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Further results on strong edge-colourings in outerplanar graphs

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Abstract

An edge-colouring is *strong* if every colour class is an induced matching. In this work we give a formula that determines either the optimal or the optimal plus one strong chromatic index of bipartite outerplanar graphs. Further, we give an improved upper bound for any outerplanar graph which is close to optimal. All our proofs yield efficient algorithms to construct such colourings.

1 Introduction

Given a simple undirected graph G, let $V(G)$ and $E(G)$, respectively, denote the vertex set and the edge set of G. A *proper k-edge-colouring* of G is a map $\mathcal{C} : E(G) \mapsto [k]$ such that adjacent edges (edges of G sharing a common vertex) receive different colours (numbers), where $[k] = \{1, 2, \ldots, k\}$. The smallest positive integer k such that G admits a proper k-edge-colouring is known as the *chromatic index* of G and is denoted by $\chi'(G)$.

An *induced matching* M in G is a matching such that $G[V(M)] = M$. That is, the subgraph of G induced by the vertices of M is M itself. A proper edge-colouring

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is a *strong edge-colouring* if every colour class is an induced matching in G. In other words, for any edge $e = uv$, the sets of colours *seen* by u and v have exactly one colour in common (in an edge-colouring we say that a vertex *sees* colour c if c is assigned to any of the edges incident to it). That is, the distance between any two edges having the same colour is at least two. The minimum positive integer k such that G admits a strong k-edge-colouring is called the *strong chromatic index* of G, denoted by $\chi'_s(G)$. The degree of a vertex v is denoted by $d(v)$. An edge incident to a vertex of degree one is called a *pendant edge*. Let $\Delta = \Delta(G)$ denote the maximum degree of vertices of G. Given a cycle $C = v_1v_2...v_k$, we say that v_i is an *odd vertex* when i is odd, otherwise we say that v_i is an *even vertex*.

A graph G is *outerplanar* if it has a planar embedding in which all vertices i are incident to the infinite face. We define a *puffer graph* as a graph obtained by adding some (possibly empty) pendant edges to each vertex of a cycle or adding a common neighbour to two consecutive vertices of the cycle. Notice that for an outerplanar graph, at most one such vertex can be added. All graphs are assumed to be connected.

The strong edge-colouring has a long history and has lead to many well known conjectures. Some of the many unsolved conjectures include $\chi'_s(G) \leq 5\Delta^2/4$ for all graphs, $\chi'_s(G) \leq \Delta^2$ for bipartite graphs and $\chi'_s(G) \leq 9$ for 3-regular planar graphs. See the open problems pages of Douglas West [11] for more details.

Molloy and Reed [9] proved a conjecture by Erdős and Nešetřil (see [3]) that for large Δ, there is a positive constant c such that $\chi'_s(G) \leq (2 - c)\Delta^2$. Mahdian [8] proved that for a C_4-free graph G, $\chi'_s(G) \leq (2 + o(1))\Delta^2/\ln \Delta$.

For integers $0 \leq \ell \leq k \leq m$, $S_m(k, \ell)$ is the bipartite graph with vertex set $\{x \subseteq [m] \colon |x| = k \text{ or } \ell\}$ and a k-subset x is adjacent to an ℓ-subset y if $y \subseteq x$. Quinn and Benjamin [1] proved that $S_m(k, \ell)$ has strong chromatic index $\binom{m}{k-\ell}$. The Θ-*graph* $\Theta(G)$ of a partial cube G (distance-invariant subgraph of some n-cube), is the intersection graph of the equivalence classes of the *Djoković-Winkler relation* Θ defined on the edges of G such that xy and uv are in relation Θ if $d(x, u) + d(y, v) \neq d(x, v) + d(y, u)$. Šumenjak [7] showed that the strong chromatic index of a tree-like partial cube graph G is at most the chromatic number of $\Theta(G)$.

Faudree, Gyárfás, Schelp and Tuza [4] proved that for graphs where all cycle lengths are multiples of four, $\chi'_s(G) \leq \Delta^2$. They mention that this result probably could be improved to a linear function of the maximum degree. Brualdi and Quinn [2] improved the upper bound to $\chi'_s(G) \leq \alpha\beta$ for such graphs, where α and β are the maximum degrees of the respective partitions. Nakprasit [10] proved that if G is bipartite and the maximum degree of one partite set is at most 2, then $\chi'_s(G) \leq 2\Delta$.

In a recent work [5] an upper bound of $3\Delta - 3$ was given for general outerplanar graphs. This result is based on two forbidden configurations of outerplanar graphs. We remark that the upper bound of $3\Delta - 3$ is tight when the graph contains a cycle of lenght three and each of the vertices on the cycle has degree Δ. In this work, using different techniques, we improve that upper bound. We also obtain either the exact value of $\chi'_s(G)$ or the exact value plus one for bipartite outerplanar graphs. The following is our main result.

Theorem 1. *Let G be an outerplanar graph. Then*

$$\chi'_s(G) = \max\{\max_{uv\in E} d(u) + d(v) - 1, \max_{H\in\mathcal{P}} \chi'_s(H)\}$$

where $\mathcal{P}$ is the set of all puffer subgraphs of G. Moreover, if G is bipartite, then $\chi'_s(G)$ is either $\max_{uv\in E} d(u) + d(v) - 1$ or $\max_{uv\in E} d(u) + d(v)$.

We also give efficient algorithms to produce strong edge-colourings of such classes of graphs satisfying the above bounds.

2 Outerplanar graphs

A *block* is a maximal connected component without a cut-vertex. A *block decomposition* of a graph G is a partition of G into its blocks. Notice that each component is either a maximally 2-connected subgraph or a single edge.

An *ear* in G to a subgraph H is a simple path P on at least three vertices with end-points in H such that (1) none of the internal vertices of P are contained in H and (2) P along with the segment between its end-points in H forms an induced cycle. An *ear decomposition* of a 2-connected subgraph is a partition of its edges into a sequence of ears where the first ear is an induced cycle. It is easily seen that for a 2-connected outerplanar graph, there is an ear decomposition where each ear contains at least one internal vertex and the endpoints of every ear are adjacent in the preceding graph (if not, the outerplanarity property is affected). Further notice that when the graph is bipartite outerplanar any added ear has an even number of internal vertices. Any such ear (which forms an induced cycle) together with the edges incident to it forms a puffer graph we defined earlier. Thus, we first show an upper bound for the puffer graphs.

2.1 Puffer graphs

Note that to compute the strong chromatic index we suppose that the puffer graph only has pendant edges (no common neighbours forming a triangle) since we can always split any common neighbour of adjacent vertices of the cycle to two pendant edges which does not affect the colouring.

The following lemma gives bounds for the puffer graphs.

Lemma 2. *Let G be a puffer graph and C its cycle. We have the following according to the cycle length $|C|$.*

1. *If $|C| = 3$, then $\chi'_s(G) = d(u) + d(v) + d(w) - 3$, $u, v, w \in C$.*

2. *If* $|C| = 4$*, then* $\chi'_s(G) = \max_{uv\in E(C)} d(u) + d(v)$*.*

3. *If* $G = C_5$*, then* $\chi'_s(G) = 5$*.*

4. *If* $|C| = 5$ *and either only a single vertex or exactly two vertices at distance* 2 *have pendant edges, then* $\chi'_s(G) = \max_{u\in C} d(u) + 2$*.*

5. *If* $|C| = 5$ *and cases 3 and 4 above do not hold, and if at least one vertex has at most* 1 *pendant edge, then* $\chi'_s(G) = \max_{uv\in E(C)} d(u) + d(v) - 1$*.*

6. *If* $|C| = 5$ *and every vertex has at least* 2 *pendant edges, let* u, v *be the vertices where* $d(u) + d(v) = \max_{u_1u_2\in E(C)} d(u_1) + d(u_2)$ *and let* x, y *and* z *be the rest of the vertices. Call* $\eta = \lceil\frac{d(x)+d(y)+d(z)-d(u)-d(v)-3}{2}\rceil$*. Then*

$$\chi'_s(G) \leq \begin{cases} d(u) + d(v) - 1 \text{ if } d(u) + d(v) \geq d(x) + d(y) + d(z) - 3 \\ d(u) + d(v) - 1 + \eta \text{ otherwise} \end{cases}$$

7. *Let* $C = C_k$*,* $k \geq 6$*. If* $G = C_k$*, then*

$$\chi'_s(G) = \begin{cases} 3 \text{ if } k \equiv 0 (\mathrm{mod} 3) \\ 4 \text{ otherwise} \end{cases}$$

8. *Let* $C = C_{2k}$*,* $k \geq 3$*, and set* $C_{2k} = v_1v_2 \dots v_{2k}v_1$*. Let* $G \neq C_{2k}$*. Let* u, v *be the vertices where* $d(u) + d(v) = \max_{u_1u_2\in E(C_{2k})} d(u_1) + d(u_2)$ *and suppose without losing generality that* $u = v_1$*,* $v = v_2$ *and at least* u *has a non-empty set of pendant edges.*

   (a) *If* $2k \equiv 0(\mathrm{mod} 3)$ *then* $\chi'_s(G) = d(u) + d(v) - 1$*.*

   (b) *If* $2k \equiv 2(\mathrm{mod} 3)$ *then*
      - $\chi'_s(G) \leq d(u) + d(v)$ *if there exists another pair of vertices* $v_jv_{j+1} \in E(C_{2k})$ *such that* $d(u)+d(v) = d(v_j)+d(v_{j+1})$*,* j *is even and* $d(v_{j+1}) = 2$ *(clearly this implies that* $d(v_{j-1}) = 2$*).*
      - $\chi'_s(G) = d(u) + d(v) - 1$ *otherwise.*

   (c) *If* $2k \equiv 1(\mathrm{mod} 3)$ *then*
      - $\chi'_s(G) \leq d(u) + d(v)$ *if for every vertex* $w \in C_{2k}$ *we have* $d(w) \leq 3$*.*
      - $\chi'_s(G) \leq d(u) + d(v)$ *if there exists another pair of vertices* $v_jv_{j+1} \in E(C_{2k})$ *such that* $d(u)+d(v) = d(v_j)+d(v_{j+1})$*,* j *is even and* $d(v_{j+1}) \leq 3$ *(clearly this implies that* $d(v_{j-1}) \leq 3$*).*
      - $\chi'_s(G) = d(u) + d(v) - 1$ *otherwise.*

9. *Let* $C = C_{2k-1}$*,* $k \geq 4$*, and set* $C_{2k-1} = v_1v_2 \dots v_{2k-1}v_1$*. Let* $G \neq C_{2k-1}$*. Let* u, v *be the vertices where* $d(u) + d(v) = \max_{u_1u_2\in E(C_{2k-1})} d(u_1) + d(u_2)$ *and let* x, y *and* z *be the consecutive vertices of* C_{2k-1} *not considering* u *and* v *where* $d(x)+d(y)+d(z) = \min_{s_1,s_2,s_3\in C_{2k-1}} d(s_1)+d(s_2)+d(s_3)$*. Suppose without losing generality that* $v_1 = x$*,* $v_2 = y$ *and* $v_3 = z$*. Let* $\eta = \lceil\frac{d(x)+d(y)+d(z)-d(u)-d(v)-2}{2}\rceil$*.*

(a) If $2k-1 \equiv 0(\mathrm{mod}3)$ *then*

$$\begin{cases} \chi'_s(G) = d(u)+d(v)-1 \text{ if } d(u)+d(v) \geq d(x)+d(y)+d(z)-2 \\ \chi'_s(G) \leq d(u)+d(v)-1+\eta \text{ otherwise} \end{cases}$$

(b) If $2k-1 \equiv 2(\mathrm{mod}3)$ *then*

- $\chi'_s(G) \leq d(u)+d(v)$ *if* $d(u)+d(v) \geq d(x)+d(y)+d(z)-2$ *and there exists another pair of vertices* $v_jv_{j+1} \in E(C_{2k-1})$ *such that* $d(u)+d(v) = d(v_j)+d(v_{j+1})$, *j is odd and* $d(v_{j+1}) = 2$ *(clearly this implies that* $d(v_{j-1}) = 2$*).*
- $\chi'_s(G) = d(u)+d(v)-1$ *if* $d(u)+d(v) \geq d(x)+d(y)+d(z)-2$ *and we are not in the previous case.*
- $\chi'_s(G) \leq d(u)+d(v)-1+\eta$ *if* $d(u)+d(v) < d(x)+d(y)+d(z)-2$.

(c) If $2k-1 \equiv 1(\mathrm{mod}3)$ *then*

- $\chi'_s(G) \leq d(u)+d(v)$ *if for every vertex* $w \in C_{2k-1}$ *we have* $d(w) \leq 3$.
- $\chi'_s(G) \leq d(u)+d(v)$ *if* $d(u)+d(v) \geq d(x)+d(y)+d(z)-2$ *and there exists another pair of vertices* $v_jv_{j+1} \in E(C_{2k-1})$ *such that* $d(u)+d(v) = d(v_j)+d(v_{j+1})$, *j is odd and* $d(v_{j+1}) \leq 3$ *(clearly this implies that* $d(v_{j-1}) \leq 3$*).*
- $\chi'_s(G) = d(u)+d(v)-1$ *if* $d(u)+d(v) \geq d(x)+d(y)+d(z)-2$ *and we are not in the previous case.*
- $\chi'_s(G) \leq d(u)+d(v)+\eta$ *if* $d(u)+d(v) < d(x)+d(y)+d(z)-2$ *and and there exists another pair of vertices* $v_jv_{j+1} \in E(C_{2k-1})$ *such that* $d(u)+d(v) = d(v_j)+d(v_{j+1})$, *j is odd and* $d(v_{j+1}) = 3$ *(clearly this implies that* $d(v_{j-1}) \leq 3$*).*
- $\chi'_s(G) \leq d(u)+d(v)-1+\eta$ *if* $d(u)+d(v) < d(x)+d(y)+d(z)-2$ *and we are not in the previous case.*

And same bounds plus 1 *if* $|C_{2k-1}| = 7$.

We give below a proof of the above lemma. We then show how to colour any outerplanar graph in a strong manner using the lemma and the ear decomposition. We start by colouring the first ear (a cycle) with its incident edges (which forms a puffer graph) and extend that colouring to the next puffer graph (next ear with its incident edges).

Proof of Lemma 2.

The proof is trivial for statements 1) through 4).

For 5), let $uvxyz$ be the vertices of the C_5 in a cyclic order and suppose that $d(u)+d(v) = \max_{u_1u_2 \in E(C)} d(u_1)+d(u_2)$. Colour the edges of the cycle with colours 1 to 5 starting at the edge uv. It is easy to see that for each vertex on the cycle we can repeat only one already used colour for its uncoloured incident edges. That is, we can use colour 3 at u, 4 at v, 5 at x, 1 at y and 2 at z. Therefore, as cases 3) and

4) do not hold, we need $d(u) - 3$ new colours at vertex u and $d(v) - 3$ new colours at vertex v. Now, as there is at least one vertex among x, y and z with at most one pendant edge, these edges (if exist) are coloured with an already used colour in the cycle as described before. Finally, if there are other uncoloured edges (at most two vertices could remain with uncoloured edges), we can repeat the $d(u)-3$ and $d(v)-3$ new colours used at u and v for them in an appropriate way to obtain a strong edge-colouring. By this we used a total of $5+d(u)-3+d(v)-3 = d(u)+d(v)-1$ colours as desired.

For 6) we note that every vertex of the cycle has at least 2 pendant edges. Let $uvxyz$ be the vertices of the C_5 in a cyclic order. We colour the cycle with colours 1 to 5 starting at the edge uv. Then we colour one uncoloured incident edge of each vertex with the only possible colour among the used ones (keeping the strong colouring property). Thus we have $d(u) + d(v) - 6$ uncoloured edges incident to u and v, and $d(x) + d(y) + d(z) - 9$ uncoloured edges incident to x, y and z. Suppose that $d(u)+d(v)-6 \geq d(x)+d(y)+d(z)-9$, i.e., $d(u)+d(v) \geq d(x)+d(y)+d(z)-3$. We use $d(u) - 3$ new colours to colour the uncoloured edges at u and $d(v) - 3$ new colours for the ones at v. We remark that this is the only possibility to keep the strong colouring property. Clearly $d(x) \leq d(u)$ and $d(z) \leq d(v)$. We colour the uncoloured edges at x and z from the set of colours used at u and v respectively. Since $d(x) + d(y) + d(z) - 3 \leq d(u) + d(v)$, we notice that there are enough colours left to colour the edges incident to y. Since we use only $d(u) + d(v) - 1$ colours, the bound is optimal in this case.

Now suppose that $d(u) + d(v) < d(x) + d(y) + d(z) - 3$. As before, we colour the $d(u) - 3$ edges at u and the $d(v) - 3$ edges at v with new colours. Now we introduce an additional η new colours and colour as many edges incident to both x and z (we can verify that both x and z have at least η uncoloured edges in this case). Then for the remaining edges at x we use at most $d(x) - 3 - \eta$ colours used at u and for the ones at z use at most $d(z) - 3 - \eta$ colours used at v. As before it is not difficult to see that we have enough colours left to colour the edges incident to y.

For statement 7), we colour the cycle in the following way. If $k \equiv 0 (\bmod 3)$, then we use colours $1, 2$ and 3 repeatedly for the cycle and we are done. If $k \equiv 1(\bmod 3)$, then we colour one edge with colour 4 and then repeatedly with colours $1, 2$ and 3. Finally, if $k \equiv 2(\bmod 3)$, then we colour the first 5 edges with colours $4, 1, 2, 3, 4$ and then repeatedly with colours $1, 2$ and 3. Again, it works since $k > 7$.

For 8a), colour the edges of the cycle repeatedly with colours $1, 2$ and 3 starting from the edge v_1v_2. Clearly the cycle is strong edge-coloured since $2k \equiv 0(\bmod 3)$. Now introduce a set of new colours A, where $|A| = d(v_1) - 2$ and for each odd vertex on the cycle colour its uncoloured incident edges with colours from A using the least permissible colour. Then do the same for each even vertex on the cycle using another set of new colours B, where $|B| = d(v_2) - 2$. If there are not more uncoloured edges we are done.

Suppose now that there exists one vertex v_j, j odd (j even is similar) such that $d(v_j) > d(v_1)$. Therefore there are $d(v_j) - d(v_1)$ edges to colour incident to v_j. We know that $d(v_1) + d(v_2) \geq d(v_j) + d(v_{j+1})$ and $d(v_1) + d(v_2) \geq d(v_j) + d(v_{j-1})$. So if we suppose (without losing generality) that $d(v_{j+1}) \geq d(v_{j-1})$ then $d(v_2) - d(v_{j+1}) \geq d(v_j) - d(v_1)$. Therefore we have $d(v_2) - d(v_{j+1})$ colours from B not used neither at v_{j-1} nor at v_{j+1} and then we can use them to colour the remaining $d(v_j) - d(v_1)$ edges at v_j. Clearly this edge-colouring is strong and we used $3 + |A| + |B| = 3 + d(u) - 2 + d(v) - 2 = d(u) + d(v) - 1$ colours, which is optimal since $d(u) + d(v) - 1$ is also a lower bound.

For 8b), we colour the cycle with colours $1, 2, 3$ starting at the edge v_1v_2 until the edge $v_{2k-4}v_{2k-3}$ and for the four remaining edges we use colours (respecting the cycle ordering) $4, 3, 2, 4$. Then, by the way we coloured the cycle we have that for each odd vertex v_i in the cycle there is one available colour among the colours $\{1, 2, 3, 4\}$ to use at its uncoloured incident edges. We proceed to colour this edges with that colour. As in 8a), introduce a set of new colours A, but now $|A| = d(v_1) - 3$ and for each odd vertex on the cycle colour its uncoloured incident edges with colours from A using the least permissible colour. For each even vertex on the cycle do the same using another set of new colours B, where $|B| = d(v_2) - 2$. If there are no more edges to colour we are done.

Suppose now that there exists one vertex v_j with uncoloured edges incident to it. Suppose also that $d(v_1) + d(v_2) = d(v_j) + d(v_{j+1})$, j is even and $d(v_{j+1}) = 2$. Therefore $d(v_j) > d(v_2)$. Now, since $d(v_{j+1}) = 2$ (and also $d(v_{j-1}) = 2$). We can use the $d(v_1) - 3$ colours from A for the remaining uncoloured edges at v_j. By this we colour $2 + d(v_1) - 3 + d(v_2) - 2 = d(v_1) + d(v_2) - 3$ edges at v_j. However, since $d(v_1) + d(v_2) = d(v_j) + d(v_{j+1})$ we have that $d(v_j) = d(v_1) + d(v_2) - d(v_{j+1}) = d(v_1) + d(v_2) - 2$ therefore we need one new colour more to finish colouring v_j in a strong manner since v_j sees the four colours used at the cycle (j is even). By this we used $4 + |A| + |B| + 1 = d(u) + d(v)$ colours and then $\chi'_s(G) \leq d(u) + d(v)$. We remark that if there are few vertices v_j with that property, we can maybe recolour the cycle with four colours such that each v_j sees only three colours among the four used at the cycle. Then this new colour would not be necessary and we would have $\chi'_s(G) = d(u) + d(v) - 1$. Nevertheless, there are graphs where we cannot do this and therefore $\chi'_s(G) = d(u) + d(v)$.

To finish this case if we are not in the previous conditions for v_j then at least one condition among these three is true: (1) $d(u) + d(v) > d(v_j) + d(v_{j+1})$, (2) j is odd or (3) $d(v_{j+1}) \geq 3$. In each of these cases the way to colour the remaining edges at v_j is similar to the case 8a) and we obtain $\chi'_s(G) = d(u) + d(v) - 1$ which is optimal.

For 8c), the case where every vertex $w \in C_{2k}$ has $d(w) \leq 3$ can be easily verified. Otherwise we colour the cycle repeatedly with colours $1, 2, 3$ starting at the edge v_1v_2 until the edge $v_{2k-6}v_{2k-5}$ and for the six remaining edges we use colours (respecting the cycle ordering) $5, 3, 4, 5, 2, 4$. Then, by the way we coloured the cycle we have now that for each odd vertex v_i in the cycle there are two available colours among the colours $\{1, 2, 3, 4, 5\}$ to use at its uncoloured incident edges. We colour them with those colours. Similar to 8b), for each odd vertex on the cycle colour its uncoloured incident edges with a new set of colours A where $|A| = d(v_1) - 4$ and for each even vertex on the cycle do the same using another set of new colours B where $|B| = d(v_2) - 2$. If there are not more uncoloured edges we are done.

Suppose now that there exists one vertex v_j with uncoloured edges incident to it. Suppose also that $d(u) + d(v) = d(v_j) + d(v_{j+1})$, j is even and $d(v_{j+1}) \leq 3$. Therefore $d(v_j) > d(v_2)$. Suppose first that $d(v_{j+1}) = 2$ (and also $d(v_{j-1}) = 2$). We can use the $d(v_1) - 4$ colours from A for the remaining uncoloured edges at v_j. By this we colour $2 + d(v_1) - 4 + d(v_2) - 2 = d(v_1) + d(v_2) - 4$ edges at v_j. However, since $d(v_1) + d(v_2) = d(v_j) + d(v_{j+1})$ we have that $d(v_j) = d(v_1) + d(v_2) - d(v_{j+1}) = d(v_1) + d(v_2) - 2$ therefore we still need to colour two edges. Now, since v_j sees four colours among the five used at the cycle and $d(v_{j+1}) = 2$, $d(v_{j-1}) = 2$, we colour one of its two remaining edges with that colour. For the last one, we need to use a new colour and we finish colouring v_j. Suppose last that $d(v_{j+1}) = 3$ (and then $d(v_{j-1}) \leq 3$). As before, if we use the colours from A for the uncoloured edges at v_j, we colour $d(v_1) + d(v_2) - 4$ edges at v_j but now $d(v_j) = d(v_1) + d(v_2) - 3$. Then we still have to colour one more edge at v_j. For this one, we need to use a new colour since the colour that v_j does not see among the five used at the cycle is used either at v_{j+1} or at v_{j-1} or at both. In both cases this we used $5 + |A| + |B| + 1 = d(u) + d(v)$ colours and then $\chi'_s(G) \leq d(u) + d(v)$. Same remark as case 8$b$) applies.

If there is no v_j satisfying these three conditions, we colour its remaining edges as in cases 8a), 8b) and we obtain $\chi'_s(G) = d(u) + d(v) - 1$.

For 9a), colour the edges on the cycle from the edge v_1v_2 with colours 1,2 and 3 repeatedly Clearly, the cycle is strong edge-coloured since $2k - 1 \equiv 0 (\text{mod} 3)$. We will colour the rest of the graph as in 8a) but considering the vertices v_1 and v_2 as a single vertex (say v_2). Observe that there are $d(u) + d(v) - 4$ uncoloured edges at u and v and $d(x)+d(y)+d(z)-6$ uncoloured ones at x, y and z. Suppose first then that $d(u) + d(v) - 4 \geq d(x) + d(y) + d(z) - 6$, that is, $d(u) + d(v) \geq d(x) + d(y) + d(z) - 2$. Then, introduce a set of new colours A where $|A| = d(u) - 2$ and another set of new colours B where $|B| = d(v) - 2$. Now colour the rest of the edges as in 8a) considering v_1 and v_2 as a single vertex. Clearly, this leads to a strong edge-colouring of G following same arguments as in 6) and 8a). We use $d(u) + d(v) - 1$ colours. Second, suppose that $d(u) + d(v) < d(x) + d(y) + d(z) - 2$. We use η (as defined earlier) new colours to colour a subset of η edges incident to each of v_1 and v_3. Again it is easily seen that $d(v_1)$ and $d(v_3)$ are at least η using the assumed inequalities. Finally to colour the rest of the edges we proceed as in the first case. As before, the colouring is strong by 6) and 8a). We use $d(u) + d(v) - 1 + \eta$ colours as desired.

For 9b), we colour the cycle repeatedly with colours $1, 2, 3$ starting at the edge v_1v_2 until the edge $v_{2k-7}v_{2k-6}$ and for the six remaining edges we use colours (respecting the cycle ordering) $4, 1, 3, 4, 2, 3$. We can observe that for each even vertex v_i in the cycle there is one available colour among the colours $\{1, 2, 3, 4\}$ to use at its uncoloured incident edges. Then combining cases 8b) and 9a) the result holds. We remark that the case that $d(u) + d(v) < d(x) + d(y) + d(z) - 2$ and there exists another pair of vertices $v_jv_{j+1} \in E(C)$ such that $d(u) + d(v) = d(v_j) + d(v_{j+1})$, j is odd and $d(v_{j+1}) = 2$, is not possible since that would contradict the fact that x, y

and z were the consecutive vertices that minimise the sum of degrees or that u and v maximised it.

For 9c), we have a similar situation as case 9b) but here we colour the cycle repeatedly with colours $1, 2, 3$ starting at the edge v_1v_2 until the edge $v_{2k-9}v_{2k-8}$ and for the seven remaining edges we use colours (respecting the cycle ordering) $5, 1, 4, 5, 3, 4, 2, 3$ (for the special case $|C| = 7$ we colour $5, 1, 4, 5, 3, 4, 2$). Now for each even vertex v_i in the cycle there are two available colours among the colours $\{1, 2, 3, 4, 5\}$ to use at its uncoloured incident edges (for the special case $|C| = 7$ we may need one more colour since by the lenght of the cycle there exists one even vertex without this property). Then we combine cases 8c) and 9b). We remark that as in the previous case, we cannot have at the same time $d(u) + d(v) < d(x) + d(y) + d(z) - 2$ and the existence of another pair of vertices $v_jv_{j+1} \in E(C)$ such that $d(u) + d(v) = d(v_j) + d(v_{j+1})$, j is odd and $d(v_{j+1}) = 2$. However, we can for $d(v_{j+1}) = 3$, therefore we might need one more colour. □

2.2 General outerplanar graphs

We finish this section proving the main theorem of the paper.

Proof of Theorem 1

We observe that given a block decomposition of an outerplanar graph and then an ear decomposition of each block, every ear together with the edges incident to it forms a puffer graph. Then, adding the ears in the order of the decomposition, each new ear joins two adjacent vertices. Since only the edges incident to two adjacent vertices of the new ear are precoloured, we note that we can simply extend the colouring to the new puffer graph (as the precoloured edges all get distinct colours in both cases). The upper bound for outerplanar graphs is now clear by maximising over all puffer graphs and over all pairs of adjacent vertices (the latter is a trivial lower bound). When the graph is bipartite, this gives either the exact value or the exact value plus one colour. □

The proof itself gives the algorithm to obtain such a colouring and it is easy to see that it takes sub-quadratic time.

3 Remarks

In this work we have considered outerplanar graphs. We gave a formulae to find either the exact value or the exact value plus one of the strong chromatic index for bipartite outerplanar graphs. We also improved the upper bound for the general outerplanar graphs from the $3\Delta - 3$ stated in [5].

A recent work [6] gives an algorithm to find the strong chromatic index of any maximal outerplanar graph. However, notice that when you extend the graph to maximal outerplanar, the maximum degree and the index can increase. As our proofs are constructive, they lead to an algorithm to colour any outerplanar graph with a number of colours close to the optimum and for bipartite outerplanar graphs with the optimum (or just one more) number of colours.

In some special cases of the general outerplanar graph (where we use η extra colours, statements 6 and 9 of Lemma 2) we were not able to show the optimality of the bounds. We believe that it is very close to the exact bound within an additive factor of a small constant. It would be interesting to prove whether our bounds are optimal, and if not, to find a way to close the gap.

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