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Pure Latin directed triple systems

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Abstract

It is well-known that, given a Steiner triple system, a quasigroup can be formed by defining an operation $\cdot$ by the identities $x \cdot x = x$ and $x \cdot y = z$ where z is the third point in the block containing the pair $\{x, y\}$. The same is true for a Mendelsohn triple system where the pair (x, y) is considered to be ordered. But it is not true in general for directed triple systems. However directed triple systems which form quasigroups under this operation do exist and we call these Latin directed triple systems. The quasigroups associated with Steiner and Mendelsohn triple systems satisfy the flexible law $x \cdot (y \cdot x) = (x \cdot y) \cdot x$ but those associated with Latin directed triple systems need not. A directed triple system is said to be pure if when considered as a twofold triple system it contains no repeated blocks. In a previous paper, [*Discrete Math.* 312 (2012), 597–607], we studied non-pure Latin directed triple systems. In this paper we turn our attention to pure non-flexible and pure flexible Latin directed triple systems.

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1 Introduction

In [5], the present authors introduced the concepts of a Latin directed triple system and a DTS-quasigroup and determined their existence spectrum. The latter, an algebraic structure, may be obtained from the former, a combinatorial structure, by a standard procedure explained below. A DTS-quasigroup does not necessarily satisfy the *flexible law*, i.e. $x \cdot (y \cdot x) = (x \cdot y) \cdot x$, and a necessary and sufficient condition for it to do so was also given in [5]. The existence spectrum of flexible DTS-quasigroups was determined in [8]. These systems also possess a certain structure in terms of their topology and this is discussed in [6] and [7]. However in both [5] and [8] the Latin directed triple systems constructed are not *pure*, i.e. when considered as a twofold triple system they contain repeated blocks. Equivalently the DTS-quasigroups are not anti-commutative, i.e. they do not satisfy $x \cdot y = y \cdot x \Rightarrow x = y$. The construction of pure Latin directed triple systems is more challenging than for non-pure systems and the purpose of this paper is to present such constructions both non-flexible and flexible. We are able to adapt some of the methods used for non-pure systems, though greater care must be taken. However most of the approach in this paper uses different techniques. For pure, non-flexible Latin directed triple systems, we are able to determine the existence spectrum completely. For pure, flexible Latin directed triple systems we leave six orders unresolved. These seem to be difficult even with the aid of a computer.

First we recall some definitions. A *Steiner triple system* of order n, STS(n), is a pair $(V, \mathcal{B})$ where V is a set of n points and $\mathcal{B}$ is a collection of triples of distinct points, also called blocks, taken from V such that every pair of distinct points from V appears in precisely one block. Such systems exist if and only if $n \equiv 1$ or $3 \pmod 6$ [14]. A *Steiner quasigroup* or *squag* or *idempotent totally symmetric quasigroup* is a pair $(Q, \cdot)$ where Q is a set and $\cdot$ is an operation on Q satisfying the identities

$$x \cdot x = x, \quad y \cdot (x \cdot y) = x, \quad x \cdot y = y \cdot x.$$

If $(V, \mathcal{B})$ is an STS(n), then a Steiner quasigroup $(Q, \cdot)$ is obtained by letting $Q = V$ and defining $x \cdot y = z$ where $\{x, y, z\} \in \mathcal{B}$. The process is reversible; if Q is a Steiner quasigroup, then a Steiner triple system is obtained by letting $V = Q$ and $\{x, y, z\} \in \mathcal{B}$ where $x \cdot y = z$ for all $x, y \in Q$, $x \neq y$. Thus there is a one-one correspondence between all Steiner triple systems and all Steiner quasigroups [19, Theorem V.1.11]. All Steiner quasigroups satisfy the flexible law.

Next consider ordered triples. There are two possibilities. A *cyclically ordered triple*, denoted by (x, y, z), contains the ordered pairs (x, y), (y, z), (z, x) and a *transitively ordered triple*, denoted by $\langle x, y, z \rangle$ contains the ordered pairs (x, y), (y, z), (x, z).

A *Mendelsohn triple system* of order n, MTS(n), is a pair $(V, \mathcal{B})$ where V is a set of n points and $\mathcal{B}$ is a collection of cyclically ordered triples of distinct points taken from V such that every ordered pair of distinct points from V appears in precisely one triple. Such systems exist if and only if $n \equiv 0$ or $1 \pmod 3$, $n \neq 6$ [18]. Quasigroups can be obtained from Mendelsohn triple systems by precisely the same procedures as described above for Steiner triple systems. Note that the law $y \cdot (x \cdot y) = x$ is usually called semi-symmetric. So the quasigroups are known as *idempotent semisymmetric quasigroups* [2, Remark 2.12] or *Mendelsohn quasigroups*; they satisfy the same properties as their Steiner counterparts with the exception of commutativity. Similarly there is a one-one correspondence between Mendelsohn triple systems and Mendelsohn quasigroups. Again, all Mendelsohn quasigroups satisfy the flexible law.

A *directed triple system* of order n, DTS(n), is a pair $(V, \mathcal{B})$ where V is a set of n points and $\mathcal{B}$ is a collection of transitively ordered triples of distinct points taken from V such that every ordered pair of distinct points from V appears in precisely one triple. Such systems exist if and only if $n \equiv 0$ or 1 (mod 3) [13]. Given a DTS(n), an algebraic structure $(V, \cdot)$ can be obtained as above by defining $x \cdot x = x$ and $x \cdot y = z$ for all $x, y \in V$, $x \neq y$ where z is the third element in the transitive triple containing the ordered pair (x, y). However the structure obtained need not necessarily be a quasigroup. If $\langle u, x, y\rangle$ and $\langle y, v, x\rangle \in \mathcal{B}$ then $u \cdot x = v \cdot x = y$. But some DTS($n$)s do yield quasigroups. Such a DTS(n) will be called a *Latin directed triple system*, denoted by LDTS(n), to reflect the fact that in this case the operation table forms a Latin square. We call the quasigroup so obtained a *DTS-quasigroup*.

In [5] the following two theorems were proved.

Theorem 1.1 *Let $D = (V, \mathcal{B})$ be a DTS(n). Then D is an LDTS(n) if and only if $\langle x, y, z\rangle \in \mathcal{B} \Rightarrow \langle w, y, x\rangle \in \mathcal{B}$ for some $w \in V$.*

Theorem 1.2 *A DTS-quasigroup obtained from an LDTS(n), $D = (V, \mathcal{B})$, satisfies the flexible law if and only if $\langle x, y, z\rangle \in \mathcal{B} \Rightarrow \langle x, z \cdot x, y \cdot x\rangle \in \mathcal{B}$.*

Let $(V, \mathcal{B})$ be a pure LDTS(n). Denote by F, the set of all unordered triples $\{x, y, z\}$, where $\langle x, y, z\rangle$ runs through all triples of $\mathcal{B}$. Now consider F as a set of faces. Each edge $\{x, y\}$ is incident to two faces and hence we get a generalized pseudosurface. By separating pinch points we obtain a set of one or more components which are an invariant of the LDTS(n) and are very useful in determining whether two DTS-quasigroups are isomorphic.

Consider a transitive triple $\langle z_1, x, z_0\rangle \in \mathcal{B}$. Then, using Theorem 1.1, there exists $k \geq 3$ and points z_0, z_1, z_2, …, z_{k-1} such that

$$\langle z_1, x, z_0\rangle, \langle z_2, x, z_1\rangle, \ldots, \langle z_{k-1}, x, z_{k-2}\rangle, \langle z_0, x, z_{k-1}\rangle \in \mathcal{B}.$$

If $(V, \mathcal{B})$ is also flexible, using Theorem 1.2,

$$\langle z_1, y, z_2\rangle, \langle z_2, y, z_3\rangle, \ldots, \langle z_{k-1}, y, z_0\rangle, \langle z_0, y, z_1\rangle \in \mathcal{B}$$

where $y = z_0 \cdot z_1 = z_1 \cdot z_2 = \cdots = z_{k-2} \cdot z_{k-1} = z_{k-1} \cdot z_0$. These $2k$ transitive triples define a k-gonal bipyramid; denoted by O_k, i.e. a graph of $k+2$ vertices with a cycle of length k, the points of which can be thought of as situated around the equator of a sphere, and two middle vertex points which are connected to all points of the cycle and which can be thought of as situated at the poles of the sphere. Thus we have the following important result.

Theorem 1.3 *A pure flexible LDTS(n) exists if and only if the complete graph K_n can be decomposed into k-gonal bipyramid graphs O_k, $k \geq 3$.*

Unlike Steiner and Mendelsohn triple systems and their algebraic counterparts, there is not a one-one correspondence between Latin directed triple systems and DTS-quasigroups. This is because if the LDTS(n) is not pure, then it will contain a pair of triples $\langle x, y, z\rangle$ and $\langle z, y, x\rangle$. Replacing these with the pair of triples $\langle y, z, x\rangle$ and $\langle x, z, y\rangle$ gives a system which yields the same DTS-quasigroup as the first and yet the two LDTS(n)s may be non-isomorphic [5, Example 2.4]. However if the LDTS(n) is pure then this situation does not arise and there is a one-one correspondence between pure Latin directed triple systems and anti-commutative DTS-quasigroups.

2 Recursive constructions

In this section we present some recursive constructions for pure Latin directed triple systems. We start with two elementary recursive constructions adapted from standard design-theoretic techniques and appropriate for our purposes.

Proposition 2.1

(i) If there exists a pure LDTS(n), then there exists a pure LDTS($3n$).

(ii) If there exists a pure LDTS(n), then there exists a pure LDTS($3n-2$).

Proof.

(i) Take three copies of the LDTS(n) on point sets $\{0_i, 1_i, \ldots, (n-1)_i\}$, where $i \in \{0, 1, 2\}$ respectively. Then take two disjoint Latin squares $L(i, j)$ and $M(i, j)$ of order n on the set $\{0, 1, \ldots, n-1\}$ and adjoin all transitive triples $\langle i_0, j_1, L(i, j)_2\rangle$ and $\langle M(i, j)_2, j_1, i_0\rangle$, $0 \leq i \leq n-1$, $0 \leq j \leq n-1$.

(ii) Take three copies of the LDTS(n) on point sets $\{\infty, 0_i, 1_i, \ldots, (n-2)_i\}$, where $i \in \{0, 1, 2\}$ respectively. Then take two disjoint Latin squares $L(i, j)$ and $M(i, j)$ of order $n-1$ on the set $\{0, 1, \ldots, n-2\}$ and adjoin all transitive triples $\langle i_0, j_1, L(i, j)_2\rangle$ and $\langle M(i, j)_2, j_1, i_0\rangle$, $0 \leq i \leq n-2$, $0 \leq j \leq n-2$.

□

We now present some recursive constructions for pure flexible LDTSs. The following is a doubling construction which employs a Hamiltonian decomposition of the complete graph K_{2n+1}. In the proof we represent the k-gonal bipyramids O_k by the notation $[N : E_1, E_2, \ldots, E_k : S]$ where N and S are the poles and $(E_1, E_2, \ldots, E_k)$ is the equator cycle.

Proposition 2.2 *If there exists a pure LDTS($2n$), then there exists a pure LDTS($4n+1$). The LDTS($4n+1$) is flexible if and only if the LDTS($2n$) is flexible.*

Proof. Let $D = (V, \mathcal{B})$ be a pure LDTS($2n$) where $V = \{1, 2, \ldots, 2n\}$ and K_{2n+1} be the complete graph on the set W, disjoint from V. Take a decomposition of K_{2n+1} into n disjoint Hamiltonian cycles H_i, $1 \le i \le n$. For each i, construct a $(2n+1)$-gonal bipyramid $[2i-1 : H_i : 2i]$ and let $\mathcal{B}'$ be the set of transitive triples obtained from these bipyramids. Then $D' = (V \cup W, \mathcal{B} \cup \mathcal{B}')$ is a pure LDTS($4n+1$). Since D' contains D as a subsystem, D' is flexible only if D is flexible. Conversely, whenever D is flexible, D' will be flexible as well, because $\mathcal{B}'$ consists of bipyramid components which satisfy the flexible law. □

Proposition 2.3 *If there exists a pure LDTS($2n$), then there exists a pure LDTS($4n+19$). The LDTS($4n+19$) is flexible if and only if the LDTS($2n$) is flexible.*

Proof. Let $(V, \mathcal{B})$ be a pure LDTS($2n$) where $V = \{\infty_1, \infty_2, \ldots, \infty_{2n}\}$. Construct a set of triples $\mathcal{B}'$ on the point set $\mathbb{Z}_{2n+19} \cup V$ from the following set of starter blocks under the action of the mapping $i \mapsto i+1$ with the elements of V as fixed points.

$$\{\langle 2,0,6\rangle, \langle 6,0,9\rangle, \langle 9,0,2\rangle, \langle 2,1,9\rangle, \langle 9,1,6\rangle, \langle 6,1,2\rangle\} \cup \{\, \langle 0, \infty_r, 9{+}r\rangle : r = 1, \ldots, 2n\}$$

Then $(\mathbb{Z}_{2n+19} \cup V, \mathcal{B} \cup \mathcal{B}')$ is a pure LDTS($4n+19$). To see that the constructed system is flexible whenever $(V, \mathcal{B})$ is flexible, it suffices to check that the triples in $\mathcal{B}'$ define a set of bipyramids satisfying the flexible law. □

Proposition 2.4 *If there exists a pure LDTS($6n+1$), then there exists a pure LDTS($12n+22$). The LDTS($12n+22$) is flexible if and only if the LDTS($6n+1$) is flexible.*

Proof. Let $(V, \mathcal{B})$ be a pure LDTS($6n+1$) where $V = \{\infty_0, \infty_1, \ldots, \infty_{6n}\}$ and let $W = \{\, i_j : \ i \in \mathbb{Z}_{2n+7},\ j = 0, 1, 2\,\}$. Construct a set of triples $\mathcal{B}'$ from the following set of starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$ with the elements of V as fixed points.

$\langle 0_0, 0_2, 1_1\rangle$, $\langle 1_1, 0_2, 2_2\rangle$, $\langle 2_2, 0_2, 0_0\rangle$, $\langle 0_0, 3_2, 2_2\rangle$, $\langle 2_2, 3_2, 1_1\rangle$, $\langle 1_1, 3_2, 0_0\rangle$, $\langle 0_0, 3_0, 1_0\rangle$, $\langle 1_0, 3_0, 0_2\rangle$, $\langle 0_2, 3_0, 2_1\rangle$, $\langle 2_1, 3_0, 0_1\rangle$, $\langle 0_1, 3_0, 0_0\rangle$, $\langle 0_0, 3_1, 0_1\rangle$, $\langle 0_1, 3_1, 2_1\rangle$, $\langle 2_1, 3_1, 0_2\rangle$, $\langle 0_2, 3_1, 1_0\rangle$, $\langle 1_0, 3_1, 0_0\rangle$, $\langle 0_1, 3_2, 0_2\rangle$, $\langle 0_2, 3_2, 2_0\rangle$, $\langle 2_0, 3_2, 0_1\rangle$, $\langle 0_1, \infty_0, 2_0\rangle$, $\langle 0_2, \infty_0, 0_1\rangle$, $\langle 2_0, \infty_0, 0_2\rangle$,

$$\begin{array}{lll}
\langle 0_0, \infty_r, (3+r)_0\rangle, & \langle 0_1, \infty_r, (3+r)_1\rangle, & \langle 0_2, \infty_r, (3+r)_2\rangle, \\
\langle 0_0, \infty_{2n+r}, (3+r)_1\rangle, & \langle 0_1, \infty_{2n+r}, (3+r)_2\rangle, & \langle 0_2, \infty_{2n+r}, (3+r)_0\rangle, \\
\langle (3+r)_1, \infty_{4n+r}, 0_0\rangle, & \langle (3+r)_2, \infty_{4n+r}, 0_1\rangle, & \langle (3+r)_0, \infty_{4n+r}, 0_2\rangle,
\end{array}$$

where $r = 1, \ldots, 2n$. Then $(V \cup W, \mathcal{B} \cup \mathcal{B}')$ is a pure LDTS($12n+22$). The triples in $\mathcal{B}'$ define a set of bipyramids satisfying the flexible law. □

Proposition 2.5 *If there exists a pure LDTS($6n+1$), then there exists a pure LDTS($12n+28$). The LDTS($12n+28$) is flexible if and only if the LDTS($6n+1$) is flexible.*

Proof. Let $(V,\mathcal{B})$ be a pure LDTS$(6n+1)$ where $V = \{\infty_0, \infty_1, \ldots, \infty_{6n}\}$ and let $W = \{\, i_j : i \in \mathbb{Z}_{2n+9},\ j = 0,1,2\,\}$. Construct a set of triples $\mathcal{B}'$ from the following set of starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$ with the elements of V as fixed points.

$\langle 1_1, 0_0, 3_1\rangle$, $\langle 3_1, 0_0, 4_1\rangle$, $\langle 4_1, 0_0, 4_2\rangle$, $\langle 4_2, 0_0, 1_1\rangle$, $\langle 0_2, 3_2, 1_2\rangle$, $\langle 1_2, 3_2, 3_0\rangle$, $\langle 3_0, 3_2, 0_2\rangle$, $\langle 1_2, 0_0, 4_0\rangle$, $\langle 4_0, 0_0, 2_2\rangle$, $\langle 2_2, 0_0, 1_2\rangle$, $\langle 0_1, 3_1, 4_2\rangle$, $\langle 4_2, 3_1, 1_0\rangle$, $\langle 1_0, 3_1, 0_1\rangle$, $\langle 1_1, 0_2, 4_2\rangle$, $\langle 4_2, 0_2, 4_1\rangle$, $\langle 4_1, 0_2, 3_1\rangle$, $\langle 3_1, 0_2, 1_1\rangle$, $\langle 0_1, 4_0, 3_0\rangle$, $\langle 0_0, 1_0, 3_0\rangle$, $\langle 3_0, 1_0, 1_1\rangle$, $\langle 4_1, 4_0, 0_1\rangle$, $\langle 3_0, 2_2, 0_1\rangle$, $\langle 0_1, 2_2, 4_1\rangle$, $\langle 2_1, 0_2, 4_0\rangle$, $\langle 4_0, 0_2, 1_0\rangle$, $\langle 0_1, \infty_0, 1_0\rangle$, $\langle 1_0, \infty_0, 4_2\rangle$, $\langle 4_2, \infty_0, 0_1\rangle$,

$$\begin{array}{lll}
\langle 0_0, \infty_r, (4+r)_0\rangle, & \langle 0_1, \infty_r, (4+r)_1\rangle, & \langle 0_2, \infty_r, (4+r)_2\rangle, \\
\langle 0_0, \infty_{2n+r}, (4+r)_1\rangle, & \langle 0_1, \infty_{2n+r}, (4+r)_2\rangle, & \langle 0_2, \infty_{2n+r}, (4+r)_0\rangle, \\
\langle (4+r)_1, \infty_{4n+r}, 0_0\rangle, & \langle (4+r)_2, \infty_{4n+r}, 0_1\rangle, & \langle (4+r)_0, \infty_{4n+r}, 0_2\rangle,
\end{array}$$

where $r = 1, \ldots, 2n$. Then $(V \cup W, \mathcal{B} \cup \mathcal{B}')$ is a pure LDTS$(12n+28)$. The triples in $\mathcal{B}'$ define a set of bipyramids satisfying the flexible law. □

Proposition 2.6 *If there exists a pure LDTS$(6n+3)$, then there exists a pure LDTS$(12n+24)$. The LDTS$(12n+24)$ is flexible if and only if the LDTS$(6n+3)$ is flexible.*

Proof. Let $(V,\mathcal{B})$ be a pure LDTS$(6n+3)$ where $V = \{\infty_0, \infty_1, \ldots, \infty_{6n+2}\}$ and let $W = \{\, i_j : i \in \mathbb{Z}_{2n+7},\ j = 0,1,2\,\}$. Construct a set of triples $\mathcal{B}'$ from the following set of starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$ with the elements of V as fixed points.

$\langle 0_1, 0_0, 3_1\rangle$, $\langle 3_1, 0_0, 1_1\rangle$, $\langle 2_1, 1_0, 0_0\rangle$, $\langle 0_0, 1_0, 3_0\rangle$, $\langle 2_0, 0_0, 2_2\rangle$, $\langle 2_2, 0_0, 3_2\rangle$, $\langle 3_2, 0_0, 0_1\rangle$, $\langle 0_1, 0_2, 3_2\rangle$, $\langle 3_2, 0_2, 2_2\rangle$, $\langle 2_2, 0_2, 2_0\rangle$, $\langle 3_0, 1_2, 0_0\rangle$, $\langle 0_0, 1_2, 2_1\rangle$, $\langle 1_1, 0_2, 3_1\rangle$, $\langle 3_1, 0_2, 0_1\rangle$, $\langle 1_1, 0_1, 2_2\rangle$, $\langle 2_2, 0_1, 3_0\rangle$, $\langle 3_0, 0_1, 1_1\rangle$, $\langle 0_1, \infty_0, 2_0\rangle$, $\langle 2_0, \infty_0, 1_2\rangle$, $\langle 1_2, \infty_0, 0_1\rangle$, $\langle 0_2, \infty_1, 2_1\rangle$, $\langle 2_1, \infty_1, 3_0\rangle$, $\langle 3_0, \infty_1, 0_2\rangle$, $\langle 0_2, \infty_2, 3_0\rangle$, $\langle 3_0, \infty_2, 2_1\rangle$, $\langle 2_1, \infty_2, 0_2\rangle$,

$$\begin{array}{lll}
\langle 0_0, \infty_{r+2}, (3+r)_0\rangle, & \langle 0_1, \infty_{r+2}, (3+r)_1\rangle, & \langle 0_2, \infty_{r+2}, (3+r)_2\rangle, \\
\langle 0_0, \infty_{2n+r+2}, (3+r)_1\rangle, & \langle 0_1, \infty_{2n+r+2}, (3+r)_2\rangle, & \langle 0_2, \infty_{2n+r+2}, (3+r)_0\rangle, \\
\langle (3+r)_1, \infty_{4n+r+2}, 0_0\rangle, & \langle (3+r)_2, \infty_{4n+r+2}, 0_1\rangle, & \langle (3+r)_0, \infty_{4n+r+2}, 0_2\rangle,
\end{array}$$

where $r = 1, \ldots, 2n$. Then $(V \cup W, \mathcal{B} \cup \mathcal{B}')$ is a pure LDTS$(12n+24)$. The triples in $\mathcal{B}'$ define a set of bipyramids satisfying the flexible law. □

The last recursive construction we present can only be used to produce non-flexible LDTSs.

Proposition 2.7 *If there exists a pure LDTS$(6n+3)$, then there exists a pure non-flexible LDTS$(12n+18)$.*

Proof. Let $(V, \mathcal{B})$ be a pure LDTS$(6n+3)$ where $V = \{\infty_0, \infty_1, \ldots, \infty_{6n+2}\}$ and let $W = \{ i_j : i \in \mathbb{Z}_{2n+5}, j = 0, 1, 2 \}$. Construct a set of triples $\mathcal{B}'$ from the following set of starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$ with the elements of V as fixed points.

$\langle \infty_2, 0_0, 0_2 \rangle$, $\langle 0_0, 0_1, 2_2 \rangle$, $\langle 2_2, 0_1, 1_0 \rangle$, $\langle 1_1, 0_2, 2_2 \rangle$, $\langle 2_1, 0_2, 1_1 \rangle$, $\langle 2_2, 0_2, 2_1 \rangle$, $\langle 1_2, 1_0, 0_2 \rangle$, $\langle 2_0, 1_1, 0_0 \rangle$, $\langle 0_1, 2_0, 0_2 \rangle$, $\langle 0_2, 2_0, \infty_2 \rangle$, $\langle 1_2, 2_0, 0_1 \rangle$, $\langle 0_0, 2_1, 2_0 \rangle$, $\langle 1_0, 2_1, 0_0 \rangle$, $\langle 0_0, \infty_0, 1_0 \rangle$, $\langle 0_1, \infty_0, 1_1 \rangle$, $\langle 0_2, \infty_0, 1_2 \rangle$, $\langle 0_0, \infty_1, 1_2 \rangle$, $\langle 0_1, \infty_1, 2_1 \rangle$, $\langle 2_2, \infty_1, 0_0 \rangle$, $\langle 2_1, \infty_2, 0_1 \rangle$,

$$\begin{array}{lll} \langle 0_0, \infty_{r+2}, (2+r)_0 \rangle, & \langle 0_1, \infty_{r+2}, (2+r)_1 \rangle, & \langle 0_2, \infty_{r+2}, (2+r)_2 \rangle, \\ \langle 0_0, \infty_{2n+r+2}, (2+r)_1 \rangle, & \langle 0_1, \infty_{2n+r+2}, (2+r)_2 \rangle, & \langle 0_2, \infty_{2n+r+2}, (2+r)_0 \rangle, \\ \langle (2+r)_1, \infty_{4n+r+2}, 0_0 \rangle, & \langle (2+r)_2, \infty_{4n+r+2}, 0_1 \rangle, & \langle (2+r)_0, \infty_{4n+r+2}, 0_2 \rangle, \end{array}$$

where $r = 1, \ldots, 2n$. Then $(V \cup W, \mathcal{B} \cup \mathcal{B}')$ is a pure LDTS$(12n+18)$. The triples in $\mathcal{B}'$ do not satisfy the flexible law. For example $(0_0 \cdot 2_1) \cdot 0_0 = 2_0 \cdot 0_0 = 1_1$, whilst $0_0 \cdot (2_1 \cdot 0_0) = 0_0 \cdot 1_0 = \infty_0$. □

3 Pure non-flexible Latin directed triple systems

In this section we determine the existence spectrum of pure non-flexible LDTS(n). It was shown in [5] that there is no pure LDTS(n) for $3 \le n \le 12$. Part of the existence proof in this section uses a standard technique known as Wilson's fundamental construction for which we need the concept of a *group divisible design* (GDD). A 3-GDD of type g^u is an ordered triple $(V, \mathcal{G}, \mathcal{B})$ where V is a base set of cardinality $v = gu$, $\mathcal{G}$ is a partition of V into u subsets of cardinality g called *groups* and $\mathcal{B}$ is a family of triples called *blocks* which collectively have the property that every pair of elements from different groups occur in precisely one block but no pair of elements from the same group occur at all. We will also need 3-GDDs of type $g^u m^1$. These are defined analogously, with the base set V being of cardinality $v = gu + m$ and the partition G being into u subsets of cardinality g and one subset of cardinality m. Necessary and sufficient conditions for 3-GDDs of type g^u were determined in [12] and for 3-GDDs of type $g^u m^1$ in [4]; a convenient reference is [9] where the existence of all the GDDs that are used can be verified.

We will assume that the reader is familiar with this construction but briefly the basic idea is as follows. Begin with a 3-GDD of cardinality $v = gu$ or $gu + m$, usually called the *master GDD*. Each point is then assigned a weight, usually the same weight, say w. In effect, each point is replaced by w points. Each block of the master GDD is then replaced by a 3-GDD of type w^3, called a *slave GDD*. We will only need to use the two values $w = 2$ and $w = 3$, and instead of slave GDDs we will use partial Latin directed triple systems. When $w = 2$ we will employ the partial LDTS(6), say $\mathcal{P}$, whose blocks are $\langle a, b, c \rangle$, $\langle a, y, z \rangle$, $\langle x, b, z \rangle$, $\langle x, y, c \rangle$, $\langle z, y, x \rangle$, $\langle c, b, x \rangle$, $\langle c, y, a \rangle$, $\langle z, b, a \rangle$ and the sets $\{a, x\}$, $\{b, y\}$, $\{c, z\}$ play the role of the groups. As a component of an LDTS(n), it satisfies the flexible law. When $w = 3$ we will use the partial LDTS(9), say $\mathcal{Q}$, whose blocks are $\langle a, p, x \rangle$, $\langle b, q, y \rangle$, $\langle c, r, z \rangle$, $\langle a, q, z \rangle$, $\langle b, r, x \rangle$, $\langle c, p, y \rangle$, $\langle a, r, y \rangle$, $\langle b, p, z \rangle$, $\langle c, q, x \rangle$, $\langle x, q, a \rangle$, $\langle y, r, b \rangle$, $\langle z, p, c \rangle$, $\langle z, r, a \rangle$,

$\langle x,p,b\rangle$, $\langle y,q,c\rangle$, $\langle y,p,a\rangle$, $\langle z,q,b\rangle$, $\langle x,r,c\rangle$ and $\{a,b,c\}$, $\{p,q,r\}$, $\{x,y,z\}$ play the role of the groups. It does not satisfy the flexible law, e.g. $(a\cdot p)\cdot a = x\cdot a = q$ but $a\cdot(p\cdot a) = a\cdot y = r$. To complete the construction we then "fill in" the groups of the expanded master GDD, sometimes adjoining an extra point, say ∞, to all of the groups. Thus we may need pure non-flexible Latin directed triple systems of orders gw, mw, $gw+1$ or $mw+1$ as appropriate. For a more elaborate explanation of this construction see, for example, the proof of Proposition 4.3 in [5].

Type of master GDD	Orders of LDTS(n) needed	Residue classes covered modulo 36	Missing values
6^s, $s\geq 3$	13	1, 13, 25	25
9^{2s+1}, $s\geq 1$	19	19	
$9^{2s}\,15^1$, $s\geq 2$	19, 31	31	67
$9^{2s}\,21^1$, $s\geq 2$	19, 43	7	79

Table 1: Schema for pure non-flexible LDTS(n), $n\equiv 1 \pmod 6$.

Schema of the master GDDs and Latin directed triple systems needed to construct pure non-flexible LDTS(n) for $n\equiv 1 \pmod 6$ is given in Table 1. We always weight with 2 and adjoin an extra point ∞. Pure non-flexible LDTS(n) for $n = 13$, 19, 25 and 31 are given as Examples NO.1, NO.3, NO.5 and NO.7 respectively in the Appendix. The LDTSs of orders $n = 43$ and 79 can be constructed using part (ii) of Proposition 2.1 from the LDTS(15) and LDTS(27) which are given as Examples NO.2 and NO.6 in the Appendix. This just leaves the value $n = 67$ which can be constructed using a master GDD of type $4^4\,6^1$ or $6^3\,4^1$, assigning weight 3, adjoining the point ∞ and using the pure non-flexible LDTS(13) and LDTS(19).

We can now use these systems to construct pure non-flexible LDTS(n) of order $n\equiv 4 \pmod 6$. By Proposition 2.5 there exists a pure non-flexible LDTS(n) for all $n\equiv 4 \pmod{12}$, $n\geq 52$. Pure non-flexible systems of orders 16, 28 and 40 are given as Examples NE.1, NE.5 and NE.9 in the Appendix. By Proposition 2.4 there exists a pure non-flexible LDTS(n) for all $n\equiv 10 \pmod{12}$, $n\geq 46$. Pure non-flexible systems of orders 22 and 34 are given as Examples NE.3 and NE.7 in the Appendix.

These systems may in turn be used to construct pure non-flexible LDTS(n) of order $n\equiv 3 \pmod 6$. By Proposition 2.2 there exists a pure non-flexible LDTS(n) for all $n\equiv 9 \pmod{12}$, $n\geq 33$. A pure non-flexible LDTS(21) is given as Example NO.4 in the Appendix. By Proposition 2.3 there exists a pure non-flexible LDTS(n) for all $n\equiv 3 \pmod{12}$, $n\geq 51$. Pure non-flexible systems of orders 15 and 27 are given as Examples NO.2 and NO.6 in the Appendix and a pure non-flexible LDTS(39) can be constructed from the LDTS(13) using part (i) of Proposition 2.1.

Finally we construct pure non-flexible LDTS(n) of order $n\equiv 0 \pmod 6$. By Proposition 2.6 there exists a pure non-flexible LDTS(n) for all $n\equiv 0 \pmod{12}$, $n\geq 48$. Pure non-flexible systems of orders 24 and 36 are given as Examples NE.4 and NE.8 in the Appendix. By Proposition 2.7 there exists a pure non-flexible LDTS(n) for all $n\equiv 6 \pmod{12}$, $n\geq 42$. Pure non-flexible systems of orders 18 and 30 are given as Examples NE.2 and NE.6 in the Appendix.

Collecting all the results together gives the following theorem.

Theorem 3.1 *The existence spectrum of pure non-flexible LDTS(n)s is* $n \equiv 0$ *or* 1 (mod 3), $n \geq 13$.

4 Pure flexible Latin directed triple systems

In this section we discuss the existence of pure flexible Latin directed triple systems. The further requirement of flexibility adds another constraint to the constructions. We are still able, and indeed do, use Wilson's fundamental construction but we cannot use weight $w = 3$ and the partial Latin directed triple system $\mathcal{Q}$ because as shown in the previous section it does not satisfy the flexible law. Another difficulty is that, as was shown in [5], there is no pure flexible LDTS(n) for $3 \leq n \leq 15$ or $n = 18$. In particular there is no pure flexible LDTS(13) which was very useful in the non-flexible case. However if the above factors are against us, then we do have a feature of pure flexible LDTS(n) to help us. This is their geometric structure as described in Theorem 1.3.

In the case where all the bipyramids have $k = 3$, this is a decomposition of K_n into graphs K_5 but missing one edge, so-called $(K_5 \setminus e)$-designs. The spectrum of n for which these designs exist has been fully determined [10, 15, 16, 20], see also [3]. It is $n \equiv 0$ or 1 (mod 9), $n \geq 19$. When all the bipyramids have $k = 4$, this is a decomposition of K_n into Pasch configurations. The spectrum of n for which this is true has also been determined [11, 1]. It is $n \equiv 1$ or 9 (mod 24), $n \geq 25$.

We first complete the existence spectrum for the residue class $n \equiv 1 \pmod 6$. Table 2 gives the schema for $n \equiv 7$ or 13 (mod 18). We use weight $w = 2$ and adjoin an extra point. The pure flexible LDTS(n)s of orders $n \equiv 1 \pmod{18}$ can

Type of master GDD	Orders of LDTS(n) needed	Residue classes covered modulo 36	Missing values
$9^{2s}\,15^1$, $s \geq 2$	19, 31	31	67
$9^{2s}\,21^1$, $s \geq 2$	19, 43	7	79
$18^{s}\,12^1$, $s \geq 3$	25, 37	25	61, 97
$18^{s}\,24^1$, $s \geq 3$	37, 49	13	85, 121

Table 2: Schema for pure flexible LDTS(n), $n \equiv 7$ or 13 (mod 18).

be constructed from $(K_5 \setminus e)$-designs, this includes, in particular, the LDTS(19) and LDTS(37). Pure flexible LDTS(n)s for $n = 31$, 43 and 67 are given as Examples FO.2, FO.4 and FO.5 respectively in the Appendix. For $n = 25$, 49, 97 and 121 we can use the decompositions of K_n into Pasch configurations. The remaining missing systems can be obtained using 3-GDD constructions. For $n = 61$ use type 10^3, for $n = 79$ use type 13^3 and for $n = 85$ use type $10^3\,12^1$. To do this we need systems of orders 21, 25 and 27. A pure flexible LDTS(21) is given as Example FO.1 and the pure flexible LDTS(27) can be constructed from a $(K_5 \setminus e)$-design.

We next consider the residue class $n \equiv 4 \pmod 6$. By Proposition 2.4 there exists a pure flexible LDTS(n) for $n = 22$ and for all $n \equiv 10 \pmod{12}$, $n \geq 58$. An LDTS(34) is given as Example FE.1 in the Appendix and an LDTS(46) can be constructed from a $(K_5 \setminus e)$-design. By Proposition 2.5 there exists a pure flexible LDTS(n) for $n = 28$ and for all $n \equiv 4 \pmod{12}$, $n \geq 64$. A pure flexible LDTS(16) is given in [5, Example 3.9]. Systems of orders $n = 40$ and 52 are given as Examples FE.2 and FE.3 respectively in the Appendix.

The results for $n \equiv 4 \pmod 6$ now enable us to deal with the residue class $n \equiv 3 \pmod 6$. By Proposition 2.2 there exists a pure flexible LDTS(n) for all $n \equiv 9 \pmod{12}$, $n \geq 33$. A pure flexible LDTS(21) is given as Example FO.1 in the Appendix. By Proposition 2.3 there exists a pure flexible LDTS(n) for all $n \equiv 3 \pmod{12}$, $n \geq 51$. A pure flexible LDTS(27) can be constructed from a $(K_5 \setminus e)$-design and a pure flexible LDTS(39) is given as Example FO.3 in the Appendix.

This just leaves the residue class $n \equiv 0 \pmod 6$ to consider. By Proposition 2.6 there exists a pure flexible LDTS(n) for all $n \equiv 0 \pmod{12}$, $n \geq 60$. A pure flexible LDTS(36) can be constructed from a $(K_5 \setminus e)$-design and a pure flexible LDTS(48) can be obtained using a 3-GDD of type 8^3. This leaves $n = 24$ unresolved. Table 3 gives

Type of master GDD	Orders of LDTS(n) needed	Residue classes covered modulo 108	Missing values
$27^{2s}\,33^1$, $s \geq 2$	54, 66	66	174
$27^{2s}\,51^1$, $s \geq 2$	54, 102	102	210
$27^{2s}\,69^1$, $s \geq 2$	54, 138	30	30, 246
$27^{2s}\,93^1$, $s \geq 3$	54, 186	78	78, 294, 402
$27^{2s}\,111^1$, $s \geq 3$	54, 222	6	114, 330, 438
$27^{2s}\,129^1$, $s \geq 3$	54, 258	42	42, 150, 366, 474

Table 3: Schema for pure flexible LDTS(n), $n \equiv 6$ or $30 \pmod{36}$.

the schema for $n \equiv 6$ or $30 \pmod{36}$. Again we use weight $w = 2$. The pure flexible LDTS(n)s of orders $n \equiv 18 \pmod{36}$ can be constructed from $(K_5 \setminus e)$-designs, this includes, in particular, the LDTS(54). Pure flexible LDTSs of the following orders can be constructed using 3-GDDs: 66 (use 11^3), 102 (use 17^3), 138 (use 23^3), 174 (use 29^3), 186 (use $11^6\,27^1$), 210 (use $11^8\,17^1$), 222 (use $11^6\,45^1$), 246 (use 41^3), 258 (use $17^6\,27^1$), 294 (use $17^6\,45^1$), 330 (use 11^{15}), 366 (use $23^6\,45^1$), 402 (use $23^6\,63^1$), 438 (use $23^6\,81^1$) and 474 (use $17^{12}\,33^1$). This leaves $n = 30$, 42, 78, 114 and 150 unresolved.

Collecting all the results together gives the following theorem.

Theorem 4.1 *A pure flexible LDTS(n) exists for all* $n \equiv 0$ *or* $1 \pmod 3$, $n \geq 16$ *and* $n \neq 18$, *possibly except* $n = 24$, 30, 42, 78, 114 *and* 150.

Appendix. Examples of pure LDTSs

The following examples were obtained by computer with the help of the model builder Mace4 [17] using an algebraic description of a DTS-quasigroup, see [6]. We denote the elements $(i, j) \in \mathbb{Z}_m \times \mathbb{Z}_n$ as i_j. For simplicity, we omit commas from the triples.

Example NO.1 *Pure non-flexible LDTS(13).*
$V = \mathbb{Z}_{13}$.
The triples are obtained from the following starter blocks under the action of the mapping $i \mapsto i + 1$.
$\langle 1\,0\,5\rangle$, $\langle 5\,0\,7\rangle$, $\langle 7\,0\,3\rangle$, $\langle 3\,0\,1\rangle$.
The system is non-flexible, for example $(0\cdot 2)\cdot 0 = 8\cdot 0 = 9$, *whilst* $0\cdot(2\cdot 0) = 0\cdot 12 = 4$.

Example NO.2 *Pure non-flexible LDTS(15).*
$V = (\mathbb{Z}_7 \times \mathbb{Z}_2) \cup \{\infty\}$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i + 1)_j$.
$\langle 2_0\,0_0\,2_1\rangle$, $\langle 2_1\,0_0\,1_1\rangle$, $\langle 1_1\,0_0\,5_1\rangle$, $\langle 5_1\,0_0\,3_1\rangle$, $\langle 3_1\,0_0\,4_1\rangle$, $\langle 4_1\,0_0\,6_1\rangle$, $\langle 6_1\,0_0\,6_0\rangle$, $\langle 6_0\,0_0\,2_0\rangle$, $\langle 0_0\,\infty\,4_0\rangle$, $\langle 0_1\,\infty\,3_1\rangle$.
The system is non-flexible, for example $(0_0\cdot 1_0)\cdot 0_0 = 3_0\cdot 0_0 = \infty$, *whilst* $0_0\cdot(1_0\cdot 0_0) = 0_0\cdot 0_1 = 5_0$.

Example NO.3 *Pure non-flexible LDTS(19).*
$V = \mathbb{Z}_{19}$.
The triples are obtained from the following starter blocks under the action of the mapping $i \mapsto i + 1$.
$\langle 1, 0, 5\rangle$, $\langle 5, 0, 11\rangle$, $\langle 11, 0, 7\rangle$, $\langle 7, 0, 9\rangle$, $\langle 9, 0, 3\rangle$, $\langle 3, 0, 1\rangle$.
The system is non-flexible, for example $(0\cdot 6)\cdot 0 = 14\cdot 0 = 15$, *whilst* $0\cdot(6\cdot 0) = 0\cdot 16 = 17$.

Example NO.4 *Pure non-flexible LDTS(21).*
$V = \mathbb{Z}_7 \times \mathbb{Z}_3$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i + 1)_j$.
$\langle 2_0\,0_0\,2_1\rangle$, $\langle 2_1\,0_0\,6_1\rangle$, $\langle 6_1\,0_0\,6_0\rangle$, $\langle 6_0\,0_0\,2_0\rangle$, $\langle 2_0\,0_1\,6_0\rangle$, $\langle 6_0\,0_1\,6_2\rangle$, $\langle 6_2\,0_1\,5_1\rangle$, $\langle 5_1\,0_1\,2_0\rangle$, $\langle 2_0\,0_2\,5_1\rangle$, $\langle 5_1\,0_2\,4_1\rangle$, $\langle 4_1\,0_2\,0_1\rangle$, $\langle 0_1\,0_2\,3_0\rangle$, $\langle 3_0\,0_2\,6_2\rangle$, $\langle 6_2\,0_2\,6_0\rangle$, $\langle 6_0\,0_2\,5_2\rangle$, $\langle 5_2\,0_2\,2_0\rangle$, $\langle 6_0\,1_2\,3_1\rangle$, $\langle 3_1\,1_2\,4_1\rangle$, $\langle 4_1\,1_2\,5_2\rangle$, $\langle 5_2\,1_2\,6_0\rangle$.
The system is non-flexible, for example $(6_2\cdot 6_0)\cdot 6_2 = 0_2\cdot 6_2 = 3_0$, *whilst* $6_2\cdot(6_0\cdot 6_2) = 6_2\cdot 0_1 = 5_1$.

Example NO.5 *Pure non-flexible LDTS(25).*
$V = \mathbb{Z}_{25}$.
The triples are obtained from the following starter blocks under the action of the mapping $i \mapsto i + 1$.
$\langle 1, 0, 5\rangle$, $\langle 5, 0, 16\rangle$, $\langle 16, 0, 12\rangle$, $\langle 12, 0, 19\rangle$, $\langle 19, 0, 8\rangle$, $\langle 8, 0, 10\rangle$, $\langle 10, 0, 3\rangle$, $\langle 3, 0, 1\rangle$.
The system is non-flexible, for example $(0\cdot 2)\cdot 0 = 17\cdot 0 = 11$, *whilst* $0\cdot(2\cdot 0) = 0\cdot 24 = 4$.

Example NO.6 *Pure non-flexible LDTS(27).*
$V = (\mathbb{Z}_{13} \times \mathbb{Z}_2) \cup \{\infty\}$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 1_0\, 0_0\, 5_0\rangle$, $\langle 5_0\, 0_0\, 0_1\rangle$, $\langle 0_1\, 0_0\, 12_1\rangle$, $\langle 12_1\, 0_0\, 3_1\rangle$, $\langle 3_1\, 0_0\, 11_1\rangle$, $\langle 11_1\, 0_0\, 1_1\rangle$, $\langle 1_1\, 0_0\, 7_1\rangle$, $\langle 7_1\, 0_0\, 9_1\rangle$, $\langle 9_1\, 0_0\, 6_1\rangle$, $\langle 6_1\, 0_0\, 2_1\rangle$, $\langle 2_1\, 0_0\, 7_0\rangle$, $\langle 7_0\, 0_0\, 3_0\rangle$, $\langle 3_0\, 0_0\, 1_0\rangle$, $\langle 5_1\, 1_0\, 6_1\rangle$, $\langle 6_1\, 1_0\, 11_1\rangle$, $\langle 11_1\, 1_0\, 5_1\rangle$, $\langle 0_0\, \infty\, 2_0\rangle$, $\langle 0_1\, \infty\, 11_1\rangle$.
The system is non-flexible, for example $(0_0 \cdot 2_0) \cdot 0_0 = \infty \cdot 0_0 = 11_0$, *whilst* $0_0 \cdot (2_0 \cdot 0_0) = 0_0 \cdot 12_0 = 4_0$.

Example NO.7 *Pure non-flexible LDTS(31).*
$V = \mathbb{Z}_{31}$.
The triples are obtained from the following starter blocks under the action of the mapping $i \mapsto i+1$.
$\langle 1, 0, 5\rangle$, $\langle 5, 0, 19\rangle$, $\langle 19, 0, 10\rangle$, $\langle 10, 0, 18\rangle$, $\langle 18, 0, 20\rangle$, $\langle 20, 0, 6\rangle$, $\langle 6, 0, 15\rangle$, $\langle 15, 0, 7\rangle$, $\langle 7, 0, 3\rangle$, $\langle 3, 0, 1\rangle$.
The system is non-flexible, for example $(0 \cdot 2) \cdot 0 = 13 \cdot 0 = 23$, *whilst* $0 \cdot (2 \cdot 0) = 0 \cdot 30 = 4$.

Example NE.1 *Pure non-flexible LDTS(16).*
$V = \mathbb{Z}_8 \times \mathbb{Z}_2$.
The triples are obtained from the following starter blocks under the action of the mappings $i_j \mapsto (i+1)_j$ *and* $i_j \mapsto i_{j+1}$.
$\langle 2_0, 0_0, 6_1\rangle$, $\langle 6_1, 0_0, 3_1\rangle$, $\langle 3_1, 0_0, 7_1\rangle$, $\langle 7_1, 0_0, 7_0\rangle$, $\langle 7_0, 0_0, 2_0\rangle$.
The system is non-flexible, for example $(7_1 \cdot 7_0) \cdot 7_1 = 0_0 \cdot 7_1 = 3_1$, *whilst* $7_1 \cdot (7_0 \cdot 7_1) = 7_1 \cdot 0_1 = 2_1$.

Example NE.2 *Pure non-flexible LDTS(18).*
$V = \mathbb{Z}_3 \times \mathbb{Z}_6$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 1_0\, 0_0\, 0_2\rangle$, $\langle 0_2\, 0_0\, 0_1\rangle$, $\langle 0_1\, 0_0\, 1_0\rangle$, $\langle 1_0\, 2_1\, 0_1\rangle$, $\langle 0_1\, 2_1\, 0_5\rangle$, $\langle 0_5\, 2_1\, 1_2\rangle$, $\langle 1_2\, 2_1\, 0_2\rangle$, $\langle 0_2\, 2_1\, 1_0\rangle$, $\langle 0_1\, 0_3\, 2_5\rangle$, $\langle 2_5\, 0_3\, 0_5\rangle$, $\langle 0_5\, 0_3\, 0_1\rangle$, $\langle 0_1\, 0_4\, 0_2\rangle$, $\langle 0_2\, 0_4\, 2_5\rangle$, $\langle 2_5\, 0_4\, 0_1\rangle$, $\langle 0_2\, 0_5\, 2_3\rangle$, $\langle 2_3\, 0_5\, 2_2\rangle$, $\langle 2_2\, 0_5\, 0_2\rangle$, $\langle 1_2\, 0_0\, 1_3\rangle$, $\langle 1_3\, 0_0\, 0_4\rangle$, $\langle 0_4\, 0_0\, 0_5\rangle$, $\langle 0_5\, 0_0\, 2_4\rangle$, $\langle 2_4\, 0_0\, 2_3\rangle$, $\langle 2_3\, 0_0\, 0_3\rangle$, $\langle 0_3\, 0_0\, 1_2\rangle$, $\langle 0_4\, 2_0\, 1_5\rangle$, $\langle 1_5\, 2_0\, 0_5\rangle$, $\langle 0_5\, 2_0\, 0_4\rangle$, $\langle 1_3\, 0_1\, 2_4\rangle$, $\langle 2_4\, 0_1\, 1_4\rangle$, $\langle 1_4\, 0_1\, 2_3\rangle$, $\langle 2_3\, 0_1\, 1_3\rangle$, $\langle 1_3\, 0_2\, 1_4\rangle$, $\langle 1_4\, 0_2\, 2_4\rangle$, $\langle 2_4\, 0_2\, 1_3\rangle$.
The system is non-flexible, for example $(2_5 \cdot 0_1) \cdot 2_5 = 0_4 \cdot 2_5 = 0_2$, *whilst* $2_5 \cdot (0_1 \cdot 2_5) = 2_5 \cdot 0_3 = 0_5$.

Example NE.3 *Pure non-flexible LDTS(22).*
$V = \mathbb{Z}_{11} \times \mathbb{Z}_2$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 1_0, 0_0, 5_0\rangle$, $\langle 5_0, 0_0, 10_1\rangle$, $\langle 10_1, 0_0, 6_1\rangle$, $\langle 6_1, 0_0, 7_1\rangle$, $\langle 7_1, 0_0, 0_1\rangle$, $\langle 0_1, 0_0, 3_0\rangle$, $\langle 3_0, 0_0, 1_0\rangle$, $\langle 2_0, 0_1, 9_0\rangle$, $\langle 9_0, 0_1, 6_1\rangle$, $\langle 6_1, 0_1, 2_0\rangle$, $\langle 8_0, 0_1, 10_0\rangle$, $\langle 10_0, 0_1, 3_1\rangle$, $\langle 3_1, 0_1, 2_1\rangle$, $\langle 2_1, 0_1, 8_0\rangle$.
The system is non-flexible, for example $(0_0 \cdot 2_0) \cdot 0_0 = 3_1 \cdot 0_0 = 5_1$, *whilst* $0_0 \cdot (2_0 \cdot 0_0) = 0_0 \cdot 10_0 = 4_0$.

Example NE.4 *Pure non-flexible LDTS(24).*
$V = \mathbb{Z}_4 \times \mathbb{Z}_6$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 0_2\, 0_0\, 2_2\rangle$, $\langle 2_2\, 0_0\, 2_3\rangle$, $\langle 2_3\, 0_0\, 3_3\rangle$, $\langle 3_3\, 0_0\, 0_4\rangle$, $\langle 0_4\, 0_0\, 3_4\rangle$, $\langle 3_4\, 0_0\, 0_5\rangle$, $\langle 0_5\, 0_0\, 3_5\rangle$, $\langle 3_5\, 0_0\, 1_5\rangle$, $\langle 1_5\, 0_0\, 2_5\rangle$, $\langle 2_5\, 0_0\, 1_4\rangle$, $\langle 1_4\, 0_0\, 2_4\rangle$, $\langle 2_4\, 0_0\, 1_3\rangle$, $\langle 1_3\, 0_0\, 0_3\rangle$, $\langle 0_3\, 0_0\, 1_2\rangle$, $\langle 1_2\, 0_0\, 0_2\rangle$, $\langle 3_1\, 0_1\, 1_4\rangle$, $\langle 1_4\, 0_1\, 3_4\rangle$, $\langle 3_4\, 0_1\, 3_3\rangle$, $\langle 3_3\, 0_1\, 3_5\rangle$, $\langle 3_5\, 0_1\, 2_3\rangle$, $\langle 2_3\, 0_1\, 0_3\rangle$, $\langle 0_3\, 0_1\, 3_1\rangle$, $\langle 0_0\, 1_1\, 3_0\rangle$, $\langle 3_0\, 1_1\, 1_0\rangle$, $\langle 1_0\, 1_1\, 2_0\rangle$, $\langle 2_0\, 1_1\, 1_2\rangle$, $\langle 1_2\, 1_1\, 1_4\rangle$, $\langle 1_4\, 1_1\, 2_2\rangle$, $\langle 2_2\, 1_1\, 3_2\rangle$, $\langle 3_2\, 1_1\, 0_0\rangle$, $\langle 1_3\, 0_2\, 1_4\rangle$, $\langle 1_4\, 0_2\, 1_5\rangle$, $\langle 1_5\, 0_2\, 1_3\rangle$, $\langle 3_2\, 1_3\, 3_5\rangle$, $\langle 3_5\, 1_3\, 3_4\rangle$, $\langle 3_4\, 1_3\, 3_2\rangle$, $\langle 3_2\, 1_4\, 2_3\rangle$, $\langle 2_3\, 1_4\, 3_5\rangle$, $\langle 3_5\, 1_4\, 3_2\rangle$, $\langle 3_1\, 0_5\, 2_2\rangle$, $\langle 2_2\, 0_5\, 1_4\rangle$, $\langle 1_4\, 0_5\, 3_1\rangle$, $\langle 1_1\, 1_5\, 2_3\rangle$, $\langle 2_3\, 1_5\, 2_2\rangle$, $\langle 2_2\, 1_5\, 3_1\rangle$, $\langle 3_1\, 1_5\, 1_1\rangle$.
The system is non-flexible, for example $(3_5 \cdot 1_4) \cdot 3_5 = 3_2 \cdot 3_5 = 1_3$, *whilst* $3_5 \cdot (1_4 \cdot 3_5) = 3_5 \cdot 2_3 = 0_1$.

Example NE.5 *Pure non-flexible LDTS(28).*
$V = \mathbb{Z}_{14} \times \mathbb{Z}_2$.
The triples are obtained from the following starter blocks under the action of the mappings $i_j \mapsto (i+1)_j$ *and* $i_j \mapsto i_{j+1}$.
$\langle 1_0, 0_0, 5_0\rangle$, $\langle 5_0, 0_0, 12_1\rangle$, $\langle 12_1, 0_0, 4_1\rangle$, $\langle 4_1, 0_0, 6_1\rangle$, $\langle 6_1, 0_0, 13_1\rangle$, $\langle 13_1, 0_0, 9_1\rangle$, $\langle 9_1, 0_0, 3_1\rangle$, $\langle 3_1, 0_0, 3_0\rangle$, $\langle 3_0, 0_0, 1_0\rangle$.
The system is non-flexible, for example $(3_1 \cdot 3_0) \cdot 3_1 = 0_0 \cdot 3_1 = 9_1$, *whilst* $3_1 \cdot (3_0 \cdot 3_1) = 3_1 \cdot 0_1 = 1_1$.

Example NE.6 *Pure non-flexible LDTS(30).*
$V = \mathbb{Z}_5 \times \mathbb{Z}_6$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 0_5\, 0_0\, 3_5\rangle$, $\langle 3_5\, 0_0\, 4_5\rangle$, $\langle 4_5\, 0_0\, 1_5\rangle$, $\langle 1_5\, 0_0\, 0_5\rangle$, $\langle 0_0\, 0_1\, 1_0\rangle$, $\langle 1_0\, 0_1\, 4_0\rangle$, $\langle 4_0\, 0_1\, 3_0\rangle$, $\langle 3_0\, 0_1\, 0_0\rangle$, $\langle 3_1\, 0_0\, 4_2\rangle$, $\langle 4_2\, 0_0\, 1_2\rangle$, $\langle 1_2\, 0_0\, 0_2\rangle$, $\langle 0_2\, 0_0\, 3_2\rangle$, $\langle 3_2\, 0_0\, 3_1\rangle$, $\langle 3_1\, 0_4\, 3_2\rangle$, $\langle 3_2\, 0_4\, 4_2\rangle$, $\langle 4_2\, 0_4\, 3_1\rangle$, $\langle 2_2\, 0_0\, 3_3\rangle$, $\langle 3_3\, 0_0\, 4_3\rangle$, $\langle 4_3\, 0_0\, 4_4\rangle$, $\langle 4_4\, 0_0\, 1_4\rangle$, $\langle 1_4\, 0_0\, 0_4\rangle$, $\langle 0_4\, 0_0\, 3_4\rangle$, $\langle 3_4\, 0_0\, 2_5\rangle$, $\langle 2_5\, 0_0\, 2_4\rangle$, $\langle 2_4\, 0_0\, 1_3\rangle$, $\langle 1_3\, 0_0\, 0_3\rangle$, $\langle 0_3\, 0_0\, 2_3\rangle$, $\langle 2_3\, 0_0\, 2_2\rangle$, $\langle 3_1\, 0_1\, 1_4\rangle$, $\langle 1_4\, 0_1\, 2_2\rangle$, $\langle 2_2\, 0_1\, 2_5\rangle$, $\langle 2_5\, 0_1\, 3_1\rangle$, $\langle 4_1\, 0_1\, 3_5\rangle$, $\langle 3_5\, 0_1\, 0_4\rangle$, $\langle 0_4\, 0_1\, 0_3\rangle$, $\langle 0_3\, 0_1\, 4_2\rangle$, $\langle 4_2\, 0_1\, 4_3\rangle$, $\langle 4_3\, 0_1\, 2_3\rangle$, $\langle 2_3\, 0_1\, 4_1\rangle$, $\langle 2_1\, 0_2\, 2_5\rangle$, $\langle 2_5\, 0_2\, 3_4\rangle$, $\langle 3_4\, 0_2\, 1_5\rangle$, $\langle 1_5\, 0_2\, 0_4\rangle$, $\langle 0_4\, 0_2\, 2_1\rangle$, $\langle 4_1\, 0_3\, 3_4\rangle$, $\langle 3_4\, 0_3\, 4_4\rangle$, $\langle 4_4\, 0_3\, 0_5\rangle$, $\langle 0_5\, 0_3\, 1_2\rangle$, $\langle 1_2\, 0_3\, 4_5\rangle$, $\langle 4_5\, 0_3\, 4_1\rangle$, $\langle 2_2\, 0_3\, 1_5\rangle$, $\langle 1_5\, 0_3\, 3_2\rangle$, $\langle 3_2\, 0_3\, 2_4\rangle$, $\langle 2_4\, 0_3\, 2_5\rangle$, $\langle 2_5\, 0_3\, 2_2\rangle$, $\langle 4_1\, 0_5\, 2_3\rangle$, $\langle 2_3\, 0_5\, 3_4\rangle$, $\langle 3_4\, 0_5\, 4_1\rangle$.
The system is non-flexible, for example $(2_5 \cdot 3_4) \cdot 2_5 = 0_2 \cdot 2_5 = 2_1$, *whilst* $2_5 \cdot (3_4 \cdot 2_5) = 2_5 \cdot 0_0 = 2_4$.

Example NE.7 *Pure non-flexible LDTS(34).*
$V = \mathbb{Z}_{17} \times \mathbb{Z}_2$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 1_0, 0_0, 5_0\rangle$, $\langle 5_0, 0_0, 7_0\rangle$, $\langle 7_0, 0_0, 3_0\rangle$, $\langle 3_0, 0_0, 1_0\rangle$, $\langle 6_0, 0_0, 1_1\rangle$, $\langle 1_1, 0_0, 8_0\rangle$, $\langle 8_0, 0_0, 2_1\rangle$, $\langle 2_1, 0_0, 7_1\rangle$, $\langle 7_1, 0_0, 5_1\rangle$, $\langle 5_1, 0_0, 0_1\rangle$, $\langle 0_1, 0_0, 6_0\rangle$, $\langle 3_0, 1_1, 7_1\rangle$, $\langle 7_1, 1_1, 4_0\rangle$, $\langle 4_0, 1_1, 14_1\rangle$, $\langle 14_1, 1_1, 10_0\rangle$, $\langle 10_0, 1_1, 9_1\rangle$, $\langle 9_1, 1_1, 3_0\rangle$, $\langle 5_0, 1_1, 8_1\rangle$, $\langle 8_1, 1_1, 9_0\rangle$, $\langle 9_0, 1_1, 15_1\rangle$, $\langle 15_1, 1_1, 0_1\rangle$, $\langle 0_1, 1_1, 5_0\rangle$.

The system is non-flexible, for example $(0_0 \cdot 2_0) \cdot 0_0 = 12_0 \cdot 0_0 = 13_0$*, whilst* $0_0 \cdot (2_0 \cdot 0_0) = 0_0 \cdot 16_0 = 4_0$.

Example NE.8 *Pure non-flexible LDTS(36).*
$V = (\mathbb{Z}_7 \times \mathbb{Z}_5) \cup \{\infty\}$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 2_0\, 0_0\, 2_1\rangle$, $\langle 2_1\, 0_0\, 6_1\rangle$, $\langle 6_1\, 0_0\, 6_0\rangle$, $\langle 6_0\, 0_0\, 2_0\rangle$, $\langle 2_0\, 0_1\, 6_0\rangle$, $\langle 6_0\, 0_1\, 1_2\rangle$, $\langle 1_2\, 0_1\, 5_2\rangle$, $\langle 5_2\, 0_1\, 5_1\rangle$, $\langle 5_1\, 0_1\, 2_0\rangle$, $\langle 3_0\, 0_3\, 4_4\rangle$, $\langle 4_4\, 0_3\, 5_2\rangle$, $\langle 5_2\, 0_3\, 3_0\rangle$, $\langle 0_2\, 0_4\, 1_3\rangle$, $\langle 1_3\, 0_4\, 4_2\rangle$, $\langle 4_2\, 0_4\, 0_2\rangle$, $\langle 4_1\, 0_0\, 4_2\rangle$, $\langle 4_2\, 0_0\, 0_3\rangle$, $\langle 0_3\, 0_0\, 5_4\rangle$, $\langle 5_4\, 0_0\, 2_4\rangle$, $\langle 2_4\, 0_0\, 0_2\rangle$, $\langle 0_2\, 0_0\, 4_1\rangle$, $\langle 2_3\, 1_1\, 3_4\rangle$, $\langle 3_4\, 1_1\, 1_4\rangle$, $\langle 1_4\, 1_1\, 4_4\rangle$, $\langle 4_4\, 1_1\, 5_4\rangle$, $\langle 5_4\, 1_1\, 0_4\rangle$, $\langle 0_4\, 1_1\, 6_4\rangle$, $\langle 6_4\, 1_1\, \infty\rangle$, $\langle \infty\, 1_1\, 2_4\rangle$, $\langle 2_4\, 1_1\, 2_3\rangle$, $\langle 2_0\, 0_2\, 5_1\rangle$, $\langle 5_1\, 0_2\, 1_1\rangle$, $\langle 1_1\, 0_2\, 0_3\rangle$, $\langle 0_3\, 0_2\, 1_2\rangle$, $\langle 1_2\, 0_2\, 5_4\rangle$, $\langle 5_4\, 0_2\, 4_0\rangle$, $\langle 4_0\, 0_2\, \infty\rangle$, $\langle \infty\, 0_2\, 2_0\rangle$, $\langle 4_2\, 2_2\, 3_4\rangle$, $\langle 3_4\, 2_2\, 0_3\rangle$, $\langle 0_3\, 2_2\, 4_2\rangle$, $\langle 0_3\, \infty\, 1_3\rangle$, $\langle 1_2\, 0_0\, 3_4\rangle$, $\langle 3_4\, 0_0\, 6_2\rangle$, $\langle 6_2\, 0_0\, 3_3\rangle$, $\langle 3_3\, 0_0\, 5_3\rangle$, $\langle 5_3\, 0_0\, 0_4\rangle$, $\langle 0_4\, 0_0\, 6_3\rangle$, $\langle 6_3\, 0_0\, 6_4\rangle$, $\langle 6_4\, 0_0\, 1_3\rangle$, $\langle 1_3\, 0_0\, 4_4\rangle$, $\langle 4_4\, 0_0\, 2_3\rangle$, $\langle 2_3\, 0_0\, 1_2\rangle$, $\langle 1_1\, 0_1\, 4_2\rangle$, $\langle 4_2\, 0_1\, 3_3\rangle$, $\langle 3_3\, 0_1\, 0_3\rangle$, $\langle 0_3\, 0_1\, 1_1\rangle$, $\langle 2_3\, 0_1\, 5_3\rangle$, $\langle 5_3\, 0_1\, 4_3\rangle$, $\langle 4_3\, 0_1\, 2_3\rangle$.
The system is non-flexible, for example $(0_3 \cdot 4_2) \cdot 0_3 = 2_2 \cdot 0_3 = 3_4$*, whilst* $0_3 \cdot (4_2 \cdot 0_3) = 0_3 \cdot 0_0 = 5_4$.

Example NE.9 *Pure non-flexible LDTS(40).*
$V = \mathbb{Z}_{20} \times \mathbb{Z}_2$.
The triples are obtained from the following starter blocks under the action of the mappings $i_j \mapsto (i+1)_j$ *and* $i_j \mapsto i_{j+1}$.
$\langle 1_0, 0_0, 5_0\rangle$, $\langle 5_0, 0_0, 1_1\rangle$, $\langle 1_1, 0_0, 11_0\rangle$, $\langle 11_0, 0_0, 18_1\rangle$, $\langle 18_1, 0_0, 8_1\rangle$, $\langle 8_1, 0_0, 12_0\rangle$, $\langle 12_0, 0_0, 3_1\rangle$, $\langle 3_1, 0_0, 5_1\rangle$, $\langle 5_1, 0_0, 14_0\rangle$, $\langle 14_0, 0_0, 14_1\rangle$, $\langle 14_1, 0_0, 7_0\rangle$, $\langle 7_0, 0_0, 3_0\rangle$, $\langle 3_0, 0_0, 1_0\rangle$.
The system is non-flexible, for example $(14_1 \cdot 14_0) \cdot 14_1 = 0_1 \cdot 14_1 = 5_0$*, whilst* $14_1 \cdot (14_0 \cdot 14_1) = 14_1 \cdot 0_0 = 7_0$.

Example FO.1 *Pure flexible LDTS(21).*
$V = (\mathbb{Z}_5 \times \mathbb{Z}_4) \cup \{\infty\}$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$*, with* ∞ *as a fixed point.*
$\langle 1_0\, 0_0\, 2_1\rangle$, $\langle 2_1\, 0_0\, 0_1\rangle$, $\langle 0_1\, 0_0\, 1_0\rangle$, $\langle 1_0\, 3_2\, 0_1\rangle$, $\langle 0_1\, 3_2\, 2_1\rangle$, $\langle 2_1\, 3_2\, 1_0\rangle$, $\langle 2_0\, 0_0\, 0_3\rangle$, $\langle 0_3\, 0_0\, \infty\rangle$, $\langle \infty\, 0_0\, 3_1\rangle$, $\langle 3_1\, 0_0\, 0_2\rangle$, $\langle 0_2\, 0_0\, 2_0\rangle$, $\langle 2_0\, 3_2\, 0_2\rangle$, $\langle 0_2\, 3_2\, 3_1\rangle$, $\langle 3_1\, 3_2\, \infty\rangle$, $\langle \infty\, 3_2\, 0_3\rangle$, $\langle 0_3\, 3_2\, 2_0\rangle$, $\langle 1_0\, 0_3\, 0_2\rangle$, $\langle 0_2\, 0_3\, 4_2\rangle$, $\langle 4_2\, 0_3\, 0_1\rangle$, $\langle 0_1\, 0_3\, 1_1\rangle$, $\langle 1_1\, 0_3\, 2_3\rangle$, $\langle 2_3\, 0_3\, 1_0\rangle$, $\langle 1_0\, 3_3\, 2_3\rangle$, $\langle 2_3\, 3_3\, 1_1\rangle$, $\langle 1_1\, 3_3\, 0_1\rangle$, $\langle 0_1\, 3_3\, 4_2\rangle$, $\langle 4_2\, 3_3\, 0_2\rangle$, $\langle 0_2\, 3_3\, 1_0\rangle$.

Example FO.2 *Pure flexible LDTS(31).*
$V = \mathbb{Z}_{31}$.
The triples are obtained from the following starter blocks under the action of the mapping $i \mapsto i+1$.
$\langle 8, 7, 13\rangle$, $\langle 13, 7, 30\rangle$, $\langle 30, 7, 19\rangle$, $\langle 19, 7, 10\rangle$, $\langle 10, 7, 8\rangle$, $\langle 8, 23, 10\rangle$, $\langle 10, 23, 19\rangle$, $\langle 19, 23, 30\rangle$, $\langle 30, 23, 13\rangle$, $\langle 13, 23, 8\rangle$.

Example FO.3 *Pure flexible LDTS(39).*
$V = \mathbb{Z}_{13} \times \mathbb{Z}_3$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 10_0\, 4_0\, 4_2\rangle$, $\langle 4_2\, 4_0\, 5_1\rangle$, $\langle 5_1\, 4_0\, 10_0\rangle$, $\langle 10_0\, 3_1\, 5_1\rangle$, $\langle 5_1\, 3_1\, 4_2\rangle$, $\langle 4_2\, 3_1\, 10_0\rangle$, $\langle 2_1\, 8_0\, 11_1\rangle$, $\langle 11_1\, 8_0\, 5_2\rangle$, $\langle 5_2\, 8_0\, 2_1\rangle$, $\langle 2_1\, 8_1\, 5_2\rangle$, $\langle 5_2\, 8_1\, 11_1\rangle$, $\langle 11_1\, 8_1\, 2_1\rangle$, $\langle 0_0\, 10_0\, 11_0\rangle$, $\langle 11_0\, 10_0\, 2_0\rangle$, $\langle 2_0\, 10_0\, 12_1\rangle$, $\langle 12_1\, 10_0\, 0_0\rangle$, $\langle 0_0\, 11_1\, 12_1\rangle$, $\langle 12_1\, 11_1\, 2_0\rangle$, $\langle 2_0\, 11_1\, 11_0\rangle$, $\langle 11_0\, 11_1\, 0_0\rangle$, $\langle 3_1\, 11_0\, 12_2\rangle$, $\langle 12_2\, 11_0\, 6_2\rangle$, $\langle 6_2\, 11_0\, 3_2\rangle$, $\langle 3_2\, 11_0\, 3_1\rangle$, $\langle 3_1\, 8_1\, 3_2\rangle$, $\langle 3_2\, 8_1\, 6_2\rangle$, $\langle 6_2\, 8_1\, 12_2\rangle$, $\langle 12_2\, 8_1\, 3_1\rangle$, $\langle 3_0\, 9_2\, 7_1\rangle$, $\langle 7_1\, 9_2\, 0_2\rangle$, $\langle 0_2\, 9_2\, 10_0\rangle$, $\langle 10_0\, 9_2\, 1_2\rangle$, $\langle 1_2\, 9_2\, 3_0\rangle$, $\langle 3_0\, 12_2\, 1_2\rangle$, $\langle 1_2\, 12_2\, 10_0\rangle$, $\langle 10_0\, 12_2\, 0_2\rangle$, $\langle 0_2\, 12_2\, 7_1\rangle$, $\langle 7_1\, 12_2\, 3_0\rangle$.

Example FO.4 *Pure flexible LDTS(43).*
$V = \mathbb{Z}_{43}$.
The triples are obtained from the following starter blocks under the action of the mapping $i \mapsto i+1$.
$\langle 6, 26, 37\rangle$, $\langle 37, 26, 12\rangle$, $\langle 12, 26, 6\rangle$, $\langle 6, 28, 12\rangle$, $\langle 12, 28, 37\rangle$, $\langle 37, 28, 6\rangle$, $\langle 4, 3, 8\rangle$, $\langle 8, 3, 16\rangle$, $\langle 16, 3, 6\rangle$, $\langle 6, 3, 4\rangle$, $\langle 4, 23, 6\rangle$, $\langle 6, 23, 16\rangle$, $\langle 16, 23, 8\rangle$, $\langle 8, 23, 4\rangle$.

Example FO.5 *Pure flexible LDTS(67).*
$V = \mathbb{Z}_{67}$.
The triples are obtained from the following starter blocks under the action of the mapping $i \mapsto i+1$.
$\langle 16, 0, 25\rangle$, $\langle 25, 0, 18\rangle$, $\langle 18, 0, 26\rangle$, $\langle 26, 0, 30\rangle$, $\langle 30, 0, 32\rangle$, $\langle 32, 0, 33\rangle$, $\langle 33, 0, 22\rangle$, $\langle 22, 0, 28\rangle$, $\langle 28, 0, 31\rangle$, $\langle 31, 0, 21\rangle$, $\langle 21, 0, 16\rangle$, $\langle 16, 45, 21\rangle$, $\langle 21, 45, 31\rangle$, $\langle 31, 45, 28\rangle$, $\langle 28, 45, 22\rangle$, $\langle 22, 45, 33\rangle$, $\langle 33, 45, 32\rangle$, $\langle 32, 45, 30\rangle$, $\langle 30, 45, 26\rangle$, $\langle 26, 45, 18\rangle$, $\langle 18, 45, 25\rangle$, $\langle 25, 45, 16\rangle$.

Example FE.1 *Pure flexible LDTS(34).*
$V = \mathbb{Z}_{17} \times \mathbb{Z}_2$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 5_0\, 8_1\, 12_0\rangle$, $\langle 12_0\, 8_1\, 16_1\rangle$, $\langle 16_1\, 8_1\, 5_0\rangle$, $\langle 5_0\, 10_1\, 16_1\rangle$, $\langle 16_1\, 10_1\, 12_0\rangle$, $\langle 12_0\, 10_1\, 5_0\rangle$, $\langle 12_0\, 2_1\, 14_1\rangle$, $\langle 14_1\, 2_1\, 1_1\rangle$, $\langle 1_1\, 2_1\, 12_0\rangle$, $\langle 12_0\, 4_1\, 1_1\rangle$, $\langle 1_1\, 4_1\, 14_1\rangle$, $\langle 14_1\, 4_1\, 12_0\rangle$, $\langle 0_0\, 8_0\, 11_0\rangle$, $\langle 11_0\, 8_0\, 9_0\rangle$, $\langle 9_0\, 8_0\, 13_0\rangle$, $\langle 13_0\, 8_0\, 8_1\rangle$, $\langle 8_1\, 8_0\, 0_0\rangle$, $\langle 0_0\, 10_1\, 8_1\rangle$, $\langle 8_1\, 10_1\, 13_0\rangle$, $\langle 13_0\, 10_1\, 9_0\rangle$, $\langle 9_0\, 10_1\, 11_0\rangle$, $\langle 11_0\, 10_1\, 0_0\rangle$.

Example FE.2 *Pure flexible LDTS(40).*
$V = \mathbb{Z}_{20} \times \mathbb{Z}_2$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 6_0\, 9_1\, 16_0\rangle$, $\langle 16_0\, 9_1\, 7_1\rangle$, $\langle 7_1\, 9_1\, 17_1\rangle$, $\langle 17_1\, 9_1\, 6_0\rangle$, $\langle 2_0\, 11_0\, 3_1\rangle$, $\langle 3_1\, 11_0\, 14_0\rangle$, $\langle 14_0\, 11_0\, 12_0\rangle$, $\langle 12_0\, 11_0\, 16_0\rangle$, $\langle 16_0\, 11_0\, 2_0\rangle$, $\langle 2_0\, 12_1\, 16_0\rangle$, $\langle 16_0\, 12_1\, 12_0\rangle$, $\langle 12_0\, 12_1\, 14_0\rangle$, $\langle 14_0\, 12_1\, 3_1\rangle$, $\langle 3_1\, 12_1\, 2_0\rangle$, $\langle 14_0\, 2_0\, 19_1\rangle$, $\langle 19_1\, 2_0\, 15_0\rangle$, $\langle 15_0\, 2_0\, 17_1\rangle$, $\langle 17_1\, 2_0\, 1_1\rangle$, $\langle 1_1\, 2_0\, 8_1\rangle$, $\langle 8_1\, 2_0\, 14_0\rangle$, $\langle 14_0\, 2_1\, 8_1\rangle$, $\langle 8_1\, 2_1\, 1_1\rangle$, $\langle 1_1\, 2_1\, 17_1\rangle$, $\langle 17_1\, 2_1\, 15_0\rangle$, $\langle 15_0\, 2_1\, 19_1\rangle$, $\langle 19_1\, 2_1\, 14_0\rangle$.

Example FE.3 *Pure flexible LDTS(52).*
$V = \mathbb{Z}_{26} \times \mathbb{Z}_2$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 15_0\, 8_0\, 12_1 \rangle$, $\langle 12_1\, 8_0\, 25_0 \rangle$, $\langle 25_0\, 8_0\, 15_0 \rangle$, $\langle 15_0\, 18_1\, 25_0 \rangle$, $\langle 25_0\, 18_1\, 12_1 \rangle$, $\langle 12_1\, 18_1\, 15_0 \rangle$, $\langle 5_0\, 20_0\, 23_0 \rangle$, $\langle 23_0\, 20_0\, 21_0 \rangle$, $\langle 21_0\, 20_0\, 25_0 \rangle$, $\langle 25_0\, 20_0\, 5_0 \rangle$, $\langle 5_0\, 21_1\, 25_0 \rangle$, $\langle 25_0\, 21_1\, 21_0 \rangle$, $\langle 21_0\, 21_1\, 23_0 \rangle$, $\langle 23_0\, 21_1\, 5_0 \rangle$, $\langle 4_0\, 12_1\, 3_1 \rangle$, $\langle 3_1\, 12_1\, 18_0 \rangle$, $\langle 18_0\, 12_1\, 9_1 \rangle$, $\langle 9_1\, 12_1\, 23_0 \rangle$, $\langle 23_0\, 12_1\, 11_0 \rangle$, $\langle 11_0\, 12_1\, 24_0 \rangle$, $\langle 24_0\, 12_1\, 10_0 \rangle$, $\langle 10_0\, 12_1\, 22_1 \rangle$, $\langle 22_1\, 12_1\, 5_0 \rangle$, $\langle 5_0\, 12_1\, 16_1 \rangle$, $\langle 16_1\, 12_1\, 17_0 \rangle$, $\langle 17_0\, 12_1\, 0_1 \rangle$, $\langle 0_1\, 12_1\, 20_0 \rangle$, $\langle 20_0\, 12_1\, 4_1 \rangle$, $\langle 4_1\, 12_1\, 23_1 \rangle$, $\langle 23_1\, 12_1\, 10_1 \rangle$, $\langle 10_1\, 12_1\, 17_1 \rangle$, $\langle 17_1\, 12_1\, 7_0 \rangle$, $\langle 7_0\, 12_1\, 13_1 \rangle$, $\langle 13_1\, 12_1\, 4_0 \rangle$.

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