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A note on the cyclic matching sequencibility of graphs

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Abstract

In this note we present answers to the open problems posed by Brualdi, Kiernan, Meyer and Schroeder in [Cyclic matching sequencibility of graphs, *Australas. J. Combin.* **53** (2012), 245–256].

1 Discussion and response

Let $G \subseteq K_n$ be a graph of order n with m edges. The *matching number* of G is the maximum number of edges in a matching. The *matching number of a linear ordering* $e_1, e_2, \ldots, e_m$ of the edges of G is the largest number d such that every d consecutive edges in the ordering form a d-matching of G. The *matching sequencibility* of G, denoted $\text{MS}(G)$, is the maximum matching number of a linear ordering of the edges of G. The *cyclic matching sequencibility* of G, denoted $\text{CMS}(G)$, is the largest integer d such that there exists a cyclic ordering of the edges so that every d consecutive edges in the ordering form a matching of G. In [1] Brualdi, Kiernan, Meyer, and Schroeder pose three questions concerning the relationship between $\text{MS}(G)$ and $\text{CMS}(G)$. In this note we use the graph Y_n in Figure 1 to provide answers to each of these questions. If G is any simple graph, kG denotes the multi-graph in which every edge of G is replicated k times.

If we consider the linear ordering α as a function

$$\alpha : E(G) \mapsto \{1, ..., m\}$$

we can define the linear distance in α between two edges e_i, e_j as:

$$d_\alpha(e_i, e_j) = |\alpha(e_i) - \alpha(e_j)|$$

Similarly if we consider the cyclic ordering β as a function:

$$\beta : E(G) \mapsto \{1, ..., m\}$$

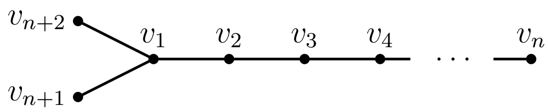


Figure 1: The graph Y_n

we can define the cyclic distance in β between two edges e_i, e_j as:

$$d_\beta(e_i, e_j) = \min\{|\beta(e_i) - \beta(e_j)|, m - |\beta(e_i) - \beta(e_j)|\}$$

Question 1: *Given a graph G with matching number p, is there a positive integer k such that* $\text{MS}(kG) = p \quad (\text{CMS}(kG) = p)$*?*

The graph Y_n has diameter n and hence because no two of the edges

$$\{v_1, v_2\}, \{v_1, v_{n+1}\}, \{v_1 v_{n+2}\}$$

are in a matching it is easy to see the matching number of Y_n is $n/2$ if n is even; $(n+1)/2$ if n is odd. However when n is odd the largest matching containing v_1v_2 is $(n-1)/2$. Hence, $\text{MS}(Y_n) \leq (n-1)/2$. Now consider $kG = kY_n$, n odd. Any of the k edges between the vertices $\{v_1\}, \{v_2\}$ can be in a matching of size at most $(n-1)/2$. Thus, $\text{MS}(kY_n) \leq (n-1)/2$ for any k.

The answer to Question 1 is no.

Question 2: *For a graph G, we have* $\text{MS}(G) \geq \text{CMS}(G)$. *How large can* $\text{MS}(G) - \text{CMS}(G)$ *be? Is* $\text{CMS}(G) \geq \text{MS}(G) - 1$*?*

Consider the graph Y_n, when n is even. Label the edges according to the following table:

Edge		Label	
$\{v_{2i}, v_{2i+1}\}$	$\mapsto$	$i+1,$	$1 \leq i < \frac{n}{2}$
$\{v_{2i+1}, v_{2i+2}\}$	$\mapsto$	$\frac{n}{2}+1+i,$	$1 \leq i < \frac{n}{2}$
$\{v_{n+2}, v_1\}$	$\mapsto$	$\frac{n}{2}+1$	
$\{v_1, v_2\}$	$\mapsto$	$n+1$	
$\{v_{n+1}, v_1\}$	$\mapsto$	1	

This labelling gives us $\text{MS}(Y_n) \geq \frac{n}{2} - 1$.

Let β be a cyclic ordering of Y_n and let e_0, e_1, e_2 be the three edges incident to the vertex v_1 ordered such that

$$1 \leq \beta(e_0) < \beta(e_1) < \beta(e_2).$$

For any $i \in \mathbb{Z}_3$ consider the set $\{e \in E(G) : \beta(e_i) \leq \beta(e) \leq \beta(e_{i+1})\}$. This is a set of size $d_\beta(e_i, e_{i+1}) + 1$ which is not a matching. Therefore the β-distance between edges e_i and e_{i+1} is an upper bound to the matching number of β. The sum of these distances is:

$$d_\beta(e_0, e_1) + d_\beta(e_1, e_2) + d_\beta(e_2, e_0) = n + 1.$$

Taking the average we obtain:

$$\frac{d_\beta(e_0, e_1) + d_\beta(e_1, e_2) + d_\beta(e_2, e_0)}{3} = \frac{n+1}{3}.$$

Hence for some i we have $d_\beta(e_i, e_{i+1}) \leq \frac{n+1}{3}$ and the matching number of β is at most $\frac{n+1}{3}$. Therefore $\text{CMS}(Y_n) \leq \frac{n+1}{3}$.

Thus,

$$\text{MS}(Y_n) - \text{CMS}(Y_n) \geq \frac{n}{2} - 1 - \frac{n+1}{3} = \frac{n-4}{6}.$$

Thus, we see that

$$\lim_{n\to\infty} \frac{\text{MS}(Y_n) - \text{CMS}(Y_n)}{n} \geq \frac{1}{6}.$$

Consequently our answer to Question 2 is that the difference $\text{MS}(G) - \text{CMS}(G)$ can be made as large as desired.

Question 3: *Given a graph G, is $\text{CMS}(2G) = \text{MS}(G)$?*

From our answer to Question 2 we know that $\text{MS}(Y_{2k}) \geq k - 1$, where $n = 2k$. Now consider $2Y_{2k}$. Let β be a cyclic ordering of $2Y_{2k}$ and let $e_0, e_1, e_2 \ldots, e_5$ be the six edges incident to the vertex v_1 ordered such that

$$1 \leq \beta(e_0) < \beta(e_1) < \cdots < \beta(e_5).$$

For any $i \in \mathbb{Z}_6$ consider the set $\{e \in E(G) : \beta(e_i) \leq \beta(e) \leq \beta(e_{i+1})\}$. This is a set of size $d_\beta(e_i, e_{i+1}) + 1$ which is not a matching. Therefore the β-distance between edges e_i and e_{i+1} is an upper bound to the matching number of β. The sum of these distances is:

$$d_\beta(e_0, e_1) + d_\beta(e_1, e_2) + \cdots + d_\beta(e_5, e_0) = 4k + 2.$$

Taking the average we obtain:

$$\frac{d_\beta(e_0, e_1) + d_\beta(e_1, e_2) + \cdots + d_\beta(e_5, e_0)}{6} = \frac{4k+2}{6}.$$

Hence for some i we have $d_\beta(e_i, e_{i+1}) \leq \frac{4k+2}{6}$ and the matching number of β is at most $\frac{4k+2}{6}$. Therefore $\text{CMS}(2Y_{2k}) \leq \frac{4k+2}{6} \leq k - 1 \leq \text{MS}(2Y_{2k})$, and the answer to the Question 3 is no.

2 Further results

Given the answer to Question 2 one might ask how small can $\text{CMS}(G)$ be? We now provide an answer.

Lower Bound for cms(G)

Theorem 2.1 $\lfloor \text{MS}(G)/2 \rfloor \leq \text{CMS}(G)$.

PROOF: Let $\text{MS}(G) = n$ and write $|E(G)| = kn + r$ for some $k \geq 1$ and $r < n$. Choose an ordering $\alpha : E(G) \mapsto \{1, ..., kn + r\}$ of the edges of G with matching number n. For $1 \leq i \leq k$, let:

$$A_{i,1} = \left\{ n(i-1)+1, ..., \left\lfloor \tfrac{n(2i-1)}{2} \right\rfloor \right\} \quad \text{and} \quad A_{i,-1} = \left\{ \left\lfloor \tfrac{n(2i-1)}{2} \right\rfloor + 1, ..., ni \right\}.$$

Also let

$$A_{k+1,1} = \left\{ kn+1, ..., kn + \left\lfloor \frac{n}{2} \right\rfloor \right\} \cap \{1, ..., kn+r\}$$

and

$$A_{k+1,-1} = \left\{ kn + \left\lfloor \frac{n}{2} \right\rfloor + 1, ..., (k+1)n \right\} \cap \{1, ..., kn+r\}.$$

Define the ordering $\beta : E(G) \mapsto \{1, ..., kn + r\}$ by

$$\beta(x) = \begin{cases} \alpha(x) - (i-1) \left\lceil \frac{n}{2} \right\rceil & \text{if } \alpha(x) \in A_{i,1} \\ nk + r - \alpha(x) + i \left\lfloor \frac{n}{2} \right\rfloor + 1 & \text{if } \alpha(x) \in A_{i,-1}. \end{cases}$$

We chose to define β in this way so that β will satisfy the following three conditions:

(i) If $\alpha(x), \alpha(y) \in \bigcup_{i=1}^{k+1} A_{i,1}$, then $\beta(x) < \beta(y)$ if and only if $\alpha(x) < \alpha(y)$.

(ii) If $\alpha(x) \in \bigcup_{i=1}^{k+1} A_{i,1}$, $\alpha(y) \in \bigcup_{i=1}^{k+1} A_{i,-1}$, then $\beta(x) < \beta(y)$ always.

(iii) If $\alpha(x), \alpha(y) \in \bigcup_{i=1}^{k+1} A_{i,-1}$, then $\beta(x) < \beta(y)$ if and only if $\alpha(x) > \alpha(y)$.

For any set B of cyclically consecutive edges with respect to the ordering β with $|B| = \left\lfloor \frac{n}{2} \right\rfloor$, we want to show that B is a matching. If $\alpha(B) \subset A_{i,\epsilon}$ the result is trivial. Otherwise we have three cases:

Case 1: $\boldsymbol{\alpha(B) \subset A_{i,\epsilon} \cup A_{i+1,\epsilon}}$

If $\alpha(B) \subset A_{i,\epsilon} \cup A_{i+1,\epsilon}$, consider $A = B \cup \alpha^{-1}\left(A_{i-\frac{-1+\epsilon}{2},-\epsilon}\right)$. As A is a set of n or fewer consecutive edges with respect to the ordering α, A is a matching. Hence B is a matching.

Case 2: $\boldsymbol{\alpha(B) \subset A_{k+1,1} \cup A_{k+1,-1} \cup A_{k,-1}}$

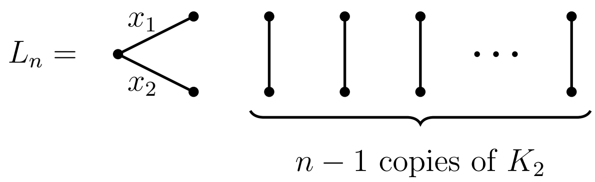
If $\alpha(B) \subset A_{k+1,1} \cup A_{k+1,-1} \cup A_{k,-1}$, consider $A = B \cup \alpha^{-1}(A_{k+1,1})$. As A is a set of n or fewer consecutive edges with respect to the ordering α, A is a matching. Hence B is a matching.

Case 3: $\boldsymbol{\alpha(B) \subset A_{1,-1} \cup A_{1,1}}$

If $\alpha(B) \subset A_{1,-1} \cup A_{1,1}$, consider $A = \alpha^{-1}(A_{1,-1} \cup A_{1,1})$. As A is a set of n or fewer consecutive edges with respect to the ordering α, A is a matching and so is B because $B \subset A$. □

We will now provide an example that shows that this bound is sharp when $\text{MS}(G) = 2k$ and almost sharp when $\text{MS}(G) = 2k+1$.

Let L_n be the disjoint union of P_2, the path of length two, with $n-1$ copies of K_2, i.e.:



If the edges of P_2 are x_1, x_2, then any ordering α such that $\alpha(x_1) = 1$ and $\alpha(x_2) = n+1$ has matching number n. This is clearly an upper bound to $\text{MS}(L_n)$, as it has $n+1$ edges. Therefore $\text{MS}(L_n) = n$.

The determination of $\text{CMS}(L_n)$ is similar to the way $\text{CMS}(Y_n)$ was determined. Here any ordering β will have $d_\beta(x_1, x_2) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1 = \left\lceil \frac{n}{2} \right\rceil$. But this number is achieved by any ordering β that satisfies $\beta(x_1) = 1$, $\beta(x_2) = \left\lceil \frac{n}{2} \right\rceil + 1$. Hence $\text{CMS}(L_n) = \left\lceil \frac{n}{2} \right\rceil = \left\lceil \frac{\text{MS}(L_n)}{2} \right\rceil$.

Notice that for $n = 2k$, $\left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor = \frac{n}{2}$, thus the bound given in Theorem 2.1 is sharp for even n. When $n = 2k+1$, $\left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor + 1$ and so if the bound is not sharp it is only off by 1.

References

[1] R.A. Brualdi, K.P. Kiernan, S.A. Meyer and M.W. Schroeder, Cyclic matching sequencibility of graphs, *Australas. J. Combin.* **53** (2012), 245–256.

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