n, G)) = \Delta(S(n, G)) + 1$.

For example, consider cycle graphs. We recall that if C_k is a cycle graph with k vertices, $k \geq 3$, then

$$\chi''(C_k) = \begin{cases} \Delta(C_k) + 1, & \text{if } k \equiv 0 \mod 3 \\ \Delta(C_k) + 2, & \text{otherwise.} \end{cases}$$

In the following theorem, we give a total coloring of Sierpiński graphs $S(n, C_k)$ of cycle graphs C_k. In the process of assigning the colors to the vertices and edges of $S(n, C_k)$, we prove that $\chi''(S(n, C_k)) = \Delta(S(n, C_k)) + 1$ for all $k \geq 3$. It is interesting to note that $\chi''(S(n, C_k)) = \Delta(S(n, C_k)) + 1$ even though $\chi''(C_k) = \Delta(C_k) + 2$ for $k \neq 3l,\ l = 1, 2, 3, \ldots$.

The Sierpiński graph $S(2, C_5)$ together with the corresponding vertex labeling is shown in Figure 2.

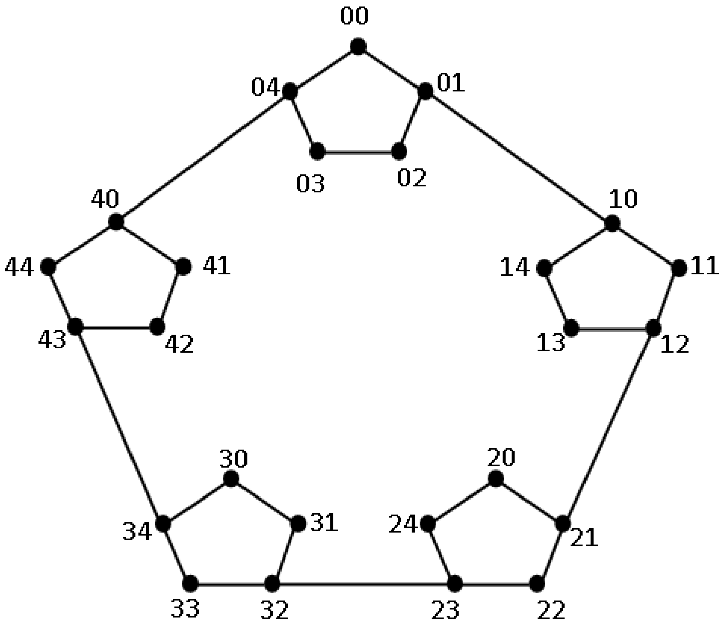

Figure 2: $S(2, C_5)$.

Theorem 2.2. *For any $n \geq 2, k \geq 3,\ k, n \in \mathbb{N}$, $\chi''(S(n, C_k)) = \Delta(S(n, C_k)) + 1$.*

Proof. Let us construct a total coloring of C_k in two different cases. First one for odd values of k and another for even values of k.

Case(i): k is odd.

First let us consider cycle graphs C_k. Assign the colors 1, 2, 3 and 4 cyclically to the sequence $v_0 e_0 v_1 e_1 \ldots v_{k-3} e_{k-3}$ in C_k. If we assign colors in this manner, we would assign the color 1 to each vertex v_{2i}, $i = 0, 1, 2, \ldots, \frac{k-3}{2}$. We cannot assign the colors 1 and 4 to the vertex v_{k-1} and also, we cannot assign colors 1, 2, and 4 to the edge e_{k-1}. Therefore, assign the color 2 to the vertex v_{k-1} and the color 3 to the edge e_{k-1}. We denote by c_1, the total coloring of C_k.

For $n = 2$, use c_1 for the first copy of $S(2, C_k)$. Assign the colors of v_i and e_i in the t^{th} copy of $S(2, C_k)$ to $v_{(i+1) \mod k}$ and $e_{(i+1) \mod k}$ in the $(t + 1)^{th}$ copy of $S(2, C_k)$, where $t, i \in [k]_0$. The vertices $v_i v_j$ and $v_j v_i$ have the same missing color.

Now, we assign the color which is missing at the vertices v_iv_j and v_jv_i to the linking edges. We denote by c_2, the total coloring of $S(2, C_k)$.

For $n = 3$, assign c_2 for odd copies of $S(3, C_k)$ except the $(k-1)^{th}$ copy. Assign c_2' for even copies of $S(3, C_k)$ and assign c_2'' for the $(k-1)^{th}$ copy of $S(3, C_k)$, where c_2' and c_2'' are obtained from c_2 using the permutations of colors (123)(4) and (13)(2)(4), respectively. Here the vertices $v_iv_jv_j$ and $v_jv_iv_i$ have the same missing color. So we can assign this missing color to the linking edges.

Finally, for $n \geq 4$, assign the colors as in $S(n-1, C_k)$ to all the k copies of $S(n, C_k)$. In this process, the vertices $v_iv_j \dots v_j$ and $v_jv_i \dots v_i$ have the same missing color. We assign the color which is missing at the vertices $v_iv_j \dots v_j$ and $v_jv_i \dots v_i$ to the linking edges.

Case(ii): k is even.

Consider cycle graphs C_k. Color odd vertices by 1 and even vertices by 2. Color the edges by 3 and 4, alternatively. Let us denote the total coloring of C_k by c_3. For $n = 2$, color odd copies of $S(2, C_k)$ by c_3 and even copies by c_3', where c_3' is obtained from c_3 using the permutation of colors (4321). The vertices v_iv_j and v_jv_i have the same missing color. Now, we assign the color which is missing at the vertices v_iv_j and v_jv_i to the linking edges. Let c_4 be the total coloring of $S(2, C_k)$.

For $n = 3$, color odd copies of $S(3, C_k)$ by c_4, and even copies of $S(3, C_k)$ by c_4', where c_4' is obtained from c_4 using the permutation of colors (13)(24). The vertices $v_iv_jv_j$ and $v_jv_iv_i$ have the same missing color. Now, we assign the color which is missing at the vertices $v_iv_jv_j$ and $v_jv_iv_i$ to the linking edges.

For $n \geq 4$, assign colors as in $S(n-1, C_k)$ to all the k copies of $S(n, C_k)$. The vertices $v_iv_j \dots v_j$ and $v_jv_i \dots v_i$ have the same missing color. Now, we assign the color which is missing at the vertices $v_iv_j \dots v_j$ and $v_jv_i \dots v_i$ to the linking edges.

So the total chromatic number of $S(n, C_k)$ is 4, which equals $\Delta(S(n, C_k)) + 1$. □

In the next theorem, we obtain a total coloring of Sierpiński graphs of hypercube graphs. Let Q_{k+1} be a hypercube graph of order $k + 1$. The hypercube graphs Q_{k+1} can be constructed from Q_k, by taking two copies Q_k and adding an edge (joining edge) from each vertex in one copy of Q_k to the corresponding vertex in the other copy of Q_k. Hence, the hypercube graphs Q_k, $k \geq 1$, are the iterated cartesian product $K_2 \square K_2 \square \cdots \square K_2$ of k copies of K_2. The Sierpiński graph $S(2, Q_3)$ is shown in Figure 3.

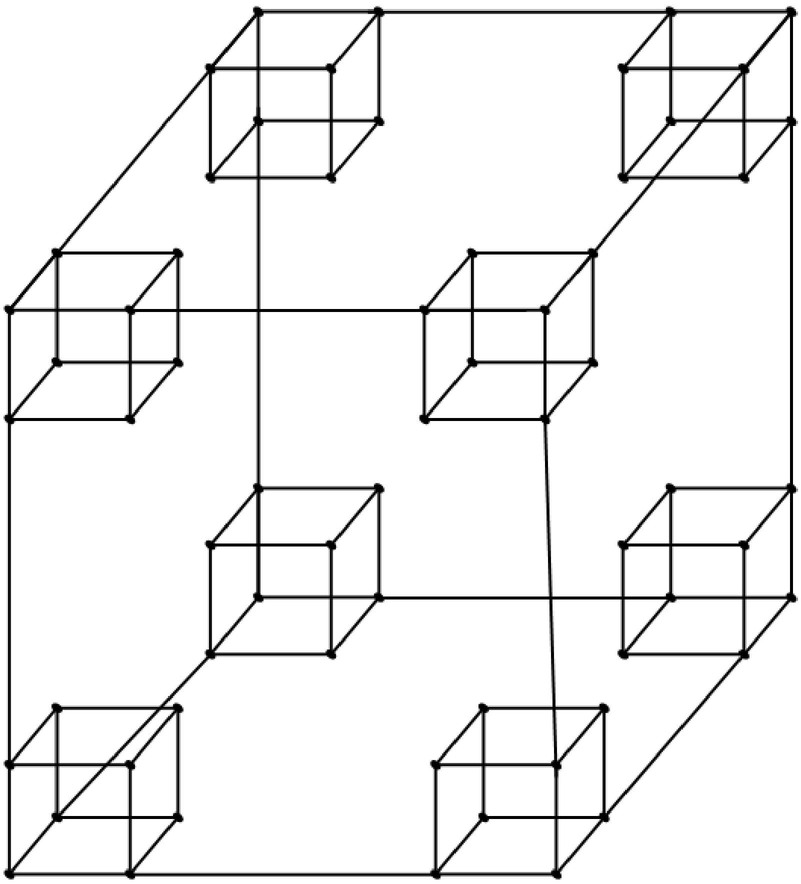

Figure 3: $S(2, Q_3)$.

Theorem 2.3. *For any* $k \geq 1,\ n \geq 2,\ n,\ k \in \mathbb{N}$, $\chi''(S(n, Q_k)) = \Delta(S(n, Q_k)) + 1$.

Proof. Consider hypercube graph Q_1, $Q_1 \cong K_2$. Sierpiński graphs of Q_1 are path graphs. The total chromatic number of path graphs is 3. For $k = 2$, $Q_2 \cong C_4$. Color the vertices of Q_2 with colors 1 and 2, alternatively. Color the edges of Q_2 with colors 3 and 4, alternatively. From Theorem 2.2, $\chi''(S(n, Q_2)) = 4$.

Now, let us consider hypercube graphs Q_k, $k \geq 3$. Color all the vertices of Q_k with colors 1 and 2, alternatively. Color the edges of two copies Q_{k-1} of Q_k as in Q_{k-1}. Color the joining edges with color $k + 2$. Let us denote the total coloring of Q_k by c_1.

For $n = 2$, assign c_1 to odd copies of Q_k and c_1' to even copies of Q_k, where c_1' is obtained from c_1 using the permutation of colors $(4321)(5)(6)\ldots(k+2)$. Assign the missing color at vertices v_iv_j and v_jv_i to the linking edges. Let c_2 denote the total coloring of $S(2, Q_k)$.

For $n = 3$, assign c_2 to odd copies of Q_k and c_2' to even copies of Q_k, where c_2' is obtained from c_2 using the permutation of colors $(13)(24)(5)(6)\ldots(k+2)$.

For $n \geq 4$, the total coloring of $S(n, Q_k)$ is obtained by assigning the colors as in $S(n-1, Q_k)$ to all the k copies of $S(n, Q_k)$.

In each step, we assign the color which is missing at the vertices $v_iv_j \ldots v_j$ and $v_jv_i \ldots v_i$ to the linking edges. Therefore the total chromatic number of $S(n, Q_k)$ is $\Delta(S(n, Q_k)) + 1$. □

3 3D-Recursive Topology

The WK-recursive topology has received much attention due to its many favorable properties such as high degree of scalability.

The WK-recursive topology can be constructed hierarchically by grouping the basic modules. We use $K(l, n, G)$ to denote a WK-recursive topology of a graph G. Here $l \geq 1$, indicates the number of layers in 3D topology and $n \geq 2$, specifies the number of levels in the recursive structure. If $l = 1$, then the topology is a 2D recursive topology. The basic module $K(1, n, G)$ is isomorphic to $S(n, G)$.

The 3D topology is formed by taking l copies of $K(1, n, G)$ and adding edges between the respective corner vertices of the adjacent layers. Figure 4. shows the 3D-recursive topology $K(3, 2, K_4)$.

We know from [8] that $\chi''(S(n, K_k)) = \Delta(S(n, K_k)) + 1,\ k,\ n\ \geq 2$. In the next theorem, we assign colors to the vertices and edges of $K(l, n, K_k)$ and we show that the total chromatic number of $K(l, n, K_k)$ is $\Delta(K(l, n, K_k)) + 1$.

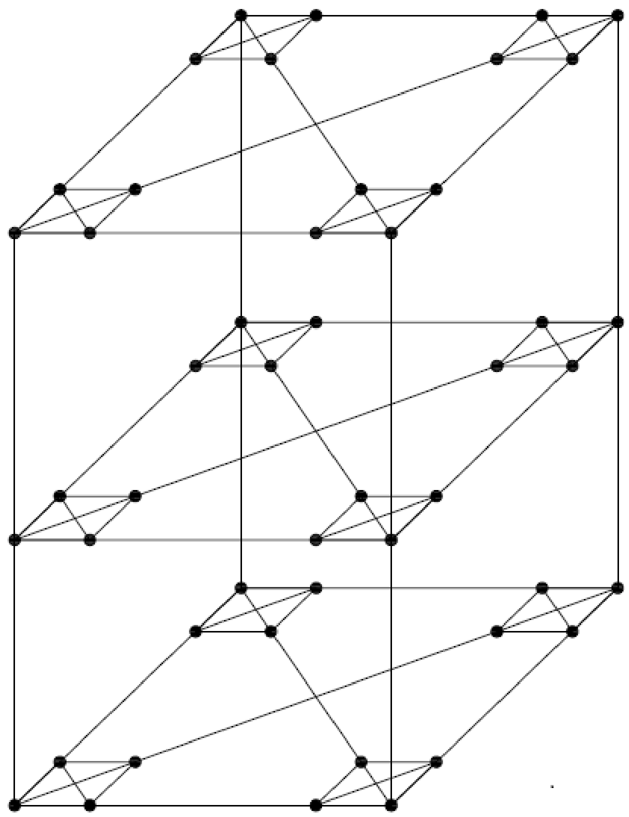

Figure 4: $K(3, 2, K_4)$.

Theorem 3.1. *For any $n,\ k \geq 2$ and $l \geq 1$, $k,\ l,\ n \in \mathbb{N}$, $\chi''(K(l,n,K_k)) = \Delta(K(l,n,K_k)) + 1$.*

Proof. The basic module $K(1,n,K_k)$ is $S(n,K_k)$. First we give a total coloring of each layer and then we color the edges between the layers.

Case(i): k is even.

Odd layers of $K(l,n,K_k)$ are colored as in [8]. $c''_k(i,j) \equiv (\tau_i(j) + \tau_j(i) + 2) \mod (k+1)$, $i \neq j$, $i,j \in [k]_0$ defines a special $(k+1)$-edge coloring of K_k with p colors and color p is missing in the line $p \in [k]_0$, where τ_p is the transposition of p and $k-1$. The vertices are colored by the canonical vertex-coloring to obtain a special total coloring of K_k.

Now, we prove this by induction on n. We use the special total coloring of K_k, where we replace the canonical vertex-coloring by $i \mapsto (i+1) \mod k$.

For the induction step, the edges $(ijj\dots j, jii\dots i)$ are colored according to the specially colored adjacency matrix $(a_{ij})_{k\times k}$ of K_k.

The diagonal entries Π_p, $p \in [k]_0$, of the adjacency matrix $(a_{ij})_{k\times k}$ of $S(n,K_k)$ are the total colorings of k copies of $S(n-1,K_k)$. The permutation Π_p is obtained by assigning color $c \mapsto c''_k(c,p)$, $c \in [k+1]_0$, $p \in [k]_0$, where $c''_k(p,p) = p$ and $c''_k(k,p) = c''_{k+1}(k,p)$.

We define the adjacency matrix $(a_{ij})_{k\times k}$ as follows:

$$a_{ij} = \begin{cases} \Pi_i, & i = j \\ c''_k(i,j), & i \neq j. \end{cases}$$

Now, consider the even layers of $K(l,n,K_k)$. Here, the total coloring of $S(n,K_k)$ is obtained from the total coloring of $S(n,K_k)$ of the odd layers of $K(l,n,K_k)$ by permuting the colors with the permutations Π_p, $p \in [k]_0$.

$$\begin{aligned} \Pi_0 &= (k(k-1)\dots 4321)(0), \\ \Pi_1 &= (k(k-1)\dots 4320)(1), \\ &\vdots \\ \Pi_p &= (k(k-1)\dots 43210)(p),\ p = 0,1,2,\dots,k-2, \text{ and} \\ \Pi_{k-1} &= ((k-2)\dots 43210)(k(k-1)). \end{aligned}$$

The linking edges of $S(n,K_k)$ are colored with the missing color at the vertices $ij\dots j$ and $ji\dots i$. The edges between odd and even layers are colored with the missing color at the corner vertices and the edges between even and odd layers are colored with a new color.

Case(ii): k is odd.

Odd layers of $K(l,n,K_k)$ are colored as in [8]. First, we obtain the total coloring of K_{k-1} as in the previous case. The color $(p+1) \mod (k-1)$ is still missing in the line $p \in [k-1]_0$. Add a new vertex named p and join all $k-1$ vertices with this new vertex. Color the new vertex with color p and the edge incident with the vertex p with color $(p+1) \mod (k-1)$, $p \in [k-1]_0$. This will give total colorings of K_k.

For the induction step, we color all the k copies of $S(n, K_k)$ as in $S(n-1, K_k)$, using the colors from $[k-1]_0$ and we color all the linking edges with color k.

The total coloring of even layers are given by $c \mapsto (c+1) \mod k$, where $c \in [k]_0$ is the color in odd layers of $K(l, n, K_k)$. The linking edges are colored with color k. The edges between odd and even layers are colored with color k and edges between even and odd layers are colored with a new color.

In both cases, we use only $(k+2)$ colors to give a total coloring. Therefore, the total chromatic number of $K(l, n, K_k)$ is $\Delta(K(l, n, K_k)) + 1$. □

The special total coloring of K_4 and $S(2, K_4)$ are given in Tables 1 and 2, respectively.

$i \backslash j$	0	1	2	3
0	1	3	4	2
1	3	2	0	4
2	4	0	3	1
3	2	4	1	0

Table 1: 5-Total coloring of K_4.

	0	1	2	3
0	Π_0	3	4	2
1	3	Π_1	0	4
2	4	0	Π_2	1
3	2	4	1	Π_3

Table 2: Total coloring of $S(2, K_4)$.

The entry Π_p in the specially colored adjacency matrix $(a_{ij})_{k\times k}$ stands for the total coloring of the subgraph $S(n-1, K_k)$ of $S(n, K_k)$.

In the next theorem, we give a total coloring of the 3D recursive topology $K(l, n, G)$, by taking G as a cycle graph.

Theorem 3.2. *For $l \geq 1$, $n \geq 2$, $k \geq 3$, and $k,\ l,\ n \in \mathbb{N}$, we have $\chi''(K(l, n, C_k)) = \Delta(K(l, n, C_k)) + 1$.*

Proof. The 3D-recursive topology $K(1, n, C_k)$ is isomorphic to $S(n, C_k)$, $n \geq 2$. We construct the total coloring of $K(l, n, C_k)$ in two cases.

Case(i): k is even.

We give the total coloring of odd layers of $K(l, n, C_k)$ as in Theorem 2.2. We denote this total coloring by c_1. For even layers, we use color c_1' to get a total coloring, where c_1' is obtained from c_1 using the permutation of colors (134)(2).

Now, the edges between odd and even layers are colored with the missing color at the corner vertices. The edges between even and odd layers are colored with a new color.

Case(ii): k is odd.

The odd layers of $K(l, n, C_k)$ are colored as in Theorem 2.2. We denote this total coloring by c_2. For the even layers, we use color c_2' to get a total coloring, where c_2' is obtained from c_2 using the permutation of colors (123)(4).

Now, the edges between odd layers and even layers are colored with the missing at the corner vertices. The edges between even layers and odd layers are colored with a new color.
Therefore using 5 colors we color the vertices and edges of $K(l, n, C_k)$.

Hence, $\chi''(K(l, n, C_k)) = \Delta(K(l, n, C_k)) + 1$. □

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