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Directed tree decompositions of Cayley digraphs with word-degenerate connection sets

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Abstract

For a tree T, the graph X is T-decomposable if there exists a partition of the edge set of X into isomorphic copies of T. In 1963, Ringel conjectured that K_{2m+1} can be decomposed by any tree with m edges. Graham and Häggkvist conjectured more generally that every $2m$-regular graph can be decomposed by any tree with m edges. Fink showed in 1994 that for any directed tree T with m arcs, the directed Cayley graph $\mathrm{DCay}(G;S)$ is T-decomposable if S is a minimal generating set of G with m elements. Building upon that technique, this paper presents an enlarged family of directed Cayley graphs that are decomposable into directed trees. In particular, a subset S of a finite group G is defined to be (k,t)-*word degenerate* if S contains exactly t elements, $s_1,\dots,s_t$, such that for each $i\in\{1,\dots,t\}$, s_i can be expressed as a product of fewer than k distinct elements from $S-\{s_i\}$ or their inverses. It is proved that if S is any (k,t)-word degenerate m-subset of a group G, and T is any directed tree having m arcs and a minimal spanning star forest F, then the directed Cayley graph $\mathrm{DCay}(G;S)$ is T-decomposable whenever $k\geq \mathrm{diam}(T)\geq 3$, and $t\leq |E(F)|$. When $\mathrm{diam}(T)=2$, additional restrictions are required. The main result of Fink and other results are obtained as immediate corollaries.

1 Introduction

This article will focus exclusively on partitioning the edge (arc) set of a finite, regular, simple graph (digraph) into various subgraphs. A *decomposition* of a graph (digraph)

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$X = (V, E)$ is a sequence of subgraphs $\mathcal{X} = [X_1, \ldots, X_b]$, such that $X_i = (V_i, E_i)$ and $E = E_1 \cup \cdots \cup E_b$ where $E_i \cap E_j = \emptyset$ for all $i \neq j$. If Y is a graph (digraph) such that $X_i \cong Y$ for all $i \in \{1, 2, \ldots, b\}$, then X has a Y*-decomposition* and X is said to be Y*-decomposable.* If X is Y-decomposable into b copies of Y, then $|E(X)|$ must be divisible by both $|E(Y)|$ and b.

The theory of graph decompositions has a vast and rich literature (see Bosák [1]). For example, the k-factorization problem decomposes the edges of a graph into k-*factors*, or k-regular spanning subgraphs. The Hamilton cycle decomposition problem is thus a special type of 2-factorization. Similarly, decomposing a graph into perfect matchings is exactly the 1-factorization problem. These types of decompositions as well as P_k-, C_k-, K_k-decompositions, and variants such as the Oberwolfach and Hamilton-Waterloo problems have been studied extensively for highly-structured graphs such as the complete graphs or complete bipartite graphs, among many others. The 1961 conjecture of Ringel and subsequent generalized version by Graham and Häggkvist has created considerable interest in the context of *tree* decompositions, which are the focus of this article.

Conjecture 1.1 (Ringel [8])**.** *If T is a tree with m edges, then the complete graph K_{2m+1} is T-decomposable.*

Conjecture 1.2 (Graham-Häggkvist [5])**.** *If T is a tree with m edges, and X is a $2m$-regular graph, then X is T-decomposable.*

In 1991, Snevily resolved Conjecture 1.2 in the following cases.

Theorem 1.3 (Snevily [10])**.** *If T is a tree with m edges and X is a $2m$-regular graph such that the girth of X is greater than the diameter of T, then X is T-decomposable.*

Theorem 1.4 ([10])**.** *If T is a tree with m edges, and X is Cartesian product of m cycles, then X is T-decomposable.*

This article will focus on analogues of Conjecture 1.2 in the context of directed graphs. We first list some terminology and definitions that will be used throughout. Unless indicated otherwise, $X = (V, E)$ will denote a directed simple graph with vertex set V and arc set E. The graph X is r*-regular* if the out-degree and the in-degree of every vertex $v \in V(X)$ is equal to r and X is *symmetric* if whenever $(u, v) \in E(X)$ then $(v, u) \in E(X)$. For example, the leftmost graph in Figure 1 is a 2-regular directed graph that is not symmetric. The *underlying graph* of X is the undirected (multi)graph obtained by making every arc of X into an undirected edge. The *projection graph* of X is the simple graph obtained from the underlying graph of X formed by collapsing every multi-edge into an edge of multiplicity one (see Figure 1). A *semi-path* P of length n in X is an alternating sequence of distinct vertices and arcs in X:

$$P : v_1, e_1, v_2, e_2, \cdots, v_n, e_n, v_{n+1}$$

where $e_i = (v_i, v_{i+1})$ or $e_i = (v_{i+1}, v_i)$ for all $i \in \{1, \ldots, n\}$.

A *star graph* is an undirected complete bipartite graph $K_{1,\alpha}$, where $\alpha \geq 0$. We shall use $DK_{1,\alpha}$ to denote a directed star graph (no specific orientation is implied). If $\alpha \geq 2$, then the *center vertex* of $K_{1,\alpha}$ (respectively $DK_{1,\alpha}$) is the unique non-leaf vertex. In 1991, Colbourn et al. considered decomposing the edges of a symmetric directed (simple) graph into (s,t)*-directed stars*, which are special directed star graphs, $DK_{1,s+t}$ such that the in-degree of the center vertex is s and the out-degree of the center vertex is t. They defined an (s,t)*-directed star decomposition* of a directed graph X as a decomposition of X into (s,t)-directed stars.

Theorem 1.5 (Colbourn et al. [2]). *Let X be a symmetric directed simple r-regular graph. Let $s,t \geq 0$ be integers such that $r \equiv 0 \pmod{s+t}$. Then:*

1. *if $s+t \neq r$, there exists an (s,t)-directed star decomposition of X.*
2. *if $s+t = r$, there exists an (s,t)-directed star decomposition of X if and only if the underlying undirected graph of X has an s-factor.*

Theorem 1.5 implies that the directed version of Conjecture 1.2 (namely, every m-regular directed graph is decomposable by any directed tree with m arcs) is false in general. For example, if T is a directed path on three vertices (a $(1,1)$-directed star), and X is a symmetric 2-regular directed graph of odd order, then despite $m = 2$, Theorem 1.5 forbids X from having a T-decomposition because the underlying graph of X does not have a 1-factor. This invites the following question, which is the focus of this article:

Question 1.6. *If T is a directed tree with m arcs, what are sufficient conditions for a directed simple m-regular graph X to be T-decomposable?*

Attention will be focused on directed Cayley graphs, which are vertex-transitive, thus regular, and engage the many tools of group theory. We now state some basic definitions and facts regarding Cayley graphs and digraphs. Throughout this article, G will denote a finite group with identity e, and S a subset of G. The inverse of $g \in G$ will be denoted g^{-1} and all notation multiplicative.

Definition 1.7. The *directed Cayley graph* (or *Cayley digraph*) of G with *connection set* S is the directed graph X, denoted $X = \mathrm{DCay}(G;S)$, with $V(X) = G$ and $E(X) = \{(x,xs) \colon s \in S\}$. Equivalently, $(x,y) \in E(X)$ if and only if $x^{-1}y \in S$.

Definition 1.8. The *undirected Cayley graph* (or *Cayley graph*) of G with connection set S is the undirected graph X, denoted $X = \mathrm{Cay}(G;S)$, with $V(X) = G$ and $E(X) = \{\{x,xs\} \colon s \in S\}$. Equivalently, $\{x,y\} \in E(X)$ if and only if either $x^{-1}y \in S$ or $y^{-1}x \in S$.

Throughout, we shall further require that $e \notin S$, so that both directed and undirected Cayley graphs do not have loops. Note, a Cayley graph or digraph is connected if and only if S is a generating set for G. The arc (x,xs) (resp. edge $\{x,xs\}$) is said to be *generated by s* and is called an *s-arc* (resp. *s-edge*). A subgraph

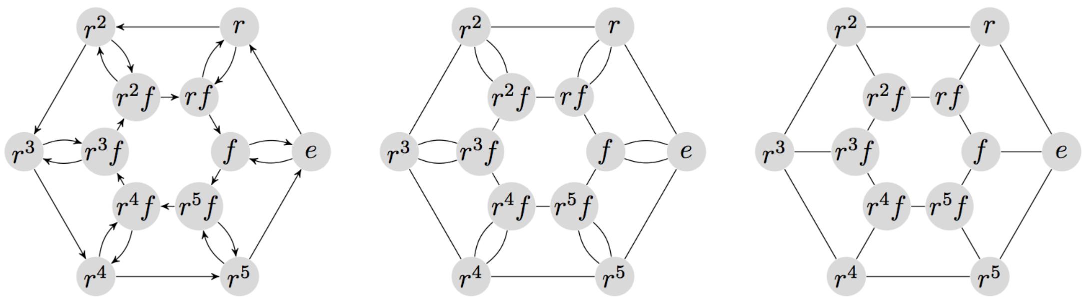

Figure 1: The 2-regular Cayley digraph $\mathrm{DCay}(D_6; \{r, f\})$ of the dihedral group D_6 with standard rotation r and reflection f (left), its underlying graph (middle), and its projection graph (right) which is simply $\mathrm{Cay}(D_6; \{r, f\})$.

Y of a Cayley graph or directed graph X is *generated by* s if $E(Y)$ consists of all s-arcs (resp. s-edges) of X.

Numerous results on Y-decompositions of Cayley graphs employ restrictions on the connection set S to gain additional control over the graph. A set S is *minimal* if for every $s \in S$, the subset $S - \{s\}$ generates a proper subgroup of the subgroup generated by S. The set S is *inverse-free* provided that whenever $s \in S$, then either $s = s^{-1}$ or $s^{-1} \notin S$. We say S is *involution-free* if $s \neq s^{-1}$ for all $s \in S$ and we say S is *inverse-closed* if $s^{-1} \in S$ for all $s \in S$.

In the directed case, regardless of whether S is inverse-free or involution-free, if $|S| = m$, then $\mathrm{DCay}(G; S)$ is always an m-regular graph and has a total of $|G| \cdot m$ arcs. If $\mathrm{DCay}(G; S)$ has a Y-decomposition into b copies, where Y is a directed graph with m arcs, then $b = |G|$ (see Figure 3).

If $\{s, s^{-1}\} \subseteq S$ (or $s = s^{-1} \in S$) then there is an s-arc from u to v, where $v = us$ and there is an s^{-1}-arc going from v to $u = vs^{-1}$. If S is inverse-closed, then $X = \mathrm{DCay}(G; S)$ is a symmetric digraph. The underlying graph of $X = \mathrm{DCay}(G; S)$ will be a multigraph if and only if S either contains involutions or is not inverse-free.

The following corollary of Theorem 1.5 is required for Theorem 2.9.

Corollary 1.9. *If S is inverse-closed and $|S| = \alpha + \beta$, for some positive integers α and β, then $X =$ $\mathrm{DCay}(G; S)$ has an (α, β)-directed star decomposition if and only if the underlying graph of X has an α-factor.*

In 1994, Fink gave a partial answer to Question 1.6.

Theorem 1.10 (Fink [4])**.** *If T is any directed tree with m arcs, and G is a group with minimal generating set S, where $|S| = m$, then* $\mathrm{DCay}(G; S)$ *is T-decomposable.*

Some of the techniques in this article use a similar approach as Fink and in fact, Theorem 1.10 can be obtained as a direct corollary of the main result, Theorem 2.9.

We shall rely on the well-known fact that the map $\phi_g : G \to G$ where $\phi_g(v) = gv$ is an automorphism of both $\mathrm{DCay}(G; S)$ and $\mathrm{Cay}(G; S)$ (see Sabidussi [9]) to

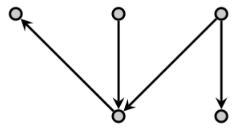

Figure 2: A directed tree T with $m = 4$ arcs.

obtain a tree decomposition via the orbit of a tree T under the natural action of the automorphism group $\Phi_G = \{\phi_g : g \in G\}$.

Example 1.11. Figure 3 shows a decomposition of

$$X = \mathrm{DCay}(S_4; \{(1234), (12), (134), (13)(24)\})$$

into trees isomorphic to T shown in Figure 2. The (1234)-arcs are black, (12)-arcs are red, (134)-arcs are green, and (13)(24)-arcs are blue. Note, Theorem 1.10 does not apply in this example because S is not a minimal generating set of S_4.

2 Main Results

In this section, we outline a framework of new results pertaining to Question 1.6 that generalize Theorem 1.10. In many of examples in this section, SAGE [11] (`http://www.sagemath.org`) was used for group computations. Therefore, all permutation composition is performed left to right. The following definition is a slight variation of the concept of a *word*, as is traditionally studied in combinatorial and free group theory.

Definition 2.1. Given a subset S of a group G, a *word* w on an *alphabet* $S = \{s_1, \ldots, s_m\}$, or S-*word*, is any finite product of distinct elements of the form:

$$w = s_{\sigma(1)}^{n_1} s_{\sigma(2)}^{n_2} \cdots s_{\sigma(m)}^{n_m} = \prod_{i=1}^{m} s_{\sigma(i)}^{n_i}$$

where $n_i \in \{-1, 0, 1\}$ for all i and $\sigma \in \mathrm{SYM}(m)$. The number of nonzero n_i's in the product is called the *size* of w, denoted $\ell_S(w)$. For any group element $g \in G$, the *length* of g is

$$\tilde{\ell}_S(g) = \begin{cases} \min\{t : g = w \text{ and } \ell_S(w) = t\} & \text{if } g \text{ is an } S\text{-word} \\ \infty & \text{if } g \text{ is not an } S\text{-word} \end{cases}$$

Note, $\ell_S(g)$ is the minimum number of nonzero n_i's needed to express g as an S-word over all possible permutations σ. The identity e is the unique word of length 0. Also, $\tilde{\ell}_S(s) = 1$ if and only if $s \in S$ or $s^{-1} \in S$. Furthermore, the only way in which *both* s and s^{-1} can possibly occur in an expression for an S-word w, is if $s \in S$ and $s^{-1} \in S$.

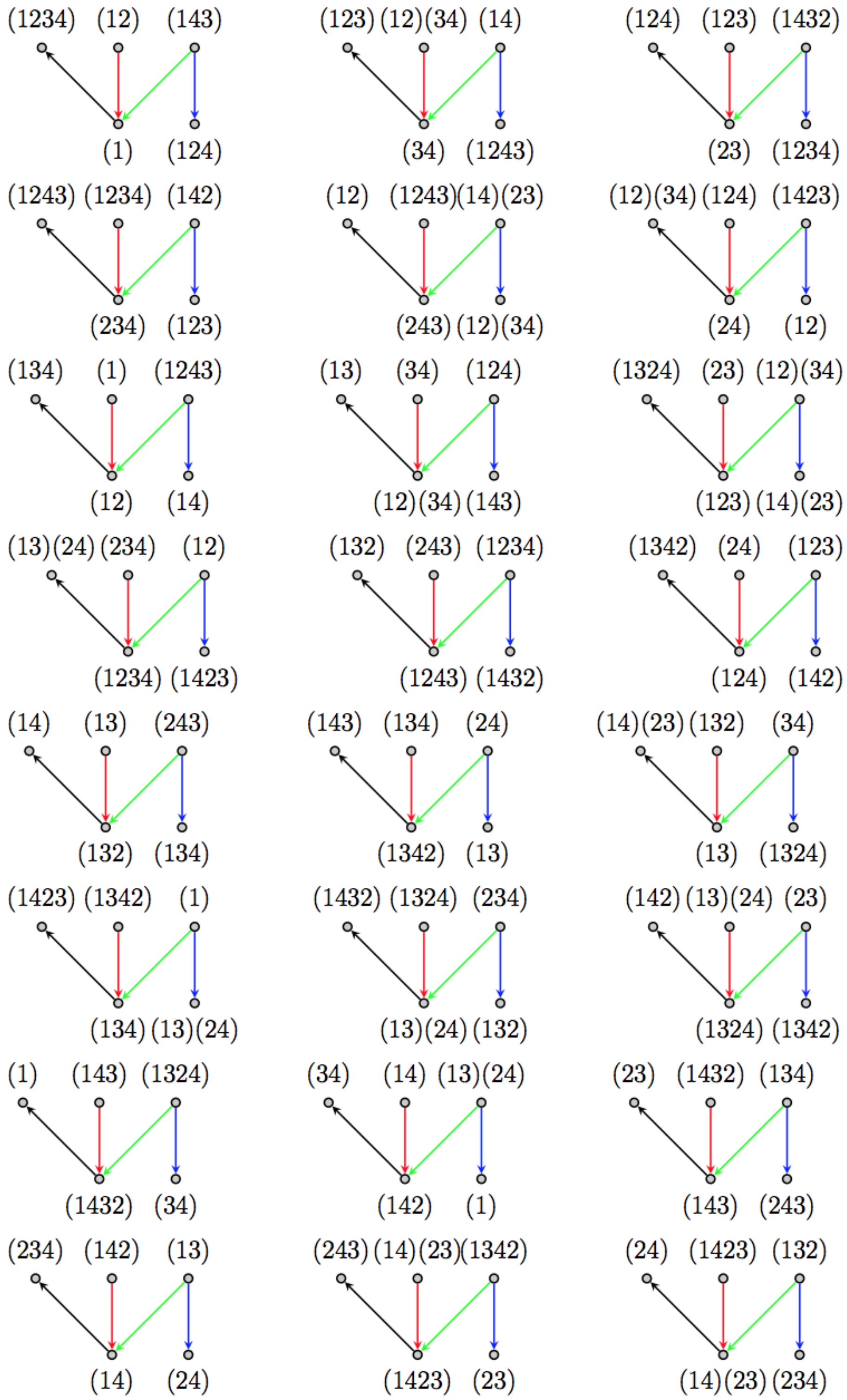

Figure 3: T-decomposition of $\mathrm{DCay}(S_4; \{(1234), (12), (134), (13)(24)\})$

Example 2.2. Let $G = S_5$ and $S = \{s_1, s_1^2, s_2, s_1s_2\}$ where $s_1 = (1234)$, $s_2 = (123)$. The element $g = (143)$ can be expressed as an S-word of size 3, e.g., $g = s_1(s_1s_2)^{-1}s_2$, and an S-word of size 2, e.g., $g = (s_1s_2)s_1$. Because $g^{-1} \notin S$, we have $\tilde{\ell}_S(g) = 2$.

The following definition provides a means to control, in some sense, how far a subset can deviate from being "minimal." This definition is crucial to the rest of this article, and is buttressed with numerous examples.

Definition 2.3. A set S is (k,t)-*word degenerate* if S contains exactly t elements, $s_1, \ldots, s_t$ such that, for each $i \in \{1, \ldots, t\}$, $\tilde{\ell}_U(s_i) < k$ where $U = S - \{s_i\}$. These t elements are called *degenerate* elements of S.

Example 2.4. The connection set $S = \{(1234), (12), (134), (13)(24)\}$ of S_4 in Figure 3 is $(3,3)$-word degenerate. In this case, the degenerate elements are (1234), (12), and (134), each having length two. Note, $\tilde{\ell}_S((13)(24)) = 3$.

By definition, if S is (k,t)-word degenerate, then $2 \leq k \leq |S|$ and either $t = 0$ or $2 \leq t \leq |S|$, because a $(k,1)$-word degenerate set cannot exist. Clearly if S is a (k,t)-word degenerate set and $k > t$, then S is also a $(k-i,t)$-word degenerate set for all $i \in \{1, \ldots, k-t\}$. Similarly, if S is a $(k,|S|)$-word degenerate set, and $k < |S|$, then S is also a $(k+i,|S|)$-word degenerate set for all $i \in \{1, \ldots, |S|-k\}$. If $\tilde{\ell}_U(s) < \infty$ for all $s \in S$, then S is automatically $(|S|,|S|)$-word degenerate. Note an $(|S|,0)$-word degenerate set is necessarily inverse-free and if S is $(k,2)$-word degenerate, then the degenerate elements are the noninvolutions a and a^{-1}. Naturally, if S is a $(2,t)$-word degenerate set, then t must be even.

It is worth noticing that if S is a minimal generating set of G , then S is $(|S|,0)$-word degenerate. The converse is false, as the following example shows.

Example 2.5. Let $G = S_7$ and $S = \{s_1, s_2, s_3, s_4, s_5\}$ where $s_1 = (123)$, $s_2 = (415)$, $s_3 = (124)$, $s_4 = (126)(53)$, and $s_5 = (162)$. S is neither minimal (e.g., $s_5 \in \langle\{s_1, s_2, s_3, s_4\}\rangle$) nor square-independent (see Definition 2.13) (e.g., $s_5 = s_4^2s_3$). It was verified using [11] that $\tilde{\ell}(s_i) = \infty$ for all $i \in \{1, \ldots, 5\}$. Therefore, S is a $(5,0)$-word degenerate set.

Example 2.6. Let $G = S_5$ and $S = \{s_1, s_2, s_3, s_4, s_5\}$ where $s_1 = (1234)$, $s_2 = (12)$, $s_3 = (134)(25)$, $s_4 = s_1^2$, and $s_5 = s_1s_2$. Clearly, S is neither minimal nor square-independent (see Definition 2.13). It was verified using [11] that $\tilde{\ell}(s_1) = \tilde{\ell}(s_2) = \tilde{\ell}(s_5) = 2$, $\tilde{\ell}(s_3) = \infty$, and $\tilde{\ell}(s_4) = 3$. Therefore, S is a $(5,4)$-word degenerate set, a $(4,4)$-word degenerate set, a $(3,3)$-word degenerate set, and a $(2,0)$-word degenerate set. If $s_6 = (134)$, then $\{s_1, s_2, s_4, s_5, s_6\}$ is a $(3,5)$-word degenerate set because $\tilde{\ell}(s_i) = 2$ for all $i \in \{1, 2, 4, 5, 6\}$.

Definition 2.7. A directed *star forest* is a directed forest whose components are directed star graphs. If T is a directed tree, then a directed *spanning star forest* (SSF) of T is a spanning subgraph F that is a directed star forest. An SSF of T is *minimal*, denoted MSSF if it contains the minimum number of star components out of all SSFs of T.

Proposition 2.8. *Any directed minimal spanning star forest of a directed tree T contains the maximum number of arcs out of all directed spanning star forests of T. In particular, all minimal spanning star forests contain the same number of arcs.*

Proof. Let $|V(T)| = m$ so $|E(T)| = m - 1$. Suppose that $F = (V, E_F)$ is an MSSF of T and for each $i \in \{0, \ldots, m-1\}$, let $c_i \geq 0$ denote the number of components of F isomorphic to $DK_{1,i}$. Let $F' = (V, E_{F'})$ be any SSF of T and for each $i \in \{0, \ldots, m-1\}$, let $d_i \geq 0$ denote the number of components of F' isomorphic to $DK_{1,i}$. By definition of F,

$$\sum_{i=0}^{n-1} c_i \leq \sum_{i=0}^{n-1} d_i$$

Suppose that $|E_F| < |E_{F'}|$, i.e.,

$$|E_F| = \sum_{i=0}^{n-1} ic_i < \sum_{i=0}^{n-1} id_i = |E_{F'}|$$

Then, because both F and F' are spanning subgraphs,

$$m = \sum_{i=0}^{n-1}(i+1)c_i = \sum_{i=0}^{n-1} ic_i + \sum_{i=0}^{n-1} c_i < \sum_{i=0}^{n-1} id_i + \sum_{i=0}^{n-1} d_i = \sum_{i=0}^{n-1}(i+1)d_i = m$$

a contradiction. Therefore, $|E_F| \geq |E_{F'}|$. □

Theorem 2.9 (Main Result). *If T is any directed tree with m arcs and a minimal spanning star forest F, and S is a (k,t)-word degenerate m-subset of G where $k \geq \mathrm{diam}(T)$ and $t \leq |E(F)|$, then $X = \mathrm{DCay}(G; S)$ is T-decomposable unless the following are all true:*

1. *S is inverse-closed and involution-free;*
2. *T is an (α, β)-directed star ($m = \alpha + \beta$);*
3. *The underlying graph of X does not have an α-factor.*

Proof. We follow a similar technique as Fink [4]. Suppose $V(T) = \{v_1, \ldots, v_{m+1}\}$ and $S = \{s_1, \ldots, s_m\}$. Let S' be the set of degenerate elements in S. Let $N \subseteq E(F)$ where $|N| = t$. Label N with the t degenerate elements in S' and label $E(T) \setminus N$ with the $m-t$ non-degenerate elements in $S \setminus S'$. This arc-labeling guarantees there exist no semi-paths in T having length three or more whose arcs are labeled only with degenerate elements. Define a vertex-labeling function $\ell : V(T) \to G$ as follows. Root a vertex v_r in T and let $\ell(v_r) = e$, the identity in G. For any $v \in V(T) \setminus \{v_r\}$, let $P_v = v_{i_1}, v_{i_2}, \ldots, v_{i_{h+1}}$ be the unique semi-path in T with initial vertex $v_{i_1} = v_r$ and terminal vertex $v_{i_{h+1}} = v$. Let S_v be the sequence of arc labels on consecutive arcs along P_v,

$$S_v = [s_{i_1}, s_{i_2}, \ldots, s_{i_h}]$$

where the arc joining v_{i_j} and $v_{i_{j+1}}$ is labeled with s_{i_j} for all $j \in \{1, \ldots, h\}$. Then define

$$\ell(v) = \prod_{j=1}^{h} s_{i_j}^t \text{ where } \begin{cases} t = 1 & \text{if } (v_{i_j}, v_{i_{j+1}}) \in E(T) \\ t = -1 & \text{if } (v_{i_{j+1}}, v_{i_j}) \in E(T) \end{cases}$$

The vertices and arcs of T have been labeled with elements of G and S respectively, and each vertex of T is an S-word.

Claim 1: **The labeling ℓ is one-to-one if and only if there exists no directed 2-path (see Figure 4) labeled with a degenerate element and its inverse.**

Proof of Claim 1: Suppose that $\ell(u) = \ell(v)$ for some $u \neq v$. Because $e \notin S$, we must have that u is not adjacent to v. Then

$$\ell(u) = \prod_{j=1}^{h_1} s_{\alpha_j}^t = \prod_{j=1}^{h_2} s_{\beta_j}^t = \ell(v)$$

where $S_u = [s_{\alpha_1}, \ldots, s_{\alpha_{h_1}}]$ and $S_v = [s_{\beta_1}, \ldots, s_{\beta_{h_1}}]$. Let w be the last vertex that appears on both of the semi-paths P_u and P_v. There are three possibilities for w.

1. $w = v_r$. In this case,

$$e = \left(\prod_{j=1}^{h_1} s_{\alpha_j}^t \right)^{-1} \left(\prod_{j=1}^{h_2} s_{\beta_j}^t \right)$$

   so that e can be expressed as a word of size $h_1 + h_2 \leq \text{diam}(T)$.

2. $w \neq v_r$ and $w \in \{u, v\}$. Without loss of generality, suppose that $w = u$. In this case,

$$e = \prod_{j=h_1+1}^{h_2} s_{\beta_j}^t$$

   so that e can be expressed as a word of size $h_2 - h_1 \leq \text{diam}(T)$.

3. $w \neq v_r$ and $w \notin \{u, v\}$. In this case,

$$\ell(w) = \prod_{j=1}^{h_3} s_{\alpha_j}^t$$

   where $0 < h_3 < \min\{h_1, h_2\}$. Therefore,

$$e = \left(\prod_{j=h_3+1}^{h_1} s_{\alpha_j}^t \right)^{-1} \left(\prod_{j=h_3+1}^{h_2} s_{\beta_j}^t \right)$$

   so that e can be expressed as a word of size $h_1 + h_2 - 2h_3 \leq \text{diam}(T)$.

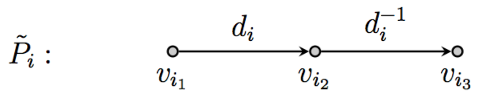

Figure 4: The directed 2-path with degenerate element d_i from Claims 1 and 2.

In each of the expressions for e in cases (1)–(3) above, the elements in the product are distinct, therefore each expression is an S-word of length at most diam(T). Furthermore, if s_{i_j} is any factor in one of the three S-words for e above, s_{i_j} may be written as a product of at most diam$(T) - 1$ others, so s_{i_j} is degenerate. By the definition of the arc-labeling of T, the length of the semi-path $P_{u,v}$ from u to v must equal two. Hence, up to isomorphism, $P_{u,v}$ is one of the following cases (4)–(6):

4. $P_{u,v} = u \to z \leftarrow v$, where (u, z) is an a-arc and (v, z) is a b-arc. Without loss of generality, either $w = u$ or $w = z$. In that former case, $ab^{-1} = e$, and in the latter case $a^{-1} = b^{-1}$, both contradictions to $a \neq b$.

5. $P_{u,v} = u \leftarrow z \to v$, where (z, u) is an a-arc and (z, v) is a b-arc. Without loss of generality, either $w = u$ or $w = z$. In that former case, $a^{-1}b = e$, and in the latter case $a = b$, both contradictions to $a \neq b$.

6. $P_{u,v} = u \to z \to v$, where (u, z) is an a-arc and (z, v) is a b-arc. Either $w = u$, $w = v$, or $w = z$. In first case, $ab = e$, in the second case, $b^{-1}a^{-1} = e$, and in the third case, $a^{-1} = b$. Either way, $b = a^{-1}$.

Therefore, the only possibility for $P_{u,v}$ is Case (6), which is isomorphic to Figure 4. This establishes that Claim 1 is true.

Claim 2: **There exists a permutation of the arc-labels of T such that ℓ is one-to-one.**

Proof of Claim 2: If ℓ is not one-to-one, then by Claim 1, there exists at least one directed 2-path of the form shown in Figure 4. Let $D = \{d_1, \ldots, d_q\}$ be an inverse-free subset of S' consisting of all degenerate elements d_i such that for each $i \in \{1, \ldots, q\}$, there exists a directed 2-path in the arc-labeled tree T where (v_{i_1}, v_{i_2}) is a d_i-arc and (v_{i_2}, v_{i_3}) is an d_i^{-1}-arc (see Figure 4). Note that these degenerate-labeled directed 2-paths must occur on arcs in $N \subseteq E(F)$ originally, i.e., arcs on directed stars. Hence for each $i \in \{1, \ldots, q\}$, the vertex v_{i_2} is always the center of a star in F and the unique d_i-arc is directed inward to the center vertex of the star and the unique d_i^{-1}-arc is directed outward away from the center vertex of the star. Let $y \in \mathbb{Z}^+$.

Case 1 ($q = 2y$): Apply the arc label permutation

$$\sigma = (d_1, d_2^{-1}, d_3, d_4^{-1}, \ldots, d_{2y}^{-1})$$

to $N \subseteq E(T)$. For each $i \in \{1, \ldots, q\}$ if i is odd (see Figure 5), then both the d_i-arc and the d_i^{-1}-arc are oriented away from the center of the corresponding star and if i is even (see Figure 6), then both the d_i-arc and the d_i^{-1}-arc are oriented towards the

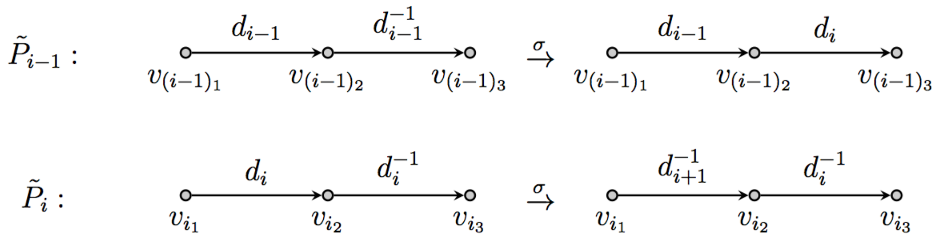

Figure 5: The effect of applying σ when i is odd.

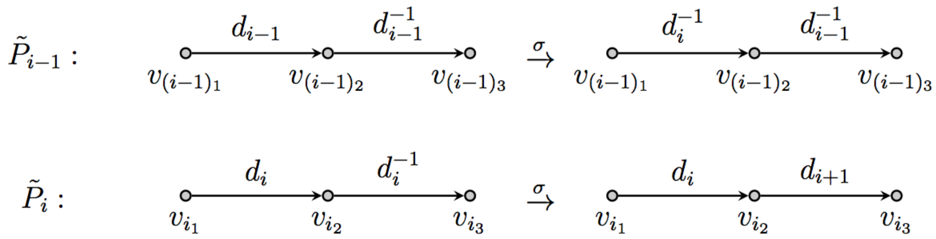

Figure 6: The effect of applying σ when i is even.

center of any corresponding star(s). This is true regardless of whether $v_{(i-2)_2} = v_{i_2}$ for some i or not. Hence by Claim 1, the new vertex-labeling ℓ will be one-to-one and so Claim 2 has been proven.

Case 2 ($q = 2y + 1$): Apply the arc label permutation

$$\tau = (d_1, d_2^{-1}, d_3, d_4^{-1}, \ldots, d_{2y}^{-1}, d_{2y+1})$$

to $N \subseteq E(T)$. Clearly, τ agrees with σ except for $\tau(d_{2y}^{-1})$ and $\tau(d_{2y+1})$. Similar to Claim 2 Case 1 (see Figures 5 and 6) for each $i \in \{2, \ldots, q\}$, the d_i-arc and the d_i^{-1}-arc are both oriented away from or both oriented towards the center of any corresponding star(s), depending on the parity of i. This is regardless of whether $v_{(i-2)_2} = v_{i_2}$ for some i or not. However by Claim 1 and Figure 7, a duplicate vertex labeling will exist if and only if $v_{1_2} = v_{(2y+1)_2}$. Let $F_1 \cong DK_{1,n_1}$, where $n_1 \geq 2$ be the component of F which contains v_{1_2}. In this case, τ produces an arc-labeling of F_1 such that the d_1-arc is oriented towards the center vertex and the d_1^{-1}-arc is oriented away from the center vertex. Furthermore, this is the *only* directed 2-path of the form shown in Figure 4 in F. We consider three cases:

(a) ***F* contains at least one component other than F_1 that is not an isolated vertex:** Let $F_2 \cong DK_{1,n_2}$ where $n_2 \geq 1$ be the component prescribed. Let g be the label of a fixed arc on F_2. If $g \neq d_{2y+1}$, then transpose the labels g and d_1. If $g = d_{2y+1}$, then transpose the labels g and d_1^{-1}.

(b) **All components of *F* other than F_1 are isolated vertices:** Let u be an isolated vertex in F. By the minimality of F, there exists no arc between u

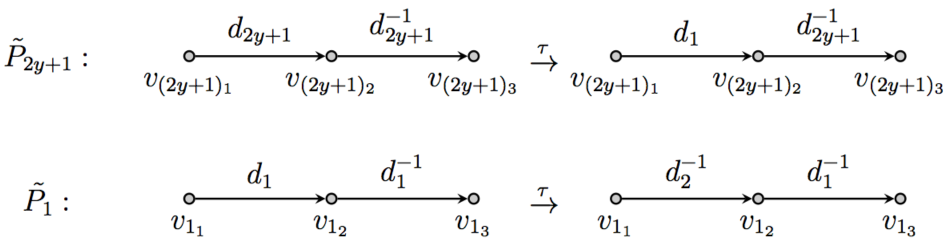

Figure 7: The effect of applying τ when $q = 2y + 1$.

and the center vertex of F_1. Therefore, in T, there exists a g-arc between u and a leaf v of F_1. Note g is not a degenerate element. Let g' be the label of the arc in F_1 that is incident with the g-arc in T. If $g' \neq d_1^{-1}$, then transpose the labels g and d_1. If $g' = d_1^{-1}$, then transpose the labels g and d_1^{-1}.

(c) ***F* is connected:** Clearly, $T = F = F_1$.

i. If there exists $g \in S$ such that either $g = g^{-1}$ or $g^{-1} \notin S$, then transpose the labels d_1 and g if the g-arc is oriented away from the center vertex of T, and transpose the labels d_1^{-1} and g otherwise.

ii. S is involution-free, inverse-closed, and thus every element is degenerate. T is an (α, β)-directed star, where $\alpha + \beta = m$ and $q \leq \min\{\alpha, \beta\}$. If the underlying graph of X has an α-factor, we are done by Corollary 1.9. Otherwise, this is the exceptional case of the hypothesis.

Clearly, there now exist no directed 2-paths labeled with a degenerate element and its inverse. Therefore, this establishes that Claim 2 is true.

At this point, T can be viewed as a subgraph of $\mathrm{DCay}(G; S)$. Finally, we show that the automorphism $\phi_g(v) = gv$ induces a decomposition of X into copies of T. For $g \in G$, let

$$gT = \{\phi_g(\ell(v)) : v \in V(T)\}$$

denote the image of the subgraph T of X under the natural action of ϕ_g. Each arc of gT is the unique s_i-arc in gT. It remains to show that gT and hT are arc-disjoint for all $g \neq h$. Suppose to the contrary that $e_1 = (x, xs_i)$ is an s_i-arc that appears in both $E(gT)$ and $E(hT)$. Because ϕ_g is an automorphism, e_1 is the unique s_i-arc in gT and hT, respectively. Hence, $\phi_g(e_1) = \phi_h(e_1)$ so that $(gx, g(xs_i)) = (hx, h(xs_i))$. Since the orientation of the arc remains the same it follows that $gx = hx$, a contradiction to $g \neq h$. Hence the orbit of T under that left action of G, $\{gT : g \in G\}$ is an arc-decomposition of X into copies of T. □

Example 2.10. Let $G = S_8$ and $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$ where $s_1 = (1234)$, $s_2 = (12)$, $s_3 = (134)$, $s_4 = s_1^2$, $s_5 = (452)$, $s_6 = (153)$, and $s_7 = (67)$. It was verified using [11] that $\tilde{\ell}(s_1) = \tilde{\ell}(s_2) = \tilde{\ell}(s_3) = \tilde{\ell}(s_5) = 2$, $\tilde{\ell}(s_4) = 3$, $\tilde{\ell}(s_6) = 4$, and $\tilde{\ell}(s_7) = \infty$. Therefore, S is a $(4, 5)$-word degenerate set. (It is also a $(6, 6)$-, a

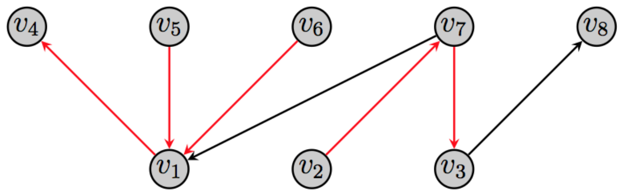

Figure 8: A directed tree T with spanning star forest with 3 components, isomorphic to directed stars $DK_{1,3}$, $DK_{1,2}$, and $DK_{1,0}$ from Example 2.10.

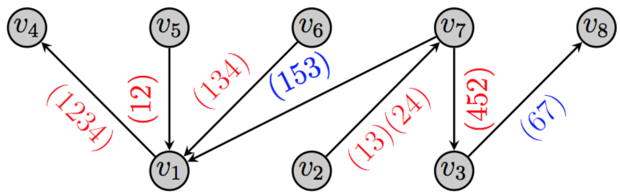

Figure 9: Arc-labeling T with degenerate elements (in red) and non-degenerate elements (in blue) of S from Example 2.10.

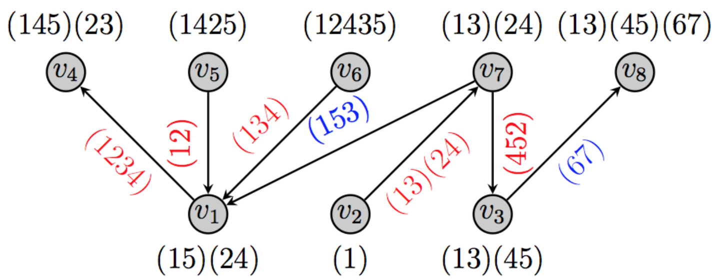

Figure 10: The vertex-labeling $\ell : V(T) \to S_8$ of T from Theorem 2.9 where $\ell(v_2) = (1) = e$. The S_8-valuation of T provides an embedding into $\text{Cay}(S_8; S)$ whose orbit decomposes the directed Cayley graph.

$(5, 6)$-, a $(3, 4)$-, and a $(2, 0)$-word degenerate set, but we shall focus on $k = 4$ and $t = 5$.) Consider the directed tree in Figure 8 with a minimal spanning star forest F having three components isomorphic to directed stars: $DK_{1,3}$, $DK_{1,2}$, $DK_{1,0}$. Consider the directed 7-regular Cayley graph $X = \text{DCay}(S_8; S)$ of order $8!$. Note X is disconnected, and S is not minimal. As $\text{diam}(T) \leq 4 = k$ and $|E(F)| \geq 5 = t$ and $S_t = \{s_1, s_2, s_3, s_4, s_5\}$ are the degenerate elements of S, we may label $E(F)$ with S_t, and remaining edges with $S \setminus S_t = \{s_5, s_6\}$ as shown in Figure 9. Root v_2 and label $\ell(v_2) = (1)$. The vertex labeling defined in Theorem 2.9 is shown in Figure 10.

As any minimal generating set S is necessarily $(|S|, 0)$-word degenerate, we obtain

Fink's result, Theorem 1.10 as a corollary.

If $\mathrm{diam}(T) \geq 3$, we have a simpler statement.

Corollary 2.11. *Let T be a directed tree with m arcs and a minimal spanning star forest F. If S is a (k,t)-word degenerate m-subset of G where $k \geq \mathrm{diam}(T) \geq 3$ and $t \leq |E(F)|$, then* $\mathrm{DCay}(G;S)$ *is T-decomposable.*

The following corollary ties the matching number of the tree to the number of degenerate elements allowed in the connection set.

Corollary 2.12. *If T is any directed tree with m arcs, and S is (k,t)-word degenerate m-subset of G, where $k \geq \mathrm{diam}(T) \geq 3$ and $t \leq \nu(T)$, the matching number of T, then* $\mathrm{DCay}(G;S)$ *is T-decomposable.*

Proof. If $M = (V_M, E_M)$ is a maximum matching in T having $|E_M| = \nu(T)$ arcs, then $V(T) \setminus V_M$ is an independent set of vertices. Therefore, we may identify M with a directed spanning star forest consisting of $c_1 = \nu(T)$ components of the form $DK_{1,1}$ and c_0 components of the form $DK_{1,0}$. By Proposition 2.8 and Theorem 2.9 the result is established. □

We briefly discuss the undirected case. In 2000, El-Zanati et al., verified that certain undirected Cayley graphs of even order are decomposable into trees with twice as many edges as prescribed by Ringel. The following technical definition implies that no element of the connection set can be expressed as a distinct product of powers from $\{-2,-1,0,1,2\}$ of the remaining elements.

Definition 2.13 (El-Zanati et al. [3]). A subset $S = \{s_1, \ldots, s_m\}$ of a group G is *square-independent* if for any $s_j \in S$

$$s_j \neq \prod_{i \in N_j} s_{\sigma(i)}^{\pm 1,2,0}$$

where $N_j \subseteq \{1,\ldots,m\} \setminus \{j\}$ and $\sigma \in \mathrm{Sym}(N_j)$.

Theorem 2.14 ([3]). *Let G be a finite group and H be a subgroup of index two. If $S \subset G - H$ is square-independent, containing n_1 non-involutions and n_2 involutions, and T is any tree with $2n_1 + n_2$ edges, then* $\mathrm{Cay}(G;S)$ *has a T-decomposition into $|G|/2$ copies of T.*

It should be noted that every square-independent subset S is necessarily $(|S|,0)$-word degenerate. This provides a simple directed analogue of Theorem 2.14:

Theorem 2.15. *If S is a square-independent m-subset of a finite group G, and T is any directed tree with m arcs, then* $\mathrm{DCay}(G;S)$ *is T-decomposable.*

From the definition, the projection graph of $\mathrm{DCay}(G;S)$ is exactly $\mathrm{Cay}(G;S)$. If $X = \mathrm{DCay}(G;S)$ is m-regular and S is inverse-free *and* involution-free, then the projection graph of X and the underlying graph of X are equal and are $2m$-regular. This implies the following proposition and corollary.

Proposition 2.16. *If S is involution-free and inverse-free, and* $\mathrm{DCay}(G;S)$ *is T-decomposable into $|G|$ copies of the directed tree T with $|S|$ arcs, then* $\mathrm{Cay}(G;S)$ *is T'-decomposable into $|G|$ copies of the underlying graph T' of T.*

Corollary 2.17. *If S is involution-free and is* $(|S|, 0)$*-word degenerate, and T is any tree with $|S|$ edges, then* $\mathrm{Cay}(G;S)$ *is T-decomposable.*

Proof. If S is $(|S|, 0)$-word degenerate, then S is necessarily inverse-free, hence exceptional Case (2) of the statement of Theorem 2.9 does not occur. The underlying graph of $X = \mathrm{DCay}(G;S)$ is just $X' = \mathrm{Cay}(G;S)$. By Theorem 2.9, X is T-decomposable where T is any directed tree on $|S|$ edges and the result follows from Proposition 2.16. □

3 Concluding Remarks

The concept of "word-degeneracy," though somewhat technical, proved to be a useful mechanism for gradually loosening the reigns on minimality of the connection set S. We have proved that the family of directed Cayley graphs which are tree-decomposable is much larger than the original result of Fink by simply saturating the arcs of a minimal spanning star forest with degenerate element labels. This is a considerable contrast to the Hamilton decomposition problem for Cayley graphs, which is resolved for minimal or strongly-minimal connection subsets of Abelian groups (see Liu [6, 7]), but remains very open when minimality is even slightly relaxed. Two natural questions to investigate regarding Question 1.6 and the concept of (k, t)-word degenerate are (1) *how small can we make k with a fixed t* and (2) *how large can we make t with a fixed k*? It seems that these methods could possibly be translated to the vertex-transitive graphs though the arc- and vertex-labelings as well as the action of automorphisms on the tree may prove quite difficult to obtain. Word-degeneracy may also prove useful in different types of decomposition problems for Cayley graphs and digraphs.

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