

Pairwise balanced designs of dimension three*

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Abstract

The dimension of a pairwise balanced design $\text{PBD}(v, K)$ is the maximum positive integer d such that any d of its points are contained in a proper flat. The standard examples are affine and projective linear spaces. Also, Teirlinck provided a nearly complete existence theory for ‘Steiner spaces’, which is the case $K = \{3\}$ and $d = 3$. A recent result of the first author and A.C.H. Ling says that there exists a $\text{PBD}(v, K)$ of dimension at least d for all sufficiently large and numerically admissible v . Here, we consider the detailed existence question for $K = \{3, 4, 5\}$ and $d = 3$. A certain problem on latin squares connects with this particular case.

1 Introduction

Let v be a positive integer and $K \subset \mathbb{Z}_{\geq 2} := \{2, 3, 4, \dots\}$. A pairwise balanced design $\text{PBD}(v, K)$ is a pair (X, \mathcal{B}) , where X is a v -set of *points* and \mathcal{B} is a family of *blocks* such that

- for each $B \in \mathcal{B}$, we have $B \subseteq X$ with $|B| \in K$; and
- any two distinct points in X appear together in exactly one block.

Note that there are numerical constraints on v given K . First, the number of pairs of distinct points must be a (nonnegative) integral linear combination of the number of distinct pairs arising from blocks with sizes in K . This leads to the *global condition*

$$v(v-1) \equiv 0 \pmod{\beta(K)}, \tag{1.1}$$

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where $\beta(K) := \gcd\{k(k - 1) : k \in K\}$. Also, deleting any point $x \in X$ from its incident blocks must induce a partition of the remaining points. That is, $v - 1$ is an integral combination of $k - 1$ for $k \in K$. In other words, we have the *local condition*

$$v - 1 \equiv 0 \pmod{\alpha(K)}, \tag{1.2}$$

where $\alpha(K) := \gcd\{k - 1 : k \in K\}$. Wilson’s theory, [11], asserts that (1.1) and (1.2) are sufficient for $v \gg 0$.

A *flat* (or *subdesign*) is a pair $(Y, \mathcal{B}|_Y)$, where $Y \subseteq X$ and $\mathcal{B}|_Y := \{B \in \mathcal{B} : B \subseteq Y\}$, which is itself a pairwise balanced design. Note that any block intersecting Y in two or more points must be fully contained in Y , and hence a member of $\mathcal{B}|_Y$. Flats in (X, \mathcal{B}) form a lattice under intersection. So any set of points $S \subseteq X$ generates a flat $\langle S \rangle_{\mathcal{B}}$ (or simply $\langle S \rangle$) equal to the intersection of all flats containing S . Alternatively, $\langle S \rangle$ can be computed algorithmically starting from S by repeatedly extending pairs by blocks.

Pairwise balanced designs are also known as ‘linear spaces’, where blocks correspond with lines and flats correspond with subspaces. This innocent difference in terminology often reflects a substantial difference in the typical research directions. Our topic is a blend of these.

The *dimension* of a PBD (or linear space) is the maximum integer d such that any set of d points generates a proper flat. For instance, the flat generated by any two points is the line containing them. So every ‘nontrivial’ $\text{PBD}(v, K)$ has dimension at least two. See [5] and Chapter 7 of [1] for nice surveys of dimension in linear spaces.

There are classical examples of designs with nontrivial dimension. Let q be a prime power and \mathbb{F}_q the finite field of order q . We can take as points the vector space $X = \mathbb{F}_q^d$, and as flats all possible translates of subspaces $x + W \subseteq X$. This forms the *affine space* $\text{AG}_d(q)$. Taking the one-dimensional flats as blocks, we obtain a $\text{PBD}(q^d, \{q\})$ of dimension d . For example, the case $q = 3, d = 4$ recovers the popular card game ‘Set’.

Consider next the vector space \mathbb{F}_q^{d+1} , where as points we take the ‘directions’ or one-dimensional subspaces. When equipped with the full lattice of subspaces, we have the *projective space* $\text{PG}_d(q)$. In particular, if \mathcal{B} is the set of two-dimensional subspaces, incidence given by inclusion, we obtain a $\text{PBD}([d]_q, \{q + 1\})$ of dimension d , where $[d]_q = \frac{q^{d+1}-1}{q-1}$. With $q = 2, d = 3$, we have $[3]_2 = 15$ points represented by all nonzero binary 4-tuples; lines correspond to zero-sum sets of three points.

Recall that a *Steiner triple system* is a $\text{PBD}(v, \{3\})$. It is well known that Steiner triple systems on v points exist if and only if $v \equiv 1$ or $3 \pmod{6}$. A *Steiner space* is defined to be a Steiner triple system of dimension at least 3. Teirlinck in [10] studied the existence of Steiner spaces, finding that, for $v \notin \{51, 67, 69, 145\}$, they exist if and only if $v = 15, 27, 31, 39$, or $v \geq 45$. The four undecided cases are still open, to the best of our knowledge.

Recently, the first author and Ling obtained an asymptotic existence theory for any K and prescribed minimum dimension d .

Theorem 1.1 ([8]) *Given $K \subseteq \mathbb{Z}_{\geq 2}$ and $d \in \mathbb{Z}_+$, there exists a $PBD(v, K)$ of dimension at least d for all sufficiently large v satisfying (1.1) and (1.2).*

Here, we consider in more detail the case $K = \{3, 4, 5\}$ and dimension three. Note that $\alpha(K) = 1$ and $\beta(K) = 2$ in this case, so (1.1) and (1.2) disappear and all positive integers are admissible.

Theorem 1.2 (Main theorem) *There exists a $PBD(v, \{3, 4, 5\})$ of dimension three if and only if $v = 15$ or $v \geq 27$ except for $v = 32$ and possibly for $v \in E := \{33, 34, 35, 38, 41, 42, 43, 47\}$.*

There is a natural division into cases: large values of v via recursion, small values of v via ad hoc constructions, and nonexistence results for small v . These define Sections 3, 4, and 5, respectively. Also, there is the need to reduce dimension to three if it happens to be constructed as larger. This is accomplished in Section 7. As a result of Theorem 7.1 to follow, we need not distinguish between ‘dimension three’ and ‘dimension at least three’ in our constructions. In Section 6, there is a discussion of other sets K and an application to latin squares in which any cell-symbol combination can be bounded inside a proper subsquare.

2 Background

We begin by stating the basic existence result for PBDs having $K = \{3, 4, 5\}$. The following has by now achieved ‘folklore’ status, but the interested reader can see [4] for more information.

Theorem 2.1 *There exists a $PBD(v, \{3, 4, 5\})$ if and only if $v \neq 2, 6, 8$.*

Unless otherwise mentioned, $K = \{3, 4, 5\}$. In what follows, we occasionally make use of structure in the smallest cases. The unique PBDs for $v = 7, 9$ are the ‘Fano plane’ $PG_2(2)$ and the affine plane $AG_2(3)$. The unique PBD for $v = 10$ is the extension of one parallel class in $AG_2(3)$ by a point. The extended blocks have size 4. The only PBD for $v = 11$ has a unique block of size 5 and all other blocks of size 3. For $v = 12$, only blocks of size 3 and 4 are possible. In the case $v = 13$, there exists a Steiner triple system, a projective plane $PG_2(3)$, and even systems with blocks of size 5. For $v = 14$, we have a PBD with one block of size 5 and all other blocks of size 3 or 4.

A *group divisible design* is a triple (X, Π, \mathcal{B}) , where X is a set of *points*, Π is a partition of X into *groups*, and \mathcal{B} is a set of *blocks* such that

- a group and a block intersect in at most one point; and
- every pair of points from distinct groups is together in exactly one block.

For short, we refer to this as a GDD or K -GDD, the latter emphasizing that the blocks have sizes in $K \subset \mathbb{Z}_{\geq 2}$. The *type* of a GDD is the list of its group sizes. When this list contains, say, u copies of the integer g , this is abbreviated with ‘exponential notation’ as g^u . Note that a K -GDD of type 1^v is just a $\text{PBD}(v, K)$. The $\text{PBD}(14, \{3, 4, 5\})$ mentioned earlier is actually equivalent to a GDD of type $3^3 5^1$, where four blocks partitioning the points are turned into groups.

A *transversal design* $\text{TD}(k, n)$ is a $\{k\}$ -GDD of type n^k . In this case, every block meets every group in one point. For our purposes, it is convenient to state the known existence theory for block size 5.

Theorem 2.2 *There exists $\text{TD}(5, n)$ if $n \neq 2, 3, 6, 10$.*

Remark. Except for the case $n = 10$, which still remains in doubt, the converse also holds.

The above transversal designs can be modified somewhat to yield a useful existence result on five-group GDDs. Most of the following are obtained by ‘truncation’ (see below), but some of the small cases require special constructions. For more details, see [9].

Lemma 2.3

- (a) *For any integers $m \geq 3$ and $i \in \{0, \dots, 5\}$, there exists a $\{3, 4, 5\}$ -GDD of type $(m - 1)^i m^{5-i}$.*
- (b) *For any $i \in \{0, \dots, 5\}$, there exists a $\{3, 4, 5\}$ -GDD of type $1^i 3^{5-i}$.*

We now review some standard design-theoretic constructions. First, we can ‘fill’ the groups of a GDD with PBDs (sometimes as small as a single block). In fact, one or more points can be added, provided the filled PBDs each have the extra points as a common flat.

Construction 2.4 (Filling groups) *Suppose there exists a K -GDD on v points with group sizes in G . If, for each $g \in G$:*

- *there exists a $\text{PBD}(g, K)$, then there exists a $\text{PBD}(v, K)$;*
- *there exists a $\text{PBD}(g + 1, K)$, then there exists a $\text{PBD}(v + 1, K)$; and*
- *there exists a $\text{PBD}(g + h, K)$ containing a flat of order h , then there exists a $\text{PBD}(v + h, K)$.*

To construct a GDD, one can *delete* a point x and all its incident blocks \mathcal{B}_x from some PBD. The resulting group partition $\Pi_x := \{B \setminus \{x\} : B \in \mathcal{B}_x\}$ is left behind from the missing blocks. In reverse, adding a point and filling in groups with new blocks, we obtain a special case of the above.

One could alternatively *truncate* a set of points, say $A \subseteq X$, replacing blocks $B \in \mathcal{B}$ by new blocks $B \setminus A$. (New blocks of size 0,1 can be ignored.) If the original space is a PBD or GDD, then so is the truncation. However, care must be taken to respect the desired set of block sizes.

In the next two sections, we make frequent use of the following.

Construction 2.5 (Wilson’s fundamental construction) *Suppose there exists a ‘master’ GDD (X, Π, \mathcal{B}) , where $\Pi = \{X_1, \dots, X_u\}$. Let $\omega : X \rightarrow \{0, 1, 2, \dots\}$, assigning nonnegative weights to each point in such a way that for every $B \in \mathcal{B}$ there exists an ‘ingredient’ K -GDD of type $\omega(B) := [\omega(x) \mid x \in B]$. Then there exists a K -GDD of type*

$$\left[\sum_{x \in X_1} \omega(x), \dots, \sum_{x \in X_u} \omega(x) \right].$$

Remark. Truncation can be viewed as a special case of Construction 2.5, where weights 0 and 1 are used, and the ingredients are shortened blocks.

In all but one of our applications of Wilson’s fundamental construction, our master GDD is simply a PBD. Let us call a weighting of points $\omega : X \rightarrow \mathbb{Z}_{\geq 0}$ *nondegenerate* if $\langle X \setminus \omega^{-1}(0) \rangle_{\mathcal{B}} = X$. That is, in a nondegenerate weighting, there is no proper flat containing all the points of nonzero weight. With this mild assumption, we conclude that dimension is preserved (in a strong sense).

Proposition 2.6 *Suppose a nondegenerate weighting is applied to a PBD (X, \mathcal{B}) of dimension d . The result of Wilson’s fundamental construction is a GDD of dimension at least d . Moreover, every set of d points is contained in a proper subsystem that intersects each group in zero or all points.*

See [8] for a proof. It is helpful to illustrate the idea in the special case of truncation, used heavily in Section 4. Suppose X is truncated to $X' = \omega^{-1}(1)$. Any collection of d points in X' generated some proper flat Y in X . Since the weighting is nondegenerate, $Y \cap X'$ remains proper in X' .

Note if we have a GDD with the ‘strong’ dimension property as written in the conclusion of Proposition 2.6, then filling groups as in Construction 2.4 preserves the dimension.

3 Recursion

The purpose of this section is to settle all large values of v with the following result.

Theorem 3.1 *There exists a PBD($v, \{3, 4, 5\}$) of dimension three for all $v \geq 85$ except possibly $v = 86, 88$ and 94 .*

The proof of Theorem 3.1 follows immediately from the following three propositions. Note the use of interval notation for sets of integers.

Proposition 3.2 *There exists a PBD($v, \{3, 4, 5\}$) of dimension three for all $v \geq 171$.*

PROOF: We regard the projective space $\text{PG}_3(4)$ as a $\text{PBD}(85, \{5\})$ of dimension three. Give every point weight m or $m - 1$ and apply Wilson’s fundamental construction, appealing to the existence of five-group GDDs in Lemma 2.3(a). The result is a $\{3, 4, 5\}$ -GDD of type $m^{85-i}(m - 1)^i$ and strong dimension at least three. Varying $i \in [0, 85]$ leads to intervals of constructible values for v ; however, it remains to fill in the groups.

First, when $m = 4, 5$ or $m \geq 10$, we can replace groups with PBDs of order m or $m - 1$, which exist by Theorem 2.1. When $m = 3$, add a point ∞ and fill groups with triples and quadruples identified at ∞ . The case $m = 9$ is similar, using PBDs of order 9 or 10. The cases $m = 7, 8$ can be handled by adding a triple of new points $\{\infty_1, \infty_2, \infty_3\}$ and filling with PBDs of order 9, 10, or 11, identifying a common block of size three. Together, these intervals give $[2 \times 85 + 1, 5 \times 85] \cup [6 \times 85 + 3, \infty)$.

The ‘gap’ above can be settled using $\text{AG}_3(5)$, which is a $\text{PBD}(125, \{5\})$ of dimension three. Give every point weight m or $m - 1$ as before, where $m \in \{4, 5\}$. Groups can be filled directly with blocks in $\{3, 4, 5\}$. This gives constructible values $[3 \times 125, 5 \times 125] \supset (5 \times 85, 6 \times 85 + 3)$. □

Proposition 3.3 *There exists a $\text{PBD}(v, \{3, 4, 5\})$ of dimension three for all odd $v \in [85, 170]$.*

PROOF: Give some of the points of $\text{PG}_3(4)$ weight three, and the rest weight one. Apply Wilson’s fundamental construction, using the GDDs in Lemma 2.3(b) as ingredients. Groups of size three in the resulting GDD get filled with blocks, and we obtain all odd orders from 85 to 3×85 . □

Proposition 3.4 *There exists a $\text{PBD}(v, \{3, 4, 5\})$ of dimension three for all even $v \in [85, 170] \setminus \{86, 88, 94\}$.*

PROOF: Give one point, say x , of $\text{PG}_3(4)$ a weight of zero, and give all other points weight one or three (to be determined later). Blocks avoiding x are replaced as in Proposition 3.3. The 21 blocks incident with x are now carefully weighted so that the number of points of weight three is either 0, 3 or 4 on each. With this restriction, we can give weight three to 3, 4, 6, 7, 8, 9, . . . , 43 points. As before, fill groups of size three with triples. We have realized a construction all for even orders in $\{90, 92\} \cup [96, 84 + 2 \times 43]$, and this agrees with the stated bound. □

4 Truncation and other constructions

Recall our set $E := \{33, 34, 35, 38, 41, 42, 43, 47\}$ of possible exceptions from the main theorem. This section settles most of the remaining values under 85. The summary result is as follows.

Theorem 4.1 *If $v \in \{15\} \cup [27, 85]$ and $v \notin E \cup \{32\}$, then there exists a $\text{PBD}(v, \{3, 4, 5\})$ of dimension three.*

Again, we divide the proof into a few pieces. First, we recall some small values which admit Steiner spaces. $\text{PG}_3(2)$, $\text{PG}_4(2)$, and $\text{AG}_3(3)$ give orders 15, 31, 27. Weighting $\text{PG}_3(2)$ by 3 yields order 45.

Proposition 4.2 ([10]) *There exists a $\text{PBD}(v, \{3\})$ of dimension three for all $v \in \{15, 27, 31, 45\}$.*

Orders 46 and 94 follow from giving weight three to $\text{PG}_3(2)$ and $\text{PG}_4(2)$, plus one new point.

Proposition 4.3 *There exists a $\text{PBD}(v, \{3, 4\})$ of dimension three for all $v \in \{46, 94\}$.*

Next we cover a wide range of small values by truncation from projective spaces of dimension three. We would like to leave behind no line of size two. For this purpose, it is helpful to recall that every line is contained in exactly $q + 1$ planes, each one being a copy of $\text{PG}_2(q)$. First, we truncate from $\text{PG}_3(3)$, which is a $\text{PBD}(40, \{4\})$ of dimension three.

Proposition 4.4 *There exists a $\text{PBD}(v, \{3, 4\})$ of dimension three for $v \in \{28, 30, 31, 36, 37, 39, 40\}$.*

PROOF: We may truncate 0, 1, 3 or 4 points on some line, leaving all remaining lines of size 3 or 4. This realizes the four largest values. For the small values, let Z be a plane and L a line on Z . Truncate the nine points of $Z \setminus L$, along with 0, 1, or 3 points of L . \square

Remark. Truncation of all of Z results in $\text{AG}_3(3)$, and so the latter construction can be viewed alternatively as extending parallel classes in this space.

Next we consider $\text{PG}_3(4)$, which recall is a $\text{PBD}(85, \{5\})$ of dimension three. Its planes are copies of $\text{PG}_2(4)$. We consider various ‘legal’ truncations in this $\text{PBD}(21, \{5\})$.

Lemma 4.5 *It is possible to truncate $\text{PG}_2(4)$ to obtain a $\text{PBD}(v, \{3, 4, 5\})$ for each $v \in [0, 21]$, $v \neq 2, 6, 8, 10$.*

PROOF: For the low values, 0, 1, 3, 4 or 5 collinear points can be chosen. For the large values, we can delete up to 9 points from two intersecting lines L_1, L_2 . It is possible to ensure that neither L_i is reduced to two points; moreover, any other line has at most one point in common with each L_i . It remains to consider the three remaining values between 5 and 12.

(7): There is a Fano plane $\text{PG}_2(2)$ lying in $\text{PG}_2(4)$ as a ‘Baer subplane’ via the containment $\mathbb{F}_2 \subset \mathbb{F}_4$.

(9): Let L_1, L_2, L_3 be lines forming a triangle and consider their symmetric difference

$S = \oplus L_i$. To be clear, this is the set of points lying on exactly one L_i . It is easy to check that S induces a copy of the affine plane $AG_2(3)$, since a generic fourth line cuts the triangle in exactly three points.

(11): This construction is essentially taken from [10] in the discussion preceding Theorem 2. An ‘oval’ of $PG_2(4)$ consists of 5 points, no three collinear. There is a unique point outside every oval, called a ‘nucleus’, such that every line through the nucleus is tangent to the oval. Choose a set T consisting of an oval, its nucleus, and an additional line L disjoint from the oval and nucleus. Pairs of points in $T \setminus L$, whether involving the nucleus or not, are covered uniquely by truncated lines of size three. □

Proposition 4.6 *There exists a PBD($v, \{3, 4, 5\}$) of dimension three for all $v \in [48, 85]$.*

PROOF: Let Z_1 and Z_2 be two planes in $PG_3(4)$. We have $|Z_1 \cup Z_2| = 21 + 21 - 5 = 37$. If $Z_1 \cup Z_2$ can be truncated down to r points so that no lines in those planes become size two, then we have a PBD($48 + r, \{3, 4, 5\}$) of dimension three. It suffices to find such subsets for all $r \in [0, 37]$.

Let $L = Z_1 \cap Z_2$. Truncating from two other lines (in each Z_i) as in the proof of Lemma 4.5 settles $r \in [21, 37]$. Alternatively, we can remove all 16 points of $Z_1 \setminus L$ and leave behind any allowed number of points from Z_2 . This settles all but $r = 2, 6, 8, 10$. The first three of these values can be realized by leaving behind just 1, 3, 4 collinear points, respectively, in each $Z_i \setminus L$. Leaving behind exactly 10 points can be accomplished with a combination of a Baer subplane in Z_1 and three additional collinear points in Z_2 . □

Remark. With a careful analysis, it may be possible to avoid using the planar truncations down to 9 and 11 points.

Finally, we conclude with three more intricate constructions.

Proposition 4.7 *There exists a PBD($44, \{3, 4, 5\}$) of dimension three.*

PROOF: In $PG_3(4)$, take a line L and consider its incident planes. Let $x \notin L$. On one of the four planes other than $\langle L, x \rangle$, leave behind an oval-nucleus PBD($11, \{3, 5\}$), say with points T . Then include all points on $\{xt : t \in T\}$, which induce oval-nucleus configurations in all four planes. This space has 45 points, but we may delete x since it is incident only with lines of size 5. □

Proposition 4.8 *There exists both a PBD($29, \{3, 5\}$) and a PBD($88, \{3, 4, 5\}$) of dimension three.*

PROOF: Start from $PG_4(2)$, on 31 points. Delete a block B and the seven Fano planes containing it. Fill the resulting groups of size four with blocks of size five incident at a common new point ∞ . The resulting space has 29 points. Any three

points either are contained in a Fano plane, or, in the case that ∞ is chosen, a flat on 13 points.

Now, for order 88, simply give weight three and add a point. □

Proposition 4.9 *There exists a $PBD(86, \{3, 4, 5\})$ of dimension three.*

PROOF: Start from $AG_3(3)$, which is also a $\{3\}$ -GDD of type 3^9 . Give every point weight 3 and apply Wilson’s fundamental construction. We obtain a GDD of type 9^9 so that the flats generated by three points are proper and intersect groups along tripled points. Add a block of 5 points and fill groups with $PBD(14, \{3, 4, 5\})$, which recall admits its own group type $3^3 5^1$. Identify the group of size 5 on the new points, and align the groups of size 3 on the tripled points. The result is a dimension three PBD on $9 \times 9 + 5 = 86$ points. □

5 Nonexistence

Here, we prove the negative results in Theorem 1.2.

Theorem 5.1 *There does not exist $PBD(v, \{3, 4, 5\})$ of dimension three for $v = 32$ or for $v < 27, v \neq 15$.*

The proofs are largely carried by the following bound on the size of proper flats. In what follows, it is convenient to say that a block *touches* a flat when it intersects that flat in exactly one point.

Lemma 5.2 *In a $PBD(v, \mathbb{Z}_{\geq 3})$ with a proper flat W of size w , we have $v \geq 2w + 1$, with equality if and only if every block intersects W , and all blocks touching W have size exactly three.*

PROOF: Consider a point x in the complement of W , and let \mathcal{B}_x denote the set of blocks incident with x . We have $|\mathcal{B}_x| \leq (v - 1)/2$, with equality if and only if every block at x has size three. Next, since no block intersects W in more than one point, it follows that there are exactly w blocks in \mathcal{B}_x which also touch W . So $|\mathcal{B}_x| \geq w$, with equality if and only if every block in \mathcal{B}_x touches W . □

This already delivers a nonexistence result for small v .

Corollary 5.3 *For $v < 22$, the only $PBD(v, \{3, 4, 5\})$ of dimension three is the Steiner space $PG_3(2)$.*

PROOF: It is reported in [10] that there do not exist Steiner spaces of order $v < 27$, except for $PG_3(2)$. So take a $PBD(v, \{3, 4, 5\})$ of dimension three with $v < 27$ and

assume there is a block B of size four or five. Two points on B and one off B must generate a proper flat of size at least 10. By Lemma 5.2, we have $v \geq 21$.

Moreover, the condition for equality in the lemma rules out $v = 21$. This is because a hypothetical flat of order 10 is touched only by triples, yet each point in the flat must get covered with 11 points outside the flat. \square

Suppose we delete a block B from a $\text{PBD}(v, \{3, 4, 5\})$, say (X, \mathcal{B}) , of dimension at least three. For each $x \in X \setminus B$, the flat $F_x = \langle B, x \rangle$ is proper in X . Hence, we can partition $X \setminus B$ according to such flats. Deleting B and these flats results in a $\{3, 4, 5\}$ -GDD whose groups are $F_x \setminus B$. The number of groups must be at least three; if it is exactly three, then all groups have the same size. Finally, this GDD has dimension three.

The above argument is particularly useful in ruling out small cases in the presence of blocks of size 5. By (1.1) and (1.2), a $\text{PBD}(v, \{3, 4, 5\})$ with $v \equiv 2 \pmod{3}$ necessarily contains a block B of size 5. Accordingly, we obtain nonexistence results for $v = 23, 26, 32$ by analysis of the above GDD.

Proposition 5.4 *There does not exist $\text{PBD}(26, \{3, 4, 5\})$ of dimension three.*

PROOF: There is no possible GDD structure here, since $(26 - 5)/3 = 7$ and there does not exist a $\text{PBD}(12, \{3, 4, 5\})$ containing a block of size 5. \square

Proposition 5.5 *There does not exist $\text{PBD}(23, \{3, 4, 5\})$ of dimension three.*

PROOF: Take a block B of size 5 and consider the $\{3\}$ -GDD of type 6^3 resulting from deletion of B and its incident proper flats of size 11. Further analysis shows that any three non-collinear points, one in each group, generates a Fano plane with one point touching B and two points in each of the three groups. In different language, this is a latin square of order six such that every selection of a row, column, and symbol is contained in a proper 2×2 subsquare. (For more on this correspondence, see Section 6.) But such latin squares can only exist for orders a power of two; see for instance [2]. \square

Proposition 5.6 *There does not exist $\text{PBD}(32, \{3, 4, 5\})$ of dimension three.*

PROOF: Again, let B be a block of size 5 and consider the GDD resulting from the deletion of B and its incident proper flats. Possible sizes for such flats are 11, 13, 14, 15. It is enough to find some three-point generated flat which exhausts an entire group of this GDD. Let's divide the remainder of the argument into two cases according to the type of the GDD. Twice we rely on the fact that a $\text{PBD}(v, \{3, 4\})$ with a block of size 4 must have $v \geq 10$.

CASE 1: (a 3-GDD of type 9^3) Let the groups be G_1, G_2, G_3 . There exist points $x \in B$, $a_1 \in G_1$, and $a_2 \in G_2$ such that block $\langle x, a_1 \rangle$ has size three and block $\langle x, a_2 \rangle$

has size four. Consider $W = \langle x, a_1, a_2 \rangle$. This flat is a PBD with blocks of size 3 and 4. It follows by the pigeonhole principle that W meets some $B \cup G_i$ in more than a block, and hence $W = V$.

CASE 2: (a $\{3, 4\}$ -GDD of type $6^3 9^1$) Let the groups be $G_i, i = 1, \dots, 4$, with $|G_4| = 9$. It is clear from parity considerations that a block of size 4 must exist in this GDD; call such a block A . Put $\{a_i\} := A \cap G_i$. For each point $x \in B$, we consider $\langle A, x \rangle$. For each $i, \{x, a_i\}$ defines a block, say A_i , in the subsystem $B \cup G_i$. Let $y_i \neq a_i, x$ be a third point in A_i . It follows that $\langle A, x \rangle$ contains at least the points $x, a_i, y_i, i = 1, \dots, 4$ and a block A of size four. Therefore, this flat must include an additional (tenth) point, say z . If $z \in G_i$ for $i = 1, 2, 3$, we obtain more than a block in $B \cup G_i$. On the other hand, suppose $z \in G_4$ for each choice of $x \in B$. This leads to five blocks of size four intersecting (only) at a_4 in the 14-point system $B \cup G_4$. Since $14 < 5 \cdot 3 + 1$, we have a contradiction. \square

Remark. Note the case of group sizes 8, 9, 10 is not possible.

It remains to prove nonexistence for 22, 24, 25. From the above GDD argument, there cannot exist a block of size 5 in a hypothetical such PBD of dimension three. Moreover, since there is no Steiner space of order 25, we conclude the existence of a block of size four in each case. A structural result is helpful here.

Lemma 5.7 *In a $PBD(v, \{3, 4\})$ of dimension three with $v \leq 27$, if every flat containing a block of size 4 has size 10, then all the blocks of size 4 intersect in a single point.*

PROOF: We will first show that any two blocks of size 4 intersect. If not, suppose B_1 and B_2 are disjoint blocks of size four. Consider the flats $\langle B_1, x_2 \rangle$, where x_2 varies in B_2 . These are four 10-point flats which pairwise intersect in exactly one of the blocks of size 4. This requires $v \geq 28$, a contradiction.

Now suppose at least three blocks of size four meet at x . That is, x belongs to a 10-point flat Y . Consider any other block B of size four. If $x \notin B$, then $|B \cap Y| \geq 3$. But $B \not\subset Y$ by the unique structure of $PBD(10, \{3, 4\})$. This contradicts that Y is a flat, and we conclude that $x \in B$. \square

Proposition 5.8 *There does not exist $PBD(v, \{3, 4, 5\})$ of dimension three for $v = 22, 24, 25$.*

PROOF: By the above remarks, it suffices to consider block sizes in $\{3, 4\}$ with at least one block of size 4. For v even, every point is incident with an odd number of blocks of size 4. This easily rules out order 24 since $24 \not\geq 2 \cdot 12 + 1$ and $24 \not\equiv 1 \pmod{3}$. For $v = 22$, deletion of the point common to all blocks of size 4 results in a Steiner space of order 21, a contradiction. Finally, for $v = 25$, every point is incident with an even number of blocks of size 4. Also, the existence of a flat W of order 12 is numerically impossible, since all other blocks are triples in view of Lemma 5.2. Therefore, all blocks of size 4 intersect in a point, and this is a contradiction. \square

6 Applications

Here, we briefly look at how our existence results on block sizes $K = \{3, 4, 5\}$ imply the existence of various related objects. Consider first the question of other block sizes.

Corollary 6.1

- (a) *There exist PBD($v, \{3, 5\}$) of dimension three for all odd integers $v \geq 97$.*
- (b) *There exist PBD($v, \{3, 4\}$) of dimension three for all $v \equiv 0, 1 \pmod{3}$, $v \geq 144$.*

PROOF: For (a), take a PBD($u, \{3, 4, 5\}$) of dimension three for $u = \frac{v-1}{2} \geq 48$. Give every point weight two and replace blocks with $\{3, 5\}$ -GDDs of type 2^i , which exist for $i = 3, 4, 5$. The result, by Wilson’s fundamental construction, is a $\{3, 5\}$ -GDD of type 2^u . Add a point and fill groups with blocks of size three to get the desired PBD.

For (b), apply a similar proof with weight three, $\{3, 4\}$ -GDDs of type 3^i , and using either zero or one extra point. □

In fact, Steiner spaces – the case $K = \{3\}$ – can also be constructed in this way. For example, we can give weight six to a PBD($v, \{3, 4, 5\}$) of dimension three and add a point, using 3-GDDs of types 6^i , $i = 3, 4, 5$, as ingredients. Of more interest, the existence of PBD($v, \{3, 4\}$) for $v = 33, 34 \in E$ would imply the existence of Steiner spaces for two undecided orders: 67 and 69. This is via weight 2 and GDDs of types $2^3, 2^4$.

Recall that a *latin square* of order n is an $n \times n$ array on n symbols such that every row and every column exhausts the symbols (with no repetition). A (latin) *subsquare* is a sub-array which is itself a latin square.

We often assume symbols (and row/column indices) are $[n] := \{1, \dots, n\}$. A latin square is *idempotent* if the entry in diagonal cell (i, i) is i for each $i \in [n]$.

Let us call a latin square L *localized* if, for every cell (i, j) and every symbol k , there exists a proper subsquare of L containing both cell (i, j) and symbol k . The smallest example of a localized latin square is the Boolean latin square of order 4, shown below.

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

This property is reminiscent of ‘rank four’ finite groups, in which no three-element set generates the entire group. But, in fact, any rank two (that is, non-cyclic) group G yields a localized latin square as its operation table. For $a, b, c \in G$, consider cell (a, b) and symbol c of its operation table. There is a proper subgroup H containing $a^{-1}cb^{-1}$. Entries in rows aH and columns Hb define a subsquare containing $c \in aHb$.

The above discussion fails to obtain localized latin squares of ‘cyclicity-forcing’ orders (primes in particular). However, we can use $\text{PBD}(v, \{3, 4, 5\})$ of dimension three to settle almost all orders.

Corollary 6.2 *There exist localized latin squares of every order $n \geq 48$.*

PROOF: Take a $\text{PBD}(n, \{3, 4, 5\})$, which exists by Theorem 1.2. We may assume the set of points is $[n] := \{1, \dots, n\}$; let the set of blocks be \mathcal{B} . For each $B \in \mathcal{B}$, there exists an idempotent latin square on the symbols of B ; call it L^B . Now, define an (idempotent) latin square L by

$$L_{ij} = \begin{cases} i & \text{if } i = j; \\ L_{ij}^B & \text{if } i \neq j \text{ with } \{i, j\} \subseteq B \in \mathcal{B}. \end{cases}$$

For any selection $1 \leq i, j, k \leq n$, the points $\{i, j, k\}$ generate a proper flat, say Y , with respect to \mathcal{B} . It follows by construction of L that its restriction to Y (row/column indices and symbols) forms a subsquare. This subsquare covers both cell (i, j) and entry k . \square

The truth is surely better than 48, although we refrain from a detailed analysis of the small cases. The only orders in question are 15, 33, 35, and primes less than 48; see <http://oeis.org/A003277>.

7 Reducing dimension

The purpose of this section is to observe that dimension can be reduced, following a ‘flat swapping’ technique of Teirlinck in [10].

Theorem 7.1 *If there exists a $\text{PBD}(v, \{3, 4, 5\})$ of dimension $d \geq 3$ then there exists one of dimension exactly three.*

Most of Section 2 in [10], which proves a similar result for $K = \{3\}$, goes through for general K . However, we must remark the existence of *planes* in our case; that is, the existence of PBDs of dimension exactly two (for all possible orders). Later, this is useful to reduce dimension in larger PBDs.

Lemma 7.2 *There exists a $\text{PBD}(v, \{3, 4, 5\})$ of dimension two if and only if $v \neq 2, 6, 8$.*

PROOF: From Theorem 5.1, no $\text{PBD}(v, \{3, 4, 5\})$ of dimension greater than 2 exists with $v \leq 26$, except for $\text{PG}_3(2)$ with $v = 15$. In that case, any of the 79 other Steiner triple systems (see [4] for instance) are planes.

Suppose, then, that $v \geq 27$. Take a circulant latin square of order n regarded as a 3-GDD of type n^3 . Without loss of generality, row 1, column 1 and symbol 1 form

a block. If we instead try to bound row 1, column 1, and symbol 2 in a subsquare, then the circulant structure forces us to take the entire square.

Add $i = 0, 1,$ or 5 points and fill the groups of this GDD with $\text{PBD}(n + i, \{3, 4, 5\})$, identifying i points (forming a block if $i = 5$). For $i = 0$ or 1 , and $n \geq 9$, we cover all $v \geq 27$ in congruence classes $0, 1 \pmod{3}$. For $i = 5$ and $n \geq 8$, we cover the remaining values in the $2 \pmod{3}$ class. Note, however, that this relies on the existence of $\text{PBD}(n + 5, \{3, 4, 5\})$ with a mandatory block of size 5 for all $n \geq 8$. This follows for $n \geq 46$ from truncating up to four points from a $\text{TD}(5, m)$, $m \geq 11$. Small values are not hard to find using various truncations of $\text{BIBD}(n, 5, 1)$ or truncations and fillings of $\text{TD}(k, m)$ for $k \in \{3, 4, 5\}$. Details are left to the interested reader.

In any case, by the circulant structure and bound in Lemma 5.2, the three above points still fail to generate a proper flat in the resulting filled PBDs. \square

Surely, there are other possible proofs of Lemma 7.2, perhaps even ‘probabilistic’ ones. Now we move on to the key replacement lemmas, adapted from [10].

Lemma 7.3 *Let (X, \mathcal{B}) be a $\text{PBD}(v, K)$. If in \mathcal{B} we replace a maximal proper flat $\mathcal{B}|_Y$ by a plane (Y, \mathcal{C}) , then the resulting $\text{PBD}(X, \overline{\mathcal{B}})$ has dimension at most three.*

PROOF: Pick three points $x_1, x_2, x_3 \in Y$ that generate Y in \mathcal{C} . Let $y \in X \setminus Y$. Put $U = \langle x_1, x_2, x_3, y \rangle_{\overline{\mathcal{B}}}$. We have $U \supset Y$ via x_1, x_2, x_3 . By maximality of Y in $(X, \overline{\mathcal{B}})$, it follows that $U = X$. Since 4 points generate X in $\overline{\mathcal{B}}$, the claim follows. \square

Recall the set of flats in a linear space forms a lattice under inclusion and intersection. Following Teirlinck, let us define the *length* of a linear space (X, \mathcal{B}) to be the length of a maximum chain in this lattice. For example, the length of $\text{AG}_d(q)$ is $d + 1$; any maximum chain looks like a point included in a line included in a plane, and so on. A space of length exactly 3 is necessarily a plane, and is called *nondegenerate* to indicate that it contains no proper planes.

We now show that swapping flats which have small length can have only mild impact on dimension.

Lemma 7.4 *Let (X, \mathcal{B}) be a $\text{PBD}(v, K)$ of dimension d containing a nondegenerate plane $(Z, \mathcal{B}|_Z)$. If in \mathcal{B} we replace $\mathcal{B}|_Z$ by any linear space (Z, \mathcal{C}) , then the resulting $\text{PBD}(X, \widehat{\mathcal{B}})$ has dimension at least $d - 1$.*

PROOF: Suppose for contradiction that some $d - 1$ points a_1, \dots, a_{d-1} generate X in $\widehat{\mathcal{B}}$. Put $W = \langle a_1, \dots, a_{d-1} \rangle_{\mathcal{B}}$ and consider $W \cap Z$. If $W \cap Z = \emptyset$, a point, or Z itself, we have a contradiction. So suppose $W \cap Z$ is a line L of Z . Pick any point z in $Z \setminus L$ and let $W' = \langle a_1, \dots, a_{d-1}, z \rangle_{\mathcal{B}}$. Since W' contains both L and z and $\mathcal{B}|_Z$ is nondegenerate, it follows that $Z \subseteq W'$. Therefore, W' is also a flat of X in $\widehat{\mathcal{B}}$. Since it contains a_1, \dots, a_{d-1} , we must have $W' = X$. This contradicts $\dim \mathcal{B} = d$. \square

Proposition 7.5 *If, in some $PBD(v, K)$, replacing every flat of length at most l by another linear space with block sizes in K fails to reduce the dimension, then replacing a flat of length $l + 1$ results in dimension at least $d - 1$.*

PROOF: [Outline] Use induction on l , with the basis case being furnished by Lemma 7.4. The proof technique is also similar to the lemma. If some $d - 1$ points generate the whole space after replacement, then we consider what those points generate in the original space \mathcal{B} and obtain a contradiction. See [9, 10] for details. \square

We are now ready to wrap up the argument on reducing dimension.

PROOF:[Proof of Theorem 7.1] Let (X, \mathcal{B}) be a $PBD(v, \{3, 4, 5\})$ of dimension $d > 3$. Starting l from 3, either replacing some flat of length l can reduce the dimension to $d - 1$, or we can increment l as in Proposition 7.5. Since by Lemmas 7.2 and 7.3 there is some replacement of a proper flat which reduces the dimension, at some value l we must be able to reduce the dimension by exactly one. The result then follows by induction on d . \square

Remark. In fact, one obtains out of this proof spaces of all dimensions between 3 and d .

8 Discussion

For PBDs having no divisibility restrictions on their order v , the first case of interest is when the allowed block sizes are in $\{3, 4, 5\}$. We have shown that such spaces exist with dimension three whenever $v \geq 48$, leaving a set E of only eight possible unsettled cases between 33 and 47. These exceptions could break in either direction. For $v = 35$, existence appears doubtful, while the large values in E are likely more promising. This compares nicely with the state of the art for Steiner spaces, in which there are four unsettled cases between 51 and 145. Computer search for the unsettled cases has not been attempted, but it is also unlikely to help due to the scarcity of examples in dimension three.

With no extra effort, we obtain a nearly complete existence theory when the same question is asked for $K = \{3, 4\}$, $\{3, 5\}$, and also for latin squares.

We have also shown the existence of planes and, by flat swapping, that there are no gaps in the possible dimensions for a given order v . It would be interesting to prove the existence of nondegenerate planes of all possible orders for $K = \{3, 4, 5\}$. A strategy similar to the proof of Lemma 7.2 might work in many cases. Alternatively, one could try to adapt Doyen's technique [6], which handles $K = \{3\}$.

Finally, one could take a slightly different viewpoint and instead ask for upper bounds on $f(v)$ such that, in some $PBD(v, \{3, 4, 5\})$, every set of three points belongs to a flat of size at most $f(v)$. From [7], we have $f(v) < 1000$, but a much better bound is to be expected. The same approach can be used in the context of localized latin squares to obtain a universal bound on their generated subsquares.

References

- [1] L.M. Batten and A. Beutelspacher, *The theory of finite linear spaces: combinatorics of points and lines*, Cambridge University Press, 1993.
- [2] P.J. Cameron, Minimal edge-colourings of complete graphs, *J. London Math. Soc.* 11 (1975), 337–346.
- [3] S. Chowla, P. Erdős and E.G. Strauss, On the maximal number of pairwise orthogonal Latin squares of a given order, *Canad. J. Math.* 12 (1960), 204–208.
- [4] C.J. Colbourn and J.H. Dinitz, eds., *The CRC Handbook of Combinatorial Designs*, 2nd ed., CRC Press, Inc., 2006.
- [5] A. Delandtsheer, Dimensional linear spaces, in *Handbook of incidence geometry*, 193–294, North-Holland, Amsterdam, 1995.
- [6] J. Doyen, Sur la structure de certains systèmes triples de Steiner, *Math. Z.* 111 (1969), 289–300.
- [7] P.J. Dukes and A.C.H. Ling, Linear spaces with small generated subspaces, *J. Combin. Theory Ser. A* 116 (2009), 485–493.
- [8] P.J. Dukes and A.C.H. Ling, Pairwise balanced designs with prescribed minimum dimension, *Discrete Comput. Geom.* 51 (2014), 485–494.
- [9] J. Niezen, Pairwise balanced designs of dimension three, M.Sc. Thesis, University of Victoria, 2013.
- [10] L. Teirlinck, On Steiner spaces. *J. Combin. Theory Ser. A* 26 (1979), 103–114.
- [11] R.M. Wilson, An existence theory for pairwise balanced designs III: Proof of the existence conjectures, *J. Combin. Theory Ser. A* 18 (1975), 71–79.

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