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The birank number of ladder, prism and Möbius ladder graphs

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Abstract

Given a graph G, a function $f : V(G) \to \{1, 2, ..., k\}$ is a k-biranking of G if $f(u) = f(v)$ implies every u-v path contains vertices x and y such that $f(x) > f(u)$ and $f(y) < f(u)$. The birank number of a graph, denoted $bi(G)$, is the minimum k such that G has a k-biranking. In this paper we determine the birank numbers for ladder, prism, and Möbius ladder graphs.

1 Introduction

A k-ranking on a graph G is a function $f : V(G) \to \{1, 2, 3, ..., k\}$ such that if $f(u) = f(v)$, then each u-v path contains an x with $f(x) > f(u)$. The integer $f(v)$ is called the rank of v. The concept of rankings was first introduced by Iyer, Ratliff, and Vijayan [4]. Jamison [5] later introduced the notion of a biranking as follows: a k-biranking on a graph G is a function $f : V(G) \to \{1, 2, 3, ..., k\}$ such that if $f(u) = f(v)$, then each u-v path contains vertices x and y where $f(x) > f(u)$ and $f(y) < f(u)$. The minimum k for which G has a valid k-biranking is the birank number of G, denoted here as $bi(G)$.

Rankings have been well-studied; the rank numbers for a variety of graphs, including paths, grid graphs, Möbius graphs, complete multipartite graphs, prism graphs, bent ladders, and split graphs, have been determined ([1] [6] [7] [8]). Birankings have not been studied as thoroughly however and currently only the birank numbers for paths and cycles have been determined [3].

A variation of ranking known as on-line ranking has been studied for cycles as well. An on-line ranking requires assigning ranks to vertices as a graph is assembled in an arbitrary order. That is, ranks are assigned as the graph grows from its subgraphs. Bruoth and Horňák found an upper bound for the on-line ranking number of a cycle graph in [2].

In Section 2 of this paper we determine the birank number for ladder graphs, $L_n = P_2 \Box P_n$. In Section 3 we determine the birank number of prism and Möbius ladder graphs by relating them to the birank number of a ladder graph.

2 The Birank Number of a Ladder Graph

We begin by finding, given k, the largest n for which L_n has a valid k-ranking. Toward this end we find a bound for the number of vertices in L_n which may be assigned the same rank. First we note that for any rank r to appear twice in a valid biranking there needs to be a set of vertices with ranks less than r blocking any path between the repeated ranks. We give such a collection vertices a name.

Definition 2.1. *Consider a graph G with a k-biranking f and an integer r with $0 < r < k$. Two vertices $u, v \in V(G)$ form a* low divider *for the rank r if the removal of the vertices in S disconnects the graph and $f(u), f(v) < r$.*

We define a high divider *for a rank r in an analogous way.*

In a ladder, if we think of L_n as consisting of two copies of P_n (the "rails" of our ladder) then we see the dividers will have one vertex on each copy. For example, consider Figure 1. In the biranking of L_6 the vertices with ranks 1,2 form a low divider for 3 through 6. In the biranking of L_5, vertices with ranks 1,2 again form a low divider. In both of these examples, any path between the two vertices with rank 3 must pass through either the vertex with rank 1 or 2.

Clearly in any valid biranking on a ladder graph, if there are two distinct vertices assigned rank r then there must be both a low divider and a high divider for r between them. In fact we generalize this reasoning for ladders in the following lemma.

Lemma 2.1. *In any biranking of a ladder graph, if a rank appears t times then there must be $t-1$ low dividers and $t-1$ high dividers for that rank.*

Using this result, we provide an upper bound on the number of times a rank can appear by counting the possible number of low/high dividers.

Lemma 2.2. *Given a k-biranking on a ladder graph and r between 1 and $\lfloor k/2 \rfloor$, the number of vertices with ranks less than $(2r+1)$ is at most $2^{r+1}-2$. The number of vertices with ranks more than $(k-2r)$ is at most $2^{r+1}-2$.*

Proof. We proceed by induction on r. Clearly the ranks 1 and 2 appear at most once each since there are no ranks less than these.

For $r \geq 2$, we are given that there are at most $2^{r+1}-2$ vertices with ranks less than $(2r+1)$. Now assume for some $t \geq 0$ the rank $(2r+2)$ appears $2^r + t$ times. By Lemma 2.1, we need $2^r + t - 1$ low dividers for $(2r+2)$ and these low dividers will require at least $2^{r+1} + 2t - 2$ vertices with ranks less than $(2r+2)$. Since there are only $2^{r+1} - 2$ vertices with ranks less than $(2r+1)$, we must use vertices with the rank $(2r+1)$ in low dividers for $(2r+2)$ at least $2t$ times. Now the vertices in a low divider cannot have the same rank, so a vertex of rank $(2r+1)$ must be paired with a vertex of smaller rank in each of these low dividers. This smaller ranked vertex is

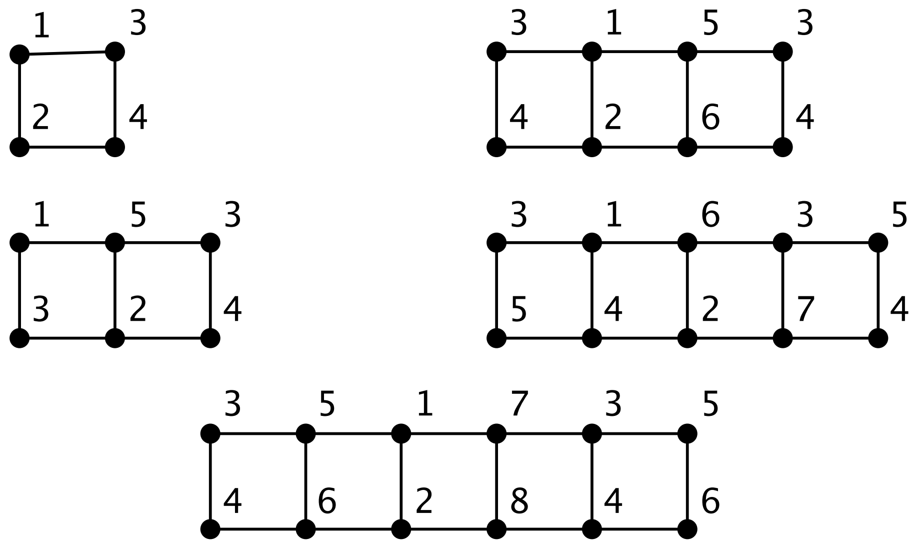

Figure 1: Examples of valid birankings on L_2 through L_6

then no longer available to act in a low divider for $(2r+1)$. So, of the $2^r + t - 1$ low dividers for $(2r+2)$, at most $2^r + t - 1 - 2t = 2^r - t - 1$ will be low dividers for $(2r+1)$. This means $(2r+1)$ appears at most $2^r - t$ times. Then these two ranks together appear at most a total of 2^{r+1} times.

Then we see the number of vertices with ranks less than $(2r+3)$ is at most:

$$2^{r+1} - 2 + 2^{r+1} = 2^{r+2} - 2$$

The other half of the proof follows from a similar argument counting high dividers. □

It is important to note that the proof of Lemma 2.2 groups consecutive ranks together in pairs. So, for example, it fairly easily gives an upper bound for the size of a ladder which has a valid 8-biranking. We apply the lemma with $r = 2$ to see the number of vertices with ranks less than 5 is at most 6. Similarly, use the high divider argument with $r = 2$ and we see that the number of vertices with ranks more than 4 is at most 6. Thus any ladder which has an 8-biranking will have at most 12 vertices. In fact, from Figure 1 we see this bound is sharp. While this works well for $k = 8$, if the number of ranks is not a multiple of 4 then more work is needed. We address this in the next lemma.

Lemma 2.3. *Given an integer $k \geq 4$, if there is a k-birank on L_n, then we have an upper bound for n as a function of k as follows:*

$$n \leq \begin{cases} 4 \cdot 2^{\frac{k}{4}-1} - 2 & \text{for } k \equiv 0 \mod 4 \\ 5 \cdot 2^{\lfloor \frac{k}{4} \rfloor - 1} - 2 & \text{for } k \equiv 1 \mod 4 \\ 6 \cdot 2^{\lfloor \frac{k}{4} \rfloor - 1} - 2 & \text{for } k \equiv 2 \mod 4 \\ 7 \cdot 2^{\lfloor \frac{k}{4} \rfloor - 1} - 2 & \text{for } k \equiv 3 \mod 4 \end{cases}$$

Proof. If $k \equiv 0 \mod 4$, then the number of vertices that can be biranked on L_n using the ranks $1, ..., k/2$ and $k/2+1, ..., k$ is at most $2(2^{\frac{k}{4}+1} - 2)$ (by Lemma 2.2 using $r = k/4$). Therefore n is bounded by half that: $2^{\frac{k}{4}+1} - 2$.

If $k \equiv 1 \mod 4$, then using Lemma 2.2 with $r = \frac{k-1}{4}$ we see the number of vertices that can be biranked using ranks less than $\left(\frac{k+1}{2}\right)$ or more than $\left(\frac{k+1}{2}\right)$ is at most $2(2^{\frac{k-1}{4}+1} - 2) = 4 \cdot 2^{\lfloor \frac{k}{4} \rfloor} - 4$.

We must still consider the rank $\left(\frac{k+1}{2}\right)$. From the previous calculation we see the number of vertices with ranks less than this is at most $2^{\lfloor \frac{k}{4} \rfloor + 1} - 2$, so there are at most $2^{\lfloor \frac{k}{4} \rfloor} - 1$ low dividers for this rank. So at most $2^{\lfloor \frac{k}{4} \rfloor}$ vertices may have the rank $\left(\frac{k+1}{2}\right)$.

If we combine these then we see that the total number of vertices that may be ranked with $1, \ldots, k$ is at most $5 \cdot 2^{\lfloor \frac{k}{4} \rfloor} - 4$ so we have our bound for n follows.

If $k \equiv 2 \mod 4$, then we apply Lemma 2.2 with $r = \frac{k+2}{4}$ to see that the number of vertices with ranks less than $\left(\frac{k+2}{2} + 1\right)$ is at most $2^{\frac{k+2}{4}+1} - 2 = 2^{\lfloor \frac{k}{4} \rfloor + 2} - 2$. We use Lemma 2.2 again this time with $r = \frac{k-2}{4}$ to see the number of vertices with ranks greater than $\left(\frac{k+2}{2}\right)$ is at most $2^{\frac{k-2}{4}+1} - 2 = 2^{\lfloor \frac{k}{4} \rfloor + 1} - 2$. Putting these together, we see the total number of vertices in this case is bounded by $6 \cdot 2^{\lfloor \frac{k}{4} \rfloor} - 4$ and so our bound for n follows.

If $k \equiv 3 \mod 4$, then by Lemma 2.2 with $r = \frac{k-3}{4}$, the number of vertices with ranks less than $\left(\frac{k-1}{2}\right)$ is at most $2^{\frac{k-3}{4}+1} - 2 = 2^{\lfloor \frac{k}{4} \rfloor + 1} - 2$. Then using $r = \frac{k+1}{4}$, the number of vertices with ranks greater than $\left(\frac{k-1}{2}\right)$ is at most $2^{\frac{k+1}{4}-1} - 2 = 2^{\lfloor \frac{k}{4} \rfloor + 2} - 2$.

Finally, consider the rank $\left(\frac{k-1}{2}\right)$. We have above a bound on the number of vertices with ranks less than this, so the number of low dividers for $\left(\frac{k-1}{2}\right)$ is at most $2^{\lfloor \frac{k}{4} \rfloor} - 1$ and so this rank occurs at most $2^{\lfloor \frac{k}{4} \rfloor}$ times.

If we combine these, we see that the total number of vertices that may be ranked with $1, \ldots, k$ is at most $7 \cdot 2^{\lfloor \frac{k}{4} \rfloor} - 4$ and so we have our bound for n follows.

□

We next consider when a k-biranking exists on L_n for appropriate n values.

Lemma 2.4. *Given an integer $k \geq 4$, there is a k-biranking on L_n whenever*

$$n = \begin{cases} 4 \cdot 2^{\frac{k}{4}-1} - 2 & \text{for } k \equiv 0 \mod 4 \\ 5 \cdot 2^{\lfloor \frac{k}{4} \rfloor - 1} - 2 & \text{for } k \equiv 1 \mod 4 \\ 6 \cdot 2^{\lfloor \frac{k}{4} \rfloor - 1} - 2 & \text{for } k \equiv 2 \mod 4 \\ 7 \cdot 2^{\lfloor \frac{k}{4} \rfloor - 1} - 2 & \text{for } k \equiv 3 \mod 4 \end{cases}$$

Proof. We proceed by induction on k. The base cases, $k = 4, 5, 6, 7$, are established in Figure 1.

Note that n is always even. Consider a ladder L_n and insert a low divider and high divider in the middle of our ladder using the ranks $1, 2, k-1, k$ (see for example the biranking of L_6 in Figure 1). We have now divided L_n into two ladders with length $\frac{n}{2} - 1$. We will show that each new $L_{\frac{n}{2}-1}$ may be biranked with the ranks $3, ..., k-2$ (the remaining $k-4$ ranks).

We begin with the case $k \equiv 0 \mod 4$ and $n = 2^{\frac{k}{4}+1} - 2$. Since $n = 2^{\frac{k}{4}+1} - 2$, $\frac{n}{2} - 1 = 2^{\frac{k}{4}} - 2 = 2^{\frac{k-4}{4}+1} - 2$. So, by induction, there is a $(k-4)$-biranking on $L_{\frac{n}{2}-1}$. Since we constructed a high divider and low divider in the middle of our L_n, the two copies $L_{\frac{n}{2}-1}$ may be biranked identically. This will give us a k-biranking for L_n.

The other cases proceed in an analagous way. □

Theorem 2.5. *For $n \geq 3$,*

$$bi(L_n) = \lfloor \log_2 \tfrac{n+1}{2} \rfloor + \lfloor \log_2 \tfrac{n+1}{3} \rfloor + \lfloor \log_2 \tfrac{n+1}{5} \rfloor + \lfloor \log_2 \tfrac{n+1}{7} \rfloor + 6.$$

Proof. Consider the sequence whose n^{th} term is $bi(L_n)$. We may easily compute the first six terms manually (see for example Figure 1). By Lemmas 2.3 and 2.4, the terms in the sequence increase if and only if n is of one of the four forms of Lemma 2.3. To compute the n^{th} term of our sequence, we count how many times a number of one of these forms appears before n. The total, $\lfloor \log_2 \frac{n+1}{2} \rfloor + \lfloor \log_2 \frac{n+1}{3} \rfloor + \lfloor \log_2 \frac{n+1}{5} \rfloor + \lfloor \log_2 \frac{n+1}{7} \rfloor + 6$, is the birank of L_n. □

3 Prism and Möbius Ladder Graphs

In this section we consider the birank number of the prism graph $Y_n = C_n \square P_2$ and the Möbius ladder graph M_n contructed by adding a twist to Y_n. We show the birank number of both Y_n and M_n are equal to $bi(L_{n-2}) + 4$.

In fact, it is easy to see that given a prism or Möbius ladder graph we may temporarily ignore four vertices that form a square in the graph. What is left then forms the graph L_{n-2}. We may then create a valid biranking on this graph using $k = bi(L_{n-2})$ ranks. Transform this biranking by adding 2 to every rank. Now we may assign to our four ignored vertices the ranks $1, 2, k+3, k+4$ in such a way as to create a low and high divider. We have then shown the following.

Lemma 3.1. *For a prism graph Y_n, $bi(Y_n) \leq bi(L_{n-2}) + 4$. For a Möbius ladder graph M_n, $bi(M_n) \leq bi(L_{n-2}) + 4$.*

To obtain this inequality in the other direction we need a few lemmas. First note that if a rank r appears t times in a valid biranking of a prism or Möbius ladder graph, then there must be t high and t low dividers for r. This is one more than was necessary in a ladder graph since we may now form paths between two vertices in two directions.

The following lemma is analogous to Lemma 2.2 and its proof is essentially the same.

Lemma 3.2. *Let G be either a prism graph Y_n or a Möbius ladder graph M_n. Given a k-biranking on G, and r between 1 and $\lfloor k/2 \rfloor$, the number of vertices with ranks less than $(2r+1)$ is at most 2^r. The number of vertices with ranks more than $(k-2r)$ is at most 2^r.*

Proof. We proceed by induction on n. First, note the ranks $1, 2$ appear at most once and we have our base case.

For $r > 0$, note that there are at most 2^r vertices with ranks less than $(2r+1)$. Now assume for some $t \geq 0$ the rank $(2r+2)$ appears $2^{r-1}+t$ times. So we need exactly this many low dividers for $(2r+2)$ which will require 2^r+2t vertices with ranks less than $(2r+2)$. Since there are only 2^r vertices with ranks less than $(2r+1)$, we must use vertices with the rank $(2r+1)$ in a low dividers for $(2r+2)$ at least $2t$ times. Now a low divider cannot have just one vertex, so a vertex of rank $(2r+1)$ must be paired with a vertex of smaller rank in each of these low dividers. This smaller ranked vertex is then no longer available to act in a low divider for $(2r+1)$. So, of the $2^{r-1}+t$ low dividers for $(2r+2)$, at most $2^{r-1}+t-2t = 2^{r-1}-t$ will be low dividers for $(2r+1)$ and so $(2r+1)$ appears at most $2^{r-1}-t$ times. Then these two ranks together appear at most a total of 2^r times.

Then we see the number of vertices with ranks less than $(2r+3)$ is at most $2^r+2^r = 2^{r+1}$.

The other half of the proof follows from a similar argument counting high dividers. □

Our next lemma is analogous to Lemma 2.3 and its proof is essentially the same using Lemma 3.2.

Lemma 3.3. *Let G be either a prism graph Y_n or a Möbius ladder graph M_n. Given an integer $k \geq 4$, if there is a k-birank on G, then we have an upper bound for n as a function of k as follows:*

$$n \leq \begin{cases} 4 \cdot 2^{\frac{k}{4}-2} & \text{for } k \equiv 0 \mod 4 \\ 5 \cdot 2^{\lfloor \frac{k}{4} \rfloor -2} & \text{for } k \equiv 1 \mod 4 \\ 6 \cdot 2^{\lfloor \frac{k}{4} \rfloor -2} & \text{for } k \equiv 2 \mod 4 \\ 7 \cdot 2^{\lfloor \frac{k}{4} \rfloor -2} & \text{for } k \equiv 3 \mod 4 \end{cases}$$

We now use this result to find an upper bound for the birank of these graphs.

Lemma 3.4. *If G is a prism graph Y_n or a Möbius ladder graph M_n then $bi(G) \geq bi(L_{n-2}) + 4$.*

Proof. Assume $bi(G) = k$; we will show there is a $(k-4)$-birank on L_{n-2}. We have four cases based on the value of $k \mod 4$.

If $k \equiv 0 \mod 4$ then by Lemma 3.2 $n \leq 2^{k/4}$ and so, $n-2 \leq 2^{k/4}-2 = 2^{\frac{k-4}{4}+1}-2$. Thus by Lemma 2.4 there is a $(k-4)$-biranking on L_{n-2}.

The other three cases are proved similarly. □

So Lemmas 3.1 and 3.4 give us our main result for this section:

Theorem 3.5. *$bi(Y_n) = bi(M_n) = bi(L_{n-2}) + 4$.*

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