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Intersecting generalised permutations

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Abstract

For any positive integers k, r, n with $r \leq \min\{k, n\}$, let $\mathcal{P}_{k,r,n}$ be the family of all sets $\{(x_1, y_1), \dots, (x_r, y_r)\}$ such that $x_1, \dots, x_r$ are distinct elements of $[k] = \{1, \dots, k\}$ and $y_1, \dots, y_r$ are distinct elements of $[n]$. The families $\mathcal{P}_{n,n,n}$ and $\mathcal{P}_{n,r,n}$ describe *permutations* of $[n]$ and *r-partial permutations* of $[n]$, respectively. If $k \leq n$, then $\mathcal{P}_{k,k,n}$ describes permutations of k-element subsets of $[n]$. A family $\mathcal{A}$ of sets is said to be *intersecting* if every two members of $\mathcal{A}$ intersect. We use Katona's elegant cycle method to show that a number of important Erdős–Ko–Rado-type results by various authors generalise as follows: the size of any intersecting subfamily $\mathcal{A}$ of $\mathcal{P}_{k,r,n}$ is at most $\binom{k-1}{r-1}\frac{(n-1)!}{(n-r)!}$, and the bound is attained if and only if $\mathcal{A} = \{A \in \mathcal{P}_{k,r,n} \colon (a, b) \in A\}$ for some $a \in [k]$ and $b \in [n]$.

1 Introduction

For an integer $n \geq 1$, the set $\{1, 2, \dots, n\}$ is denoted by $[n]$. For a set X, the *power set* $\{A \colon A \subseteq X\}$ of X is denoted by 2^X, and the *uniform* family $\{Y \subseteq X \colon |Y| = r\}$ is denoted by $\binom{X}{r}$. We call a set of size n an *n-set*.

If $\mathcal{F}$ is a family of sets and x is an element of the union of all sets in $\mathcal{F}$, then we call the family of all the sets in $\mathcal{F}$ that contain x the *star of $\mathcal{F}$* with *centre x*. A

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family $\mathcal{A}$ is said to be *intersecting* if $A \cap B \neq \emptyset$ for every $A, B \in \mathcal{A}$. Note that a star of a family is intersecting.

The classical Erdős–Ko–Rado (EKR) Theorem [11] says that if $r \leq n/2$, then an intersecting subfamily $\mathcal{A}$ of $\binom{[n]}{r}$ has size at most $\binom{n-1}{r-1}$, which is the size of a star of $\binom{[n]}{r}$. If $r < n/2$, then, by the Hilton–Milner Theorem [15], $\mathcal{A}$ attains the bound if and only if $\mathcal{A}$ is a star of $\binom{[n]}{r}$. Two alternative proofs of the EKR Theorem that are particularly short and beautiful were obtained by Katona [16] and Daykin [8]. In his proof, Katona introduced an elegant technique called the *cycle method*. Daykin's proof is based on a fundamental result known as the Kruskal–Katona Theorem [17, 18, 23]. The EKR Theorem inspired a wealth of results and continues to do so; see [3, 10, 12, 13].

For positive integers k, r, n with $r \leq \min\{k, n\}$, let

$$\begin{aligned}\mathcal{P}_{k,r,n} := \{\{(x_1, y_1), \dots, (x_r, y_r)\} \colon & \, x_1, \dots, x_r \text{ are distinct elements of } [k],\\ & \, y_1, \dots, y_r \text{ are distinct elements of } [n]\}.\end{aligned}$$

We shall call $\mathcal{P}_{k,r,n}$ a family of *generalised permutations*. This is due to the fact that the elements of $\mathcal{P}_{n,n,n}$ are permutations of the set $[n]$; the permutation $y_1 y_2 \dots y_n$ of $[n]$ corresponds uniquely to the set $\{(1, y_1), (2, y_2), \dots, (n, y_n)\}$ in $\mathcal{P}_{n,n,n}$. In the more general case where $k \leq n$, the family $\mathcal{P}_{k,k,n}$ describes permutations of k-subsets of $[n]$; a permutation $y_1 y_2 \dots y_k$ of a k-subset of $[n]$ corresponds uniquely to the set $\{(1, y_1), (2, y_2), \dots, (k, y_k)\}$ in $\mathcal{P}_{k,k,n}$. The family $\mathcal{P}_{k,k,n}$ also describes injections from $[k]$ to $[n]$. The family $\mathcal{P}_{n,r,n}$ describes *r-partial permutations* of $[n]$ (see [19]). The ordered pairs formulation we are using follows [2] and also [4, 5], in which very general frameworks are considered.

In the case $r = k$, if two sets $\{(1, y_1), (2, y_2), \dots, (k, y_k)\}$ and $\{(1, z_1), (2, z_2), \dots, (k, z_k)\}$ in $\mathcal{P}_{k,k,n}$ intersect, then $y_i = z_i$ for some $i \in [k]$, and this is exactly what we mean by saying that the permutations $y_1 y_2 \dots y_k$ and $z_1 z_2 \dots z_k$ (of two k-subsets of $[n]$) intersect. In general, two generalised permutations intersect if and only if they have at least one ordered pair in common.

In this paper, we are concerned with the EKR problem for generalised permutations. We need only to consider the problem with $k \leq n$. To see this, define $\lambda \colon [k] \times [n] \to [n] \times [k]$ by $\lambda(x, y) := (y, x)$, then $\Lambda \colon \mathcal{P}_{k,r,n} \to \mathcal{P}_{n,r,k}$ by

$$\Lambda(\{(x_1, y_1), \dots, (x_r, y_r)\}) := \{\lambda(x_1, y_1), \dots, \lambda(x_r, y_r)\} = \{(y_1, x_1), \dots, (y_r, x_r)\}.$$

The functions λ and Λ are clearly both bijections. Moreover, any $P, Q \in \mathcal{P}_{k,r,n}$ are intersecting if and only if $\Lambda(P), \Lambda(Q) \in \mathcal{P}_{n,r,k}$ are intersecting. Therefore, throughout the rest of the paper it is to be assumed that $k \leq n$.

The origins of our problem lie in [9], in which Deza and Frankl proved that the size of an intersecting family of permutations of $[n]$ is at most the size $(n-1)!$ of a star of $\mathcal{P}_{n,n,n}$. Cameron and Ku [7] extended this result by establishing that only the stars of $\mathcal{P}_{n,n,n}$ attain the bound (other proofs of this result are found in [6, 14, 20, 24]). This result was also proved independently by Larose and Malvenuto [21], who established the stronger result that the stars of $\mathcal{P}_{k,k,n}$ are the largest intersecting subfamilies of $\mathcal{P}_{k,k,n}$ (see [21, Theorem 5.1]). These results summarise as follows.

Theorem 1.1 ([7, 9, 21]) *The size of any intersecting subfamily of* $\mathcal{P}_{k,k,n}$ *is at most* $\frac{(n-1)!}{(n-k)!}$, *and the bound is attained only by the stars of* $\mathcal{P}_{k,k,n}$.

Ku and Leader [19] solved the EKR problem for r-partial permutations of $[n]$ using Katona's cycle method. Moreover, they showed that for $8 \leq r \leq n-3$, the largest intersecting subfamilies of $\mathcal{P}_{n,r,n}$ are the stars. They conjectured that only the stars are extremal for the few remaining values of r too. A proof of this conjecture, also based on the cycle method, was obtained by Li and Wang [22].

Theorem 1.2 ([19, 22]) *For* $r \in [n-1]$, *the size of any intersecting subfamily of* $\mathcal{P}_{n,r,n}$ *is at most* $\binom{n-1}{r-1}\frac{(n-1)!}{(n-r)!}$, *and the bound is attained only by the stars of* $\mathcal{P}_{n,r,n}$.

The scope of this paper is to show that the methods used in [19, 22] allow us to generalise Theorems 1.1 and 1.2 as follows.

Theorem 1.3 *If* $r \leq k \leq n$ *and* $\mathcal{A}$ *is an intersecting subfamily of* $\mathcal{P}_{k,r,n}$, *then*

$$|\mathcal{A}| \leq \binom{k-1}{r-1}\frac{(n-1)!}{(n-r)!} = \binom{n-1}{r-1}\frac{(k-1)!}{(k-r)!},$$

and equality holds if and only if $\mathcal{A}$ *is a star of* $\mathcal{P}_{k,r,n}$.

2 Proof of the result

We will prove Theorem 1.3 by extending the arguments in [19, 22] to our more general setting. Recall that we are assuming $k \leq n$ and that Theorem 1.1 settles our problem for the case $r = k$, so we will only consider $r \leq k-1$. We will abbreviate $\mathcal{P}_{k,r,n}$ to $\mathcal{P}$.

Let mod be the usual *modulo operation*. We will use $\bmod^*$ to represent the modulo operation with the exception that for any non-zero integers a and b, the value of $ba \bmod^* a$ will be a rather than 0.

Let X be a set, and let $m = |X|$. A bijection $\sigma : X \to [m]$ is called an *ordering of* X. An element x of X is the $\sigma(x)$-th element in the ordering. If σ is an ordering of X and the elements of a subset A of X are numbered consecutively by σ, in the cyclic sense, then we say that A *meets* σ. Thus, an r-subset A of X meets σ if and only if we can label its elements $a_1, \ldots, a_r$ such that $\sigma(a_{i+1}) = (\sigma(a_i) + 1) \bmod^* m$ for each $i \in [r-1]$.

Katona's cycle method is based on the following fundamental result.

Lemma 2.1 *Let* X *be a set of size at least* $2r$, *and let* σ *be an ordering of* X. *Let* $\mathcal{B} := \{B \in \binom{X}{r} : B \text{ meets } \sigma\}$, *and let* $\mathcal{A}$ *be an intersecting subfamily of* $\mathcal{B}$. *Then* $|\mathcal{A}| \leq r$. *Moreover, if* $|X| > 2r$, *then* $|\mathcal{A}| = r$ *if and only if* $\mathcal{A}$ *is a star of* $\mathcal{B}$.

The proof of the bound was given in [16] and can be extended to a proof of the whole result (see [1]).

The union of all sets in $\mathcal{P}$ is the Cartesian product $[k] \times [n]$. We say that an ordering σ of $[k] \times [n]$ is r-*good* if every r elements $(x_1, y_1), \ldots, (x_r, y_r)$ of $[k] \times [n]$ that are numbered consecutively by σ, in the cyclic sense, are such that $x_1, \ldots, x_r$ are distinct and $y_1, \ldots, y_r$ are distinct. In an r-good ordering, any r consecutive elements form a generalised permutation in $\mathcal{P}$.

We will define an ordering of $[k] \times [n]$ that is r-good for all $r \in [k-1]$. (It is interesting to note that no such ordering exists if $r = k = n$.) Let $\tau \colon [k] \times [n] \to [kn]$ be defined by

$$\tau(x, y) := k((y - x) \bmod n) + x.$$

The following is an example with $k = 5$ and $n = 7$, where each element (x, y) of $[k] \times [n]$ is given the label $\tau(x, y)$ shown in bold superscript.

$(1,7)^{\mathbf{31}}$	$(2,7)^{\mathbf{27}}$	$(3,7)^{\mathbf{23}}$	$(4,7)^{\mathbf{19}}$	$(5,7)^{\mathbf{15}}$
$(1,6)^{\mathbf{26}}$	$(2,6)^{\mathbf{22}}$	$(3,6)^{\mathbf{18}}$	$(4,6)^{\mathbf{14}}$	$(5,6)^{\mathbf{10}}$
$(1,5)^{\mathbf{21}}$	$(2,5)^{\mathbf{17}}$	$(3,5)^{\mathbf{13}}$	$(4,5)^{\mathbf{9}}$	$(5,5)^{\mathbf{5}}$
$(1,4)^{\mathbf{16}}$	$(2,4)^{\mathbf{12}}$	$(3,4)^{\mathbf{8}}$	$(4,4)^{\mathbf{4}}$	$(5,4)^{\mathbf{35}}$
$(1,3)^{\mathbf{11}}$	$(2,3)^{\mathbf{7}}$	$(3,3)^{\mathbf{3}}$	$(4,3)^{\mathbf{34}}$	$(5,3)^{\mathbf{30}}$
$(1,2)^{\mathbf{6}}$	$(2,2)^{\mathbf{2}}$	$(3,2)^{\mathbf{33}}$	$(4,2)^{\mathbf{29}}$	$(5,2)^{\mathbf{25}}$
$(1,1)^{\mathbf{1}}$	$(2,1)^{\mathbf{32}}$	$(3,1)^{\mathbf{28}}$	$(4,1)^{\mathbf{24}}$	$(5,1)^{\mathbf{20}}$

Lemma 2.2 *For $r \leq k-1$, τ is an r-good ordering of $[k] \times [n]$.*

Proof. Suppose that $\tau(x, y) = \tau(u, v)$. Then $k((y - x) \bmod n) + x = k((v - u) \bmod n) + u$, so $u = kt + x$ for some integer t. Since $u, x \in [k]$, $t = 0$. Thus $u = x$, and hence $(y - x) \bmod n = (v - x) \bmod n$. It follows that $y = v$.

Therefore, τ is injective. Since the domain and the co-domain of τ are of equal size, τ is a bijection. Thus τ is an ordering of $[k] \times [n]$.

For each $i \in \{0, \ldots, n-1\}$, consider the $2k$-tuple

$$\begin{aligned} I_i := (&(1, (i+1) \bmod^* n), (2, (i+2) \bmod^* n), \ldots, (k, (i+k) \bmod^* n), \\ &(1, (i+2) \bmod^* n), (2, (i+3) \bmod^* n), \ldots, (k, (i+k+1) \bmod^* n)). \end{aligned}$$

The entries of I_i are the $(ki+1)$-th element through to the $(ki+2k)$-th element in the ordering τ.

Let $r \leq k-1$, and let A be an r-set that meets τ. Then for some $i \in \{0, \ldots, n-1\}$, A consists of r consecutive entries of I_i. Since $r \leq k-1$, any r consecutive entries of I_i are pairs which have distinct first entries and distinct second entries, thus A is a generalised permutation. Hence the result. $\square$

Let S_n denote the set of all bijections from $[n]$ to $[n]$. For any $(\phi, \psi) \in S_k \times S_n$, define $\tau_{\phi,\psi} : [k] \times [n] \to [kn]$ by

$$\tau_{\phi,\psi}(x, y) := \tau(\phi^{-1}(x), \psi^{-1}(y))$$

(that is, $\tau_{\phi,\psi}(\phi(i),\psi(j)) := \tau(i,j)$). Note that $\tau_{\phi,\psi}$ is an ordering of $[k]\times[n]$. Let

$$T_{k,n} := \{\tau_{\phi,\psi} \colon (\phi,\psi) \in S_k \times S_n\}.$$

Further, for any $(\phi,\psi) \in S_k \times S_n$, define $f_{\phi,\psi} : [k]\times[n] \to [k]\times[n]$ by

$$f_{\phi,\psi}(x,y) := (\phi(x),\psi(y)).$$

Lemma 2.3 *For $r \le k-1$ and $(\phi,\psi) \in S_k \times S_n$, $\tau_{\phi,\psi}$ is an r-good ordering of $[k]\times[n]$.*

Proof. Suppose that $\tau_{\phi,\psi}$ is not an r-good ordering. Thus there exist two distinct elements (a_1,b_1) and (a_2,b_2) of $[k]\times[n]$ such that

$$\tau_{\phi,\psi}(a_2,b_2) = (\tau_{\phi,\psi}(a_1,b_1)+p) \operatorname{mod}^* kn$$

for some $p \in [r-1]$, with either $a_1 = a_2$ or $b_1 = b_2$. If $a_1 = a_2$, then

$$\tau(\phi^{-1}(a_1),\psi^{-1}(b_2)) = \big(\tau(\phi^{-1}(a_1),\psi^{-1}(b_1))+p\big) \operatorname{mod}^* kn,$$

contradicting Lemma 2.2. Similarly, we cannot have $b_1 = b_2$. □

Let Z be a set, and let σ be an ordering of Z. Let m be an integer with $2 \le m \le |Z|$, and suppose that $z_1,\dots,z_m$ are distinct elements of Z. If $\sigma(z_{i+1}) = (\sigma(z_i)+1) \operatorname{mod}^* |Z|$ for each $i \in [m-1]$, then we say that the tuple $(z_1,\dots,z_m)$ is an *m-interval of* σ, and we call $\{z_1,\dots,z_m\}$ the *set corresponding to* $(z_1,\dots,z_m)$. If $1 \le m_1 \le m_2 \le m$ and $\ell = m_2 - m_1 + 1$, then we call the ℓ-interval $(z_{m_1},\dots,z_{m_2})$ of σ an *ℓ-subinterval of* $(z_1,\dots,z_m)$. If a generalised permutation meets an ordering, then the elements of the generalised permutation form an interval of the ordering.

Lemma 2.4 *Each member of $\mathcal{P}$ meets exactly $r!(k-r)!(n-r)!kn$ members of $T_{k,n}$.*

Proof. Let $P,Q \in \mathcal{P}$. Clearly, $Q = \{f_{\pi,\rho}(x,y) \colon (x,y) \in P\}$ for some $(\pi,\rho) \in S_k \times S_n$.

Let $\tau_{\phi,\psi} \in T_{k,n}$. For any $(x,y) \in S_k \times S_n$,

$$\begin{aligned}\tau_{\phi,\psi} \circ f_{\pi^{-1},\rho^{-1}}(x,y) &= \tau_{\phi,\psi}(\pi^{-1}(x),\rho^{-1}(y))\\ &= \tau(\phi^{-1}\circ\pi^{-1}(x),\psi^{-1}\circ\rho^{-1}(y))\\ &= \tau((\pi\circ\phi)^{-1}(x),(\rho\circ\psi)^{-1}(y)).\end{aligned}$$

Thus, since $\pi\circ\phi \in S_k$ and $\rho\circ\psi \in S_n$, we have

$$\tau_{\phi,\psi} \circ f_{\pi^{-1},\rho^{-1}} = \tau_{\pi\circ\phi,\rho\circ\psi} \in T_{k,n}.$$

Note that if P meets $\tau_{\phi,\psi}$, then Q meets $\tau_{\phi,\psi} \circ f_{\pi^{-1},\rho^{-1}}$. Thus Q meets at least as many members of $T_{k,n}$ as P does. Conversely, we can do this for every ordering that Q meets, thus P and Q meet the same number of members of $T_{k,n}$.

Each of the $k!n!$ members of $T_{k,n}$ contains exactly kn r-intervals, and, by Lemma 2.3, the sets corresponding to these r-intervals are members of $\mathcal{P}$. Thus, for each $\tau_{\phi,\psi} \in T_{k,n}$, the number of members of $\mathcal{P}$ that meet $\tau_{\phi,\psi}$ is kn. Since $|\mathcal{P}| = \binom{k}{r}\frac{n!}{(n-r)!}$, each member of $\mathcal{P}$ meets exactly

$$\frac{k!n!kn}{\binom{k}{r}\frac{n!}{(n-r)!}} = r!(k-r)!(n-r)!kn$$

members of $T_{k,n}$. □

Proof of Theorem 1.3. Let $\mathcal{A}$ be an intersecting subfamily of $\mathcal{P}$ of maximum size. Let $T := T_{k,n}$. For an ordering σ in T, a family $\mathcal{F} \subseteq \mathcal{P}$, and a set $P \in \mathcal{P}$, let $\mathcal{F}_\sigma := \{F \in \mathcal{F} \colon F \text{ meets } \sigma\}$ and

$$\Phi(\sigma, P) := \begin{cases} 1, & \text{if } P \text{ meets } \sigma; \\ 0, & \text{otherwise.} \end{cases}$$

Let $q := r!(k-r)!(n-r)!kn$. By Lemma 2.4, $\sum_{\sigma \in T} \Phi(\sigma, P) = q$. By Lemma 2.1, $\sum_{A \in \mathcal{A}_\sigma} \Phi(\sigma, A) \leq r$ for each $\sigma \in T$. We therefore have

$$q|\mathcal{A}| = \sum_{A \in \mathcal{A}} q = \sum_{A \in \mathcal{A}} \sum_{\sigma \in T} \Phi(\sigma, A) = \sum_{\sigma \in T} \sum_{A \in \mathcal{A}} \Phi(\sigma, A) = \sum_{\sigma \in T} \sum_{A \in \mathcal{A}_\sigma} \Phi(\sigma, A) \leq \sum_{\sigma \in T} r = r|T|, \tag{1}$$

and hence

$$|\mathcal{A}| \leq \frac{r|T|}{q} = \binom{k-1}{r-1}\frac{(n-1)!}{(n-r)!}.$$

This establishes the bound in the theorem.

The intersecting family $\{P \in \mathcal{P} \colon (1,1) \in P\}$ meets the bound, so the size of $\mathcal{A}$ is $\binom{k-1}{r-1}\frac{(n-1)!}{(n-r)!}$. Thus, equality holds in (1), and hence $|\mathcal{A}_{\phi,\psi}| = r$ for each $\tau_{\phi,\psi} \in T$, where $\mathcal{A}_{\phi,\psi} := \mathcal{A}_{\tau_{\phi,\psi}}$. By Lemma 2.1, for each $\tau_{\phi,\psi} \in T$, the r sets in $\mathcal{A}_{\phi,\psi}$ contain a fixed element $(x_{\phi,\psi}, y_{\phi,\psi})$. Thus, for each $\tau_{\phi,\psi} \in T$,

$$\mathcal{A}_{\phi,\psi} = \{A \colon A \text{ corresponds to an } r\text{-subinterval of } L_{\phi,\psi}\}, \tag{2}$$

where $L_{\phi,\psi}$ is the $(2r-1)$-interval of $\tau_{\phi,\psi}$ with middle entry $(x_{\phi,\psi}, y_{\phi,\psi})$.

Let β be the identity function from $[k]$ to $[k]$, and let γ be the identity function from $[n]$ to $[n]$. Thus $\tau = \tau_{\beta,\gamma}$. We may assume that $(x_{\beta,\gamma}, y_{\beta,\gamma}) = (k,k)$. Thus $\mathcal{A}_{\beta,\gamma}$ consists of the r sets corresponding to the r-subintervals of the $(2r-1)$-interval

$$L_{\beta,\gamma} = ((k-r+1, k-r+1), \dots, (k,k), (1,2), \dots, (r-1,r)).$$

Define

$$I := \{(i,i) \colon i \in [k-1]\}, \qquad \bar{I} := ([k] \times [n]) \backslash (I \cup \{(k,k)\}).$$

If $P \subseteq I$, then P does not intersect the set $\{(k,k), (1,2), \dots, (r-1,r)\} \in \mathcal{A}_{\beta,\gamma}$; similarly, if $P \subseteq \bar{I}$, then P does not intersect the set $\{(k-r+1, k-r+1), \dots, (k,k)\} \in \mathcal{A}_{\beta,\gamma}$. Thus, for each $A \in \mathcal{A}$ with $(k,k) \notin A$, it is the case that $A \not\subseteq I$ and $A \not\subseteq \bar{I}$, so

$$1 \leq |A \cap I| \leq r-1, \qquad 1 \leq |A \cap \bar{I}| \leq r-1. \tag{3}$$

Define the sets

$$T' := \{\tau_{\pi,\rho} \in T \colon \pi(k) = \rho(k) = k\}, \qquad T^* := \{\tau_{\pi,\rho} \in T' \colon \pi(i) = \rho(i), i = 1, \dots, k\}.$$

We will first show that $(x_{\pi,\rho}, y_{\pi,\rho}) = (k,k)$ for each $\tau_{\pi,\rho} \in T^*$. From this we can show that the same holds for each $\tau_{\pi,\rho} \in T'$.

Note that for each $\tau_{\pi,\rho} \in T^*$,

$$\{(\pi(i), \rho(i)) \colon (i,i) \in I\} = I, \qquad \{(\pi(i), \rho(j)) \colon (i,j) \in \bar{I}\} = \bar{I}. \tag{4}$$

If $(x_{\pi,\rho}, y_{\pi,\rho}) \in I$, then, by (4), I contains an r-subset R that corresponds to an r-subinterval of $L_{\pi,\rho}$, and hence $R \in \mathcal{A}$ by (2), but this contradicts the first inequality in (3). Similarly, $(x_{\pi,\rho}, y_{\pi,\rho}) \in \bar{I}$ contradicts the second inequality in (3). So $(x_{\pi,\rho}, y_{\pi,\rho}) = (k,k)$ for each $\tau_{\pi,\rho} \in T^*$.

Now suppose $(x_{\pi,\rho}, y_{\pi,\rho}) \neq (k,k)$ for some $\tau_{\pi,\rho} \in T'$. Then $L_{\pi,\rho}$ has an r-subinterval which does not have (k,k) as one of its entries. Let B be the set corresponding to this interval; by (2), $B \in \mathcal{A}$. By (3), $1 \leq s := |B \cap I| \leq r-1$. Let $(a_1,a_1), \dots, (a_s,a_s)$ be the s distinct elements of $B \cap I$. Let $a_{s+1}, \dots, a_k$ be the $k-s$ distinct elements of $[k] \backslash \{a_1, \dots, a_s\}$. Since $(k,k) \notin B \cap I$, we may assume that $a_k = k$.

Choose $(\pi^*, \rho^*) \in S_k \times S_n$ such that $\pi^*(i) = \rho^*(i) = a_i$ for each $i \in [k]$. Thus $\tau_{\pi^*,\rho^*} \in T^*$, and hence $(x_{\pi^*,\rho^*}, y_{\pi^*,\rho^*}) = (k,k) = (a_k,a_k)$ (as shown above). Therefore,

$$L_{\pi^*,\rho^*} = ((a_{k-r+1}, a_{k-r+1}), \dots, (a_k,a_k), (a_1,a_2), \dots, (a_{r-1},a_r)),$$

and the r-set

$$C := \{(a_{k-r+s}, a_{k-r+s}), \dots, (a_k,a_k), (a_1,a_2), \dots, (a_{s-1},a_s)\}$$

corresponds to an r-subinterval of L_{π^*,ρ^*}; by (2), $C \in \mathcal{A}$. Since $k-r+s > s$, the pairs $(a_{k-r+s}, a_{k-r+s}), \dots, (a_{k-1},a_{k-1}), (a_k,a_k)$ are not in B. Further, for each $i \in [s]$, $(a_i, a_{i+1}) \notin B$ since $(a_i,a_i) \in B$. Thus B and C do not intersect, but this contradicts $B, C \in \mathcal{A}$.

Therefore, for every $\tau_{\pi,\rho} \in T'$,

$$(x_{\pi,\rho}, y_{\pi,\rho}) = (k,k). \tag{5}$$

Finally, let A be a set $\{(x_1,y_1), \dots, (x_r,y_r)\}$ in $\mathcal{P}$ that contains (k,k). We may assume that $(x_r,y_r) = (k,k)$. Let $(\pi,\rho) \in S_k \times S_n$ be such that $\pi(i+k-r) = x_i$ and $\rho(i+k-r) = y_i$ for each $i \in [r]$. Then $\tau_{\pi,\rho} \in T'$ and A meets $\tau_{\pi,\rho}$. By (5) and (2), $A \in \mathcal{A}$. Thus $\{P \in \mathcal{P} \colon (k,k) \in P\} \subseteq \mathcal{A}$. Since $|\mathcal{A}| \leq \binom{k-1}{r-1}\frac{(n-1)!}{(n-r)!}$, it follows that $\mathcal{A} = \{P \in \mathcal{P} \colon (k,k) \in P\}$. $\square$

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