AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **60(3)** (2014), Pages 314–324

Connected pseudoachromatic index of complete graphs

LOWELL ABRAMS

Department of Mathematics
George Washington University
Washington, DC 20052
U.S.A.
labrams@gwu.edu

YOSEF BERMAN

1705 East-West Highway
Silver Spring, MD 20910
U.S.A.
berman.yosef@gmail.com

Abstract

A *connected pseudocomplete n-coloring* of a graph G is a (non-proper) n-coloring of the vertices of G such that each color class induces a connected subgraph and for each pair of color classes there is an edge with one end of each color; this can be viewed as a kind of "inverse image" of a clique minor. The *connected pseudoachromatic index* of a graph G is the largest n for which the line graph of G has a connected pseudocomplete n-coloring. For all j, k we show that the connected pseudoachromatic index of the complete graph on $5k+j+1$ vertices is at least $9k+j$. We also provide several results on connections between connected pseudoachromatic index of complete graphs and the Erdős-Faber-Lovasz conjecture.

1 Introduction

Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. A *pseudocomplete k-coloring* of G is an assignment $\alpha\colon V(G) \to [k]$, where $[k]$ denotes the set $\{1, \ldots, k\}$, such that for each $i, j \in [k]$ there is an edge in $E(G)$ having one end in $\alpha^{-1}(i)$ and the other in $\alpha^{-1}(j)$. The *pseudoachromatic number* $\psi(G)$ of G is the maximum k for which there is a pseudocomplete k-coloring of G [9]. Basic results on pseudoachromatic number and the related notion of achromatic number (which presumes that no two adjacent vertices have the same color) were presented in [3, 5, 4, 11]. Along these lines also is [12], which discusses achromatic number of the line graph of K_n. Of more recent note is the calculation of ψ for complete multipartite graphs [14, 16] and for the line graph of K_n for special values of n [1, 2].

Suppose H is a minor of G obtained from a subgraph G' of G by contracting some edges, and that $V(H) = [k]$. Then there is a naturally corresponding pseudocomplete k-coloring $\alpha\colon G' \to [k]$ for which $\alpha^{-1}(i)$ is exactly the set of vertices of G' which contract to vertex i in H. In this case, the classes $\alpha^{-1}(i)$ have the additional property that for each i the induced subgraph $G[\alpha^{-1}(i)]$ is connected. Since this is so, without loss of generality we may presume that $G' = G$. Define the *connected pseudoachromatic number* $\psi_c(G)$ to be the maximum k for which there is a *connected pseudocomplete k-coloring* of G, i.e. , a pseudocomplete coloring in which each color-class induces a connected subgraph. With this definition, we see that $\psi_c(G)$ is the size of the largest complete-graph minor of G; this value is also called the *Hadwiger number* of G. Since for any graph G we have $\psi(G) \geq \psi_c(G)$, study of the pseudoachromatic number has been useful for bounding the Hadwiger number, as in [13] and, from a probabilistic perspective, [6].

The pseudoachromatic number of the line graph LG for any graph G is also referred to as the *pseudoachromatic index of G* [4]. We focus in this work on the line graph LK_n of the complete graph K_n. Note that any connected pseudocomplete k-coloring of LK_n may be viewed as an edge coloring of K_n in which each edge color class induces a connected subgraph, and each pair of edge color classes share at least one vertex. We will make use of this point of view below when it is convenient.

There are a few existing results on $\psi(LK_n)$. Bosák and Nešetřil provide the following values.

n	1	2	3	4	5	6	7
$\psi(LK_n)$	0	1	3	4	7	8	11

Araujo-Pardo et al. prove that if $n = 2^{2\beta} + 2^{\beta} + 1$ then $\psi(LK_n) \geq 2^{3\beta} + 2^{\beta}$, and if $n = 2^{2\beta} + 2^{\beta+1} + 2$ then $\psi(LK_n) = 2^{3\beta} + 2^{2\beta} + 3{\cdot}2^{\beta}$.

In Section 2 we provide several results on lower bounds for $\psi_c(LK_n)$ for various values of n. Together they imply Theorem 2.4: *For $j, k \geq 1$ we have $\psi_c(LK_{5k+j+1}) \geq 9k + j$.*

Our results on LK_n have an interesting implication for the relationship between two famous conjectures.

Conjecture 1.1 (Hadwiger [8, 10])**.** *If graph G does not contain K_{n+1} as a minor, then G is n-colorable.*

Of course, Hadwiger's Conjecture is the reason for the term "Hadwiger number."

Conjecture 1.2 (Erdős-Faber-Lovász [7])**.** *If graph G can be constructed by joining n copies of K_n so that no two copies of K_n share more than one vertex, then G is n-colorable.*

We refer to a graph which meets the hypothesis of the Erdős-Faber-Lovász Conjecture as an EFL graph. In Section 4 we exhibit an infinite family of EFL graphs which contain K_{n+1} as a minor, demonstrating that Hadwiger's Conjecture does not imply the Erdős-Faber-Lovász Conjecture.

2 Connected Pseudoachromatic Number

Throughout this section we let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and write $v_{i,j}$ for the vertex of LK_n corresponding to the undirected edge v_iv_j of K_n. We begin with two basic cases.

Proposition 2.1. *We have $\psi_c(LK_4) = 4$ and $\psi_c(LK_5) = 6$.*

Proof. Note first that LK_4, which is the 1-skeleton of the octahedron, is planar and hence does not contain K_5 as a minor. Given the connection between ψ_c and clique-minors, we see that $\psi_c(LK_4) \leq 4$. On the other hand, it is not difficult to construct a connected pseudocomplete 4-coloring α_4 of LK_4; here is one:

$$\begin{array}{ll} \alpha_4^{-1}(1) = \{v_{1,2}\} & \alpha_4^{-1}(3) = \{v_{2,3}\} \\ \alpha_4^{-1}(2) = \{v_{1,3}\} & \alpha_4^{-1}(4) = \{v_{1,4}, v_{2,4}, v_{3,4}\} \end{array}$$

It is easy to check that each of these vertex classes is connected and between each pair of vertex classes there is an edge connecting a vertex in one of the classes to a vertex in the other class. This completes the proof of the first assertion in the proposition.

For the second assertion, here is one way to construct a connected pseudocomplete 6-coloring α_6 of LK_5. Define α_6 by

$$\begin{array}{ll} \alpha_6^{-1}(1) = \{v_{1,2}, v_{2,3}\} & \alpha_6^{-1}(4) = \{v_{1,5}, v_{4,5}\} \\ \alpha_6^{-1}(2) = \{v_{1,3}, v_{3,5}\} & \alpha_6^{-1}(5) = \{v_{2,4}, v_{2,5}\} \\ \alpha_6^{-1}(3) = \{v_{1,4}\} & \alpha_6^{-1}(6) = \{v_{3,4}\} \end{array}$$

Again, it is not difficult to verify the validity of this connected pseudocomplete 6-coloring.

Now suppose for the sake of contradiction that $\alpha\colon V(LK_5) \to [7]$ is a connected pseudocomplete 7-coloring. Since $|V(LK_5)| = 10$, at least 4 classes $\alpha^{-1}(i)$ contain a single vertex. It is not difficult to see that the corresponding singleton edge classes in K_5 must share a single vertex x of K_5 in order to satisfy the mutual adjacency requirement. But now, since all edges incident to x in K_5 have been used, each additional edge class in K_5 must have at least three edges in order to simultaneously satisfy the connectedness and mutual adjacency requirements; see Figure 1. This implies that there are at most two non-singleton classes, contradicting the assumption that we have a 7-coloring. □

The next result shows that any lower bound on $\psi_c(LK_n)$ for some n gives lower bounds for all larger values of n as well.

Proposition 2.2. *If for some n we have $\psi_c(LK_n) \geq m$ for some m, then $\psi_c(LK_{n+j}) \geq m + j$ for all $j \geq 0$.*

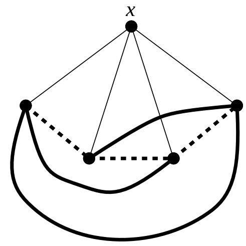

Figure 1: Edge classes in K_5. Each of the four edges incident to x comprises its own class, the bold edges comprise a single class, and the dashed edges comprise a single class.

Proof. Proceeding by induction we assume the existence of a connected pseudocomplete m-coloring α_m of LK_{n+j-1} for some $j > 0$ and extend it to construct a connected pseudocomplete $(m+1)$-coloring α_{m+1} of LK_{n+j}. Explicitly, for $1 \leq i \leq m$ let $\alpha_{m+1}^{-1}(i) = \alpha_m^{-1}(i)$ and let $\alpha_{m+1}^{-1}(m+1) = \{v_{1,n+j}, \ldots, v_{n+j,n+j}\}$. Clearly, the subgraph induced on $\alpha_{m+1}^{-1}(m+1)$ is connected. To see that α_{m+1} is pseudocomplete we can interpret $\{v_{1,n+j}, \ldots, v_{n+j-1,n+j}\}$ as the edges $v_1v_{n+j}, \ldots, v_{n+j-1}v_{n+j}$ and note that each vertex of K_{n+j-1} is incident to one of these. It then follows that each vertex in LK_{n+j} is adjacent to some vertex in $\alpha_{m+1}^{-1}(m+1)$, so we are done. □

Propositions 2.1 and 2.2 combine to yield a proof by induction that $\psi_c(LK_n) \geq n+1$ for $n \geq 5$. Nevertheless, Proposition 2.3 provides a far better result, formulated below as Theorem 2.4.

Proposition 2.3. *For $k \geq 1$ we have $\psi_c(LK_{5k+2}) \geq 9k+1$.*

Proof. We first verify the result for $k = 1$. Construct a connected pseudocomplete 10-coloring α_{10} of LK_7 as follows.

$$\begin{array}{ll} \alpha_{10}^{-1}(1) = \{v_{3,7}\} & \alpha_{10}^{-1}(6) = \{v_{5,6}, v_{6,7}\} \\ \alpha_{10}^{-1}(2) = \{v_{1,7}, v_{1,3}\} & \alpha_{10}^{-1}(7) = \{v_{3,5}, v_{2,5}\} \\ \alpha_{10}^{-1}(3) = \{v_{2,6}, v_{2,7}\} & \alpha_{10}^{-1}(8) = \{v_{3,6}, v_{4,6}\} \\ \alpha_{10}^{-1}(4) = \{v_{2,4}, v_{4,7}\} & \alpha_{10}^{-1}(9) = \{v_{2,3}, v_{1,2}, v_{1,5}\} \\ \alpha_{10}^{-1}(5) = \{v_{4,5}, v_{5,7}\} & \alpha_{10}^{-1}(10) = \{v_{1,6}, v_{1,4}, v_{3,4}\} \end{array}$$

It is easy to check that each of these vertex classes is connected and between each pair of vertex classes there is an edge connecting a vertex in one of the classes to a vertex in the other class.

We now provide a construction, working in terms of edges colorings of complete graphs, that verifies the result for general k. Take k disjoint copies $P_1, \ldots, P_k$ of the edge-colored K_7 specified above, and identify all vertices v_3 and all vertices v_7, respectively, as well as all edges v_3v_7 and their color classes. We maintain the distinct identities of the colors, other than color 1, for each P_i. The resulting graph has $5k+2$

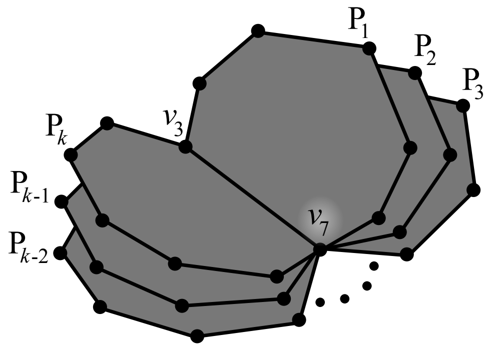

Figure 2: A schematic of the construction in the proof of Proposition 2.3. Each panel represents a copy of K_7.

vertices (a schematic is shown in Figure 3); add in edges to obtain the complete graph K_{5k+2}.

Now consider a pair P_r, P_s. Refer to the vertices in P_r as $v_1, v_2, \ldots, v_7$, to the edge colors in P_r as $1, 2, \ldots, 10$, and to the corresponding vertices and colors in P_s as $v_{1'}, v_{2'}, v_3, v_{4'}, v_{5'}, v_{6'}, v_7$, and $1, 2', \ldots, 10'$, respectively. Let α_{9k+1} be partially defined by

$$\begin{array}{llll} v_{1,2'} \mapsto 9 & v_{2,2'} \mapsto 3 & v_{4,4'} \mapsto 4 & v_{5,5'} \mapsto 5 \\ v_{1,4'} \mapsto 10 & v_{2,4'} \mapsto 3 & v_{4,5'} \mapsto 8 & v_{6,2'} \mapsto 8 \\ v_{1,5'} \mapsto 9 & v_{2,5'} \mapsto 7 & v_{5,2'} \mapsto 7 & v_{6,5'} \mapsto 6 \\ v_{1,6'} \mapsto 10 & v_{4,2'} \mapsto 4 & v_{5,4'} \mapsto 5 & v_{6,6'} \mapsto 6 \end{array}$$

It is not difficult to check that each color class in $P_r \cup P_s$ is now incident to each other color class. To assist in this check, here is a table showing the classes incident to each vertex:

vertex : incident classes	
$v_3 : 1, 2, 7, 8, 9, 10, 1', 2', 7', 8', 9', 10'$	
$v_7 : 1, 2, 3, 4, 5, 6, 1', 2', 3', 4', 5', 6'$	
$v_1 : 2, 9, 10$	$v_1' : 2', 9', 10'$
$v_2 : 3, 4, 7, 9$	$v_2' : 3', 4', 7', 9', 3, 4, 7, 8, 9$
$v_4 : 4, 5, 8, 10$	$v_4' : 4', 5', 8', 10', 3, 4, 5, 10$
$v_5 : 5, 6, 7, 9$	$v_5' : 5', 6', 7', 9', 5, 6, 7, 8, 9$
$v_6 : 3, 6, 8, 10$	$v_6' : 3', 6', 8', 10', 6, 10$

Follow the analogous procedure for all other pairs P_r, P_s. Finally, define α_{9k+1} on the remaining edges in any way that preserves the connectivity of the color classes. Since each P_i contributes nine color classes in addition to the class 1, we indeed have a connected pseudocomplete $(9k+1)$-coloring of LK_{5k+2}. □

Combining Propositions 2.3 and 2.2 we obtain Theorem 2.4.

Theorem 2.4. *For $j, k \geq 1$ we have $\psi_c(LK_{5k+j+1}) \geq 9k + j$.*

3 Computer calculation and what is known

Proposition 2.1 tells us that $\psi_c(LK_4) = 4$ and $\psi_c(LK_5) = 6$, and exhaustive computer calculation confirms that $\psi_c(LK_6) = 7$ and $\psi_c(LK_7) = 10$ (the computer calculations are described below). Beyond this, we know only the lower bounds derived from Theorem 2.4. This information is depicted in Figure 3.

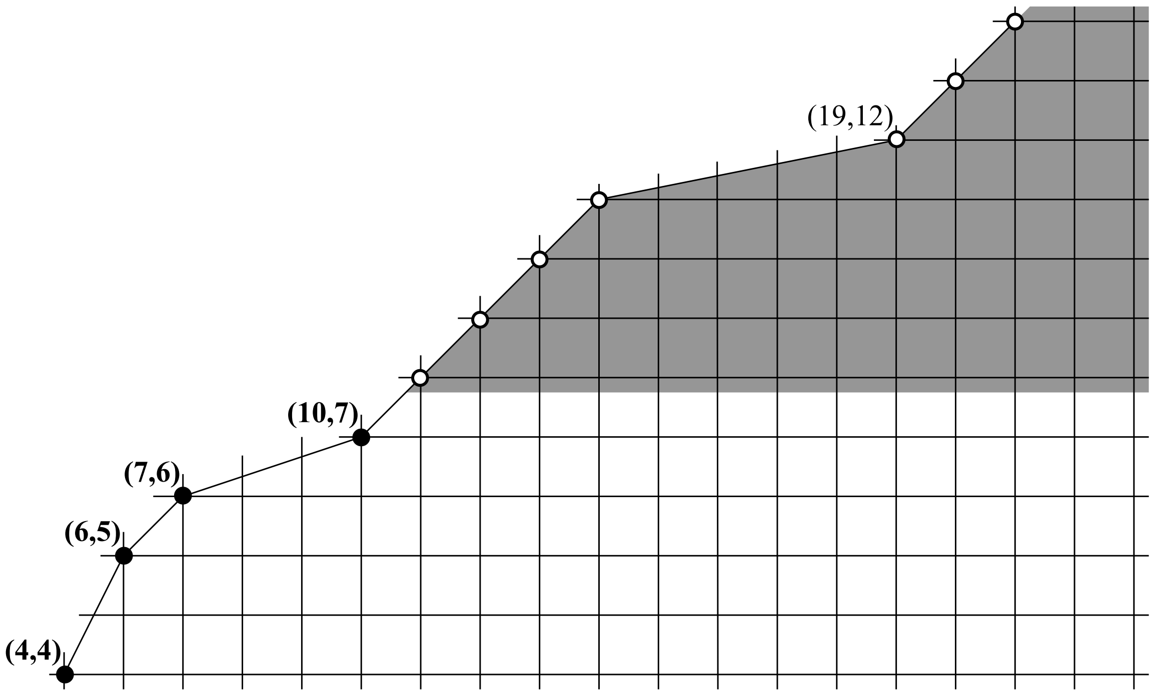

Figure 3: A schematic indicating what is known. A solid point at (x, y) indicates that $\psi_c(LK_y) = x$, whereas a hollow point indicates that $\psi_c(LK_y) \geq x$. The shading emphasizes that the actual values of the various pseudoachromatic indices may lie to the right.

We now describe how the computer calculations were done. Without loss of generality we can assume that the vertex classes in LK_n that constitute a connected pseudcomplete coloring are trees in K_n, since all other edges may be colored in any way that preserves the connectedness of the individual classes. Using Prüfer sequences it is straightforward to iterate over all spanning trees in a clique, and by considering all possible sizes of subsets of the vertices of K_n we can find all subgraphs of K_n which are trees. Let $\mathcal{T}_n$ denote the set of all such trees and define a graph structure on $\mathcal{T}_n$ by declaring two trees to be adjacent if they share at least one vertex but share no edges. A clique in $\mathcal{T}_n$ represents a partial covering of the edges of K_n with a family of edge-disjoint trees such that each pair of trees shares at least one vertex. By extending the trees as described above, we obtain a connected pseudocomplete coloring of LK_n.

Thus we have reduced the problem of finding connected pseudocomplete colorings of LK_n to the problem of finding cliques in $\mathcal{T}_n$. Of course, $\mathcal{T}_n$ grows in n faster than any exponential. Nevertheless, for sufficiently small n, this approach yielded new results, specifically that $\psi_c(LK_6) = 7$ and $\psi_c(LK_7) = 10$. Note that finding maximum cliques was done using the publicly available software package Cliquer [15].

4 Implications for EFL Graphs

An *n-EFL graph* is a connected graph G produced by joining n copies of K_n, which we call *panels*, so that no two panels share more than one vertex. We refer to vertices contained in more than one panel as *vertices of attachment*. For each n, define the *standard n-EFL graph* E_n to be the graph obtained from LK_n by adding new vertices $\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_n$ and an edge connecting $\hat{v}_i$ and $v_{i,j}$ for each $i, j \in [n]$. Note that, for each i, the subgraph of E_n induced on the vertices $\{\hat{v}_i\} \cup \{v_{i,j} \mid j \neq i\}$ is an n-clique, so E_n is indeed an n-EFL-graph. Figure 4 shows a drawing of E_4.

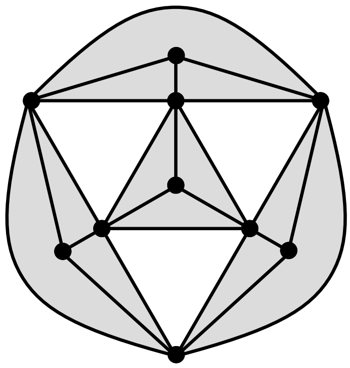

Figure 4: A plane drawing of E_4; the shading indicates copies of K_4.

The following result indicates a sense in which the graphs E_n are universal.

Theorem 4.1. *If G is an n-EFL graph and K_m is a minor of G, then K_m is a minor of E_n as well.*

Proof. Certainly, if $m \leq n$ then K_m is a subgraph of any panel in E_n, so is a minor of E_n.

Suppose, then, that $m > n$ and that K_m is a minor of G. We may realize this minor with a pair (T, β) where $T = \{t_1, \ldots, t_m\}$ is a family of vertex-disjoint trees in G and $\beta\colon \{\{i,j\} | i, j \in [m], i \neq j\} \to E(G)$ is such that for $i \neq j$ the edge $\beta(i,j)$ has one vertex in t_i and the other in t_j. The trees can be viewed as sitting in distinct color classes of a connected pseudocomplete m-coloring of G; contracting them yields the vertices of K_m and the edges in $\mathrm{Im}\beta$ are the edges of K_m.

Suppose $t \in T$ contains no vertices of attachment. In that case, t must be contained entirely in a single panel P. Moreover, all edges in $\mathrm{Im}\beta$ which have one vertex in t must have their other vertex in P as well, and thus there can be at most $n-1$ such edges. Since $m > n$, this contradicts the assumption that (T, β) represents a K_m minor. Thus every tree t in T contains vertices of attachment, and indeed must have a vertex of attachment in each panel in which it has any vertex at all.

We now produce a new pair (T', β') which realizes K_m as a minor of G but which has the additional property that for each $t \in T'$ all vertices of t are vertices of attachment. Suppose v is a vertex in panel P which is not a vertex of attachment but is contained in tree $t_i \in T$. As shown above, there is some vertex v_i in P which is contained in t_i and is a vertex of attachment. Modify t_i by deleting v and, for each neighbor w of v which is contained in t_i but is not connected to v_i in $t_i - v_i$, adding in the edge $v_i w$. Because v is not a vertex of attachment, w must be contained in panel P, and therefore the edge $v_i w$ exists. Because the only path in t_i between neighbors of v is a path of length two through v, the result of this deletion and addition is itself a tree. Corresponding to this new tree, we also modify β. For every $j \in [m]$ such that $\beta(\{i, j\}) = v w_j$ for some w_j in t_j, redefine β to map $\{i, j\} \mapsto v_i w_j$. As before, because v is not a vertex of attachment, w_j must be contained in panel P, and therefore the edge $v_i w_j$ exists. See Figure 5 for an illustration of this two step process. Repeating the process for every such vertex v and panel P yields the desired pair (T', β').

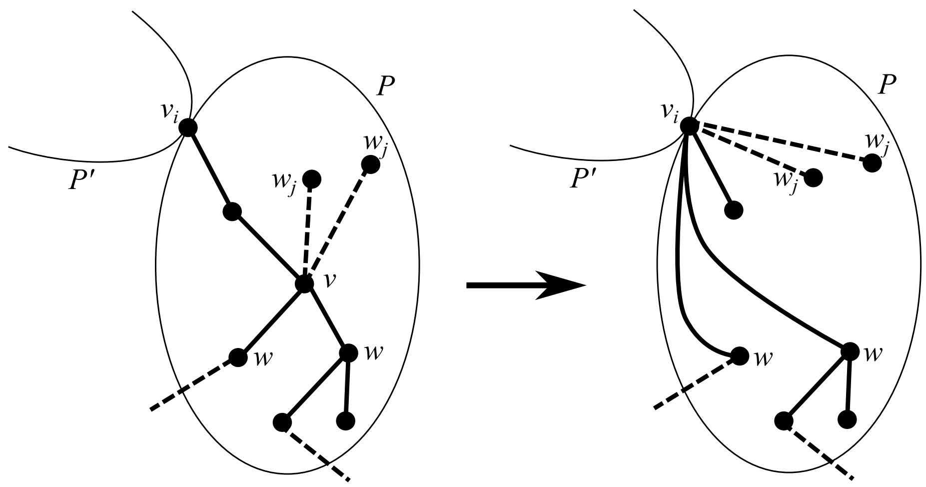

Figure 5: An illustration of the two step process to ensure that all vertices in t_i are vertices of attachment. Edges in the image of β are shown dashed, and solid edges belong to t_i. P' is some panel other than P.

We now modify G so as to convert it to E_n; adjusting the pair (T', β') appropriately through this process yields the desired K_m minor of E_n. Define the *attachment weight* of G to be the sum $W(G) := \sum_v (d_v - 2)$ where the summation is over vertices of attachment v and d_v is the number of panels containing v. If $W(G) = 0$, then we already have $G = E_n$ and we are done. Otherwise, suppose that $W(G) > 0$, that v is a vertex of attachment with $d_v > 2$, and that P_1, P_2 are two of the panels containing v. Since there are a total of n panels, and P_2 has n vertices, and $d_v > 2$, there is some vertex w of P_2 which is not a vertex of attachment and which therefore is not contained in any tree of T'. Modify G by detaching panel P_1 at v from the other panels at v, splitting off a new copy v_1 of vertex v in panel P_1 and renaming the "original" copy of v to v_2, then identify v_1 and w. See Figure 6 for an illustration of this.

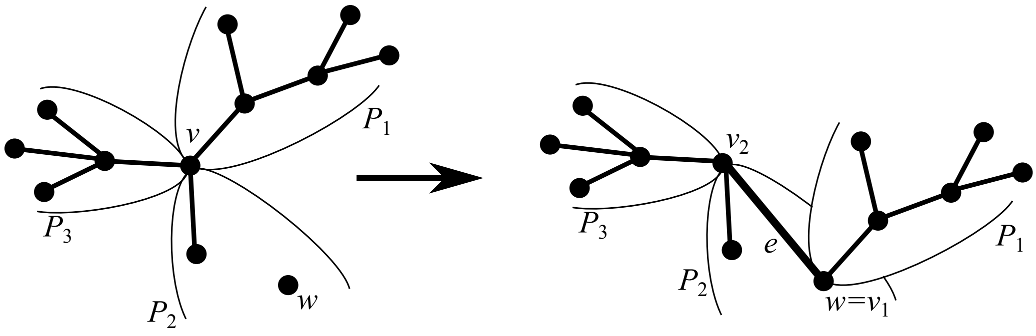

Figure 6: An illustration of the process of detaching panel P_1, reattaching it, and then relabeling vertices. The diagram on the left depicts a tree t_i containing v, and on the right is the tree t' replacing it.

If v was contained in a tree t_i of T' (as illustrated in Figure 6), now adjust t_i to include the edge $e = v_1v_2$; refer to this newly adjusted t_i as t'. To see that t' is connected, consider any vertices x and y in t'. Suppose first that both x and y are also vertices of t_i, so that there is an x-y path $P_{x,y}$ in t_i. If $P_{x,y}$ does not contain v, then $P_{x,y}$ is also an x-y path in t', so x and y are connected in t'. If $P_{x,y}$ does contain v, express $P_{x,y}$ as the concatenation $P_{x,v}P_{v,y}$ where $P_{x,v}$ is the x-v path in t_i and $P_{v,y}$ is the v-y path. For some $v', v'' \in \{v_1, v_2\}$, $P_{x,v}$ and $P_{v,y}$ correspond to an x-v' path $P'_{x,v'}$ in t' and a v''-y path $P'_{v'',y}$ in t', respectively. If $v' = v''$, then the concatenation $P'_{x,v'}P'_{v'',y}$ is an x-y path in t', and otherwise $P'_{x,v'}eP'_{v'',y}$, for some orientation on e, is an x-y path in t'.

If $\{x, y\} = \{v_1, v_2\}$, then the edge v_1v_2 itself provides the desired x-y path. Suppose therefore that exactly one of x and y is either v_1 or v_2. Without loss of generality, suppose $y = v_1$. There is an x-v path in t_i, and when the panel P_1 is detached at v this path becomes either an x-v_1 path or an x-v_2 path in t'. In the latter case, concatenating with the edge v_2v_1 yields an x-v_1 path, so in either case we see that x and v_1 are connected in t'. We thus conclude that t' is connected.

Finally, up to renaming of vertices, no changes are made to β'. This process decreases the attachment weight by 1, and the adjusted (T', β') still realizes a K_m minor. Since this process can be repeated until $W(G) = 0$, the proof is complete. □

Since any minor of LK_n is automatically a minor of E_n, the results of Section 2 readily imply that for $n \geq 5$ the graph E_n contains a K_{n+1} minor. This demonstrates that Hadwiger's Conjecture does not imply the EFL conjecture.

There is also an interesting relationship between E_n and LK_n in the other direction.

Theorem 4.2. *For each $m > 1$ and $n > 2$, if K_m is a minor of E_n then K_m is a minor of LK_n.*

Proof. The induced subgraph of LK_n with vertex set $\{v_{1,j} \mid j \neq 1\}$ forms an $(n-1)$-clique, so certainly K_m is a minor of LK_n for $m < n$.

We can find K_n as a minor of LK_n by defining a pseudocomplete n-coloring $\alpha_n \colon V(LK_n) \to [n]$ as follows: Let

$$\alpha_n^{-1}(1) = \{v_{2,3}, v_{2,4}, \ldots, v_{2,n}\}$$

and for $i = 2, 3, \ldots, n$ let $\alpha_n^{-1}(i) = v_{1,i}$. Note that $\cup_{i=2}^{n} \alpha_n^{-1}(i)$ induces an $(n-1)$-clique in LK_n and that for each i there is an edge from $\alpha_n^{-1}(i)$ to $\alpha_n^{-1}(1)$. We see that contracting $\alpha_n^{-1}(1)$ to a single vertex yields the desired K_n minor.

Suppose now that $m > n$ and that K_m is a minor of E_n, but that K_m is not a minor of LK_n. Then there is some vertex class $W \subseteq V(E_n)$ for the K_m minor which contains a vertex $w = \hat{v}_j$ for some j. Since the degree of $\hat{v}_j$ is $n-1$ but W contracts to a vertex of degree $m - 1 \geq n$, there must be an additional vertex v in W which is adjacent to w. The vertex v is necessarily adjacent to all neighbors of w so any path in E_n starting at w passes through a neighbor of v. It follows that removing w from W leaves a suitable vertex class for the desired K_m minor, and that no edges incident to w are needed. Applying this reasoning to each $\hat{v}_j$ in each vertex class in E_n for K_m shows that K_m is indeed a minor of LK_n. □

References

[1] M. G. Araujo-Pardo, J. J. Montellano-Ballesteros, C. Rubio-Montiel and R. Strausz, On the pseudoachromatic index of the complete graph II, *Bol. Soc. Mat. Mexicana* 20(1) (2014), 17–28.

[2] M. G. Araujo-Pardo, J. J. Montellano-Ballesteros and R. Strausz, On the pseudoachromatic index of the complete graph, *J. Graph Theory* 66(2) (2011), 89–97.

[3] F. Bories and J.-L. Jolivet, On complete colorings of graphs, In *Recent advances in graph theory (Proc. Second Czechoslovak Sympos., Prague, 1974)*, pp. 75–87. Academia, Prague, 1975.

[4] J. Bosák and J. Nešetřil, Complete and pseudocomplete colourings of a graph, *Math. Slovaca* 26(3) (1976), 171–184.

[5] A. Bouchet, Indice achromatique des graphes multiparti complets et réguliers, *Cahiers Centre Études Rech. Opér.* 20(3-4) (1978), 331–340, Colloque Mathématiques Discrètes: Codes et Hypergraphes (Brussels, 1978).

[6] Y. Drier and N. Linial, Minors in lifts of graphs, *Random Structures Algorithms* 29(2) (2006), 208–225.

[7] P. Erdős, On the combinatorial problems which I would most like to see solved, *Combinatorica* 1(1) (1981), 25–42, doi: 10.1007/BF02579174.

[8] R. L. Graham, M. Grötschel and L. Lovász, eds., *Handbook of Combinatorics* Vol. 1, 2, Elsevier Science B.V., Amsterdam, 1995.

[9] R. P. Gupta, Bounds on the chromatic and achromatic numbers of complementary graphs, In *Recent Progress in Combinatorics (Proc. Third Waterloo Conf. Combinatorics, 1968)*, pp. 229–235, Academic Press, New York, 1969.

[10] H. Hadwiger, Über eine Klassifikation der Streckenkomplexe, *Vierteljschr. Naturforsch. Ges. Zürich* 88 (1943), 133–142.

[11] F. Harary and S. Hedetniemi, The achromatic number of a graph, *J. Combin. Theory* 8 (1970), 154–161.

[12] R. E. Jamison, On the edge achromatic numbers of complete graphs, *Discrete Math.* 74(1-2) (1989), 99–115.

[13] A. V. Kostochka, The minimum Hadwiger number for graphs with a given mean degree of vertices, *Metody Diskret. Analiz.* (38) (1982), 37–58.

[14] M. Liu and B. Liu, On pseudoachromatic number of graphs, *Southeast Asian Bull. Math.* 35(3) (2011), 431–438.

[15] S. Niskanen and P. Östergård, Cliquer, http://users.tkk.fi/pat/cliquer.html, Accessed: 8/5/2013.

[16] V. Yegnanarayanan, The pseudoachromatic number of a graph, *Southeast Asian Bull. Math.* 24(1) (2000), 129–136.

(Received 23 Dec 2013; revised 17 June 2014)