

On radial colorings of annuli

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Abstract

We consider the chromatic numbers of unit-distance graphs of various annuli. In particular, we consider radial colorings, which are “nice” colorings, and completely determine the radial chromatic numbers of various annuli.

1 Introduction

The *Chromatic Number of the Plane*, denoted by $\chi(\mathbb{R}^2)$, is the least integer N such that the points of the plane can be colored in N colors such that no two points exactly unit distance apart are the same color. Well-known elementary arguments show that $4 \leq \chi(\mathbb{R}^2) \leq 7$, but no improvement on these bounds appears to be forthcoming.

The difficulty of improving these bounds has led several authors to consider various modifications to the problem. For instance, one might restrict the type of coloring. In [3], Falconer shows that if the color classes must be measurable, then at least five colors are required. (See [5], Ch. 9, for a nice exposition.) In [6], Townsend announces the result that every “map-coloring” of the plane requires at least six colors, and provides a proof in [7]. In a basically identical result couched in different language, Coulson [2] considers colorings in which the color classes are composed of “tiles”, and shows that six colors are required.

Another approach is to consider colorings of proper subsets of the plane. In [1], Axenovich et al. consider but do not determine the chromatic number of $\mathbb{Q} \times \mathbb{R}$. They also determine the chromatic number of infinite strips of different widths, and

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of the union of two parallel lines. In [4], Kruskal considers bounded simply connected subsets of the plane. In particular, he shows that for a closed disk of radius r ,

- the disk is 2-colorable if and only if $r \leq \frac{1}{2}$;
- the disk is 3-colorable if and only if $r \leq \frac{1}{\sqrt{3}}$;
- the disk is 4-colorable if $r \leq \frac{1}{\sqrt{2}}$.

In the present paper, we consider the annulus with inner radius $\frac{1}{2} - r$ and outer radius $\frac{1}{2} + r$, where $0 < r < \frac{1}{2}$, and prove some surprising results in the case that the colorings are required to be “nice.”

2 Arbitrary colorings of the annulus.

Let $0 < r < \frac{1}{2}$, and let $A_r = \{p \in \mathbb{R}^2 : \frac{1}{2} - r \leq \|p\| \leq \frac{1}{2} + r\}$. We recall two lemmas from Kruskal’s paper [4].

Lemma 1. A *rod* is a line segment of unit length. If a region R is 2-colored, and we slide a rod continuously so that its endpoints stay within the interior of R , then the set of points passed over by a given endpoint is monochromatic.

Lemma 2. A *tri-rod* is an equilateral triangle of unit side length. If a region R is 3-colored, and we slide a tri-rod continuously so that its vertices stay within the interior of R , then the set of points passed over by a given vertex is monochromatic.

If $r > 0$, then a rod can be placed with endpoints interior to A_r . By rotating the rod by 180° about its center, we see that if A_r were 2-colored, then by Lemma 1 the two endpoints would have to be colored the same color. But the endpoints are unit distance apart, so A_r is not 2-colorable. See Figure 1. An alternate method of proof is to embed an odd cycle. See Figure 2.

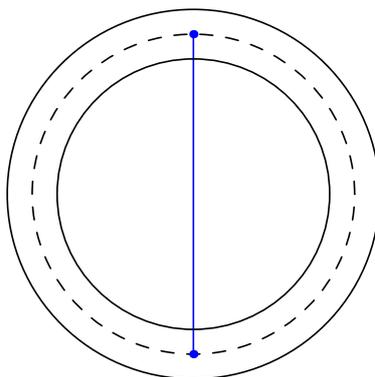


Figure 1: An embedding of a rod

If $0 < r \leq \frac{2-\sqrt{3}}{2\sqrt{3}}$, then A_r is 3-colorable. Figure 3 shows a proper 3-coloring of A_r with $r = \frac{2-\sqrt{3}}{2\sqrt{3}}$ (so that the outer radius of A_r is $\frac{1}{\sqrt{3}}$). Each boundary between sectors is colored the same color as the sector clockwise from it.

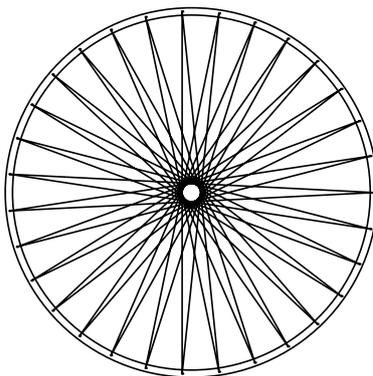


Figure 2: An embedding of an odd cycle

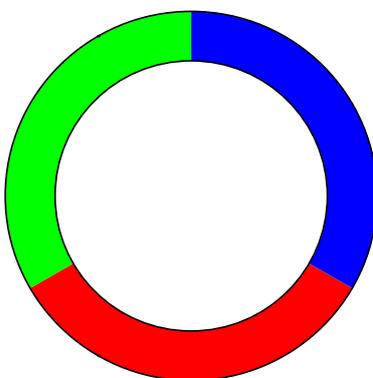


Figure 3: A proper 3-coloring for $r = \frac{2-\sqrt{3}}{2\sqrt{3}}$

If $r > \frac{2-\sqrt{3}}{2\sqrt{3}}$, then the outer radius of A_r is greater than $\frac{1}{\sqrt{3}}$, so a tri-rod can be placed such that its endpoints lie in the interior of A_r . Thus by rotating 120° and applying Lemma 2, we see that A_r is not 3-colorable. See Figure 4.

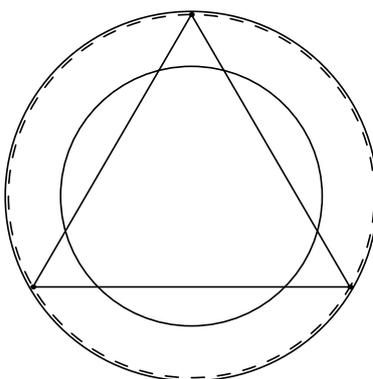


Figure 4: An embedding of a tri-rod

Figure 5 shows a proper 4-coloring of A_r for $r = \frac{2-\sqrt{2}}{2\sqrt{2}}$ (so that the outer radius of A_r is $\frac{1}{\sqrt{2}}$). Again, the boundaries between sectors are colored the same color as the adjacent sector in the clockwise direction.

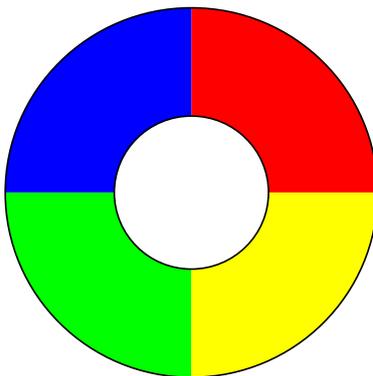


Figure 5: A proper 4-coloring for $r = \frac{2-\sqrt{2}}{2\sqrt{2}}$

Kruskal leaves open the question of whether a closed disk of radius greater than $\frac{1}{\sqrt{2}}$ can be properly 4-colored. One might think that an annulus with outer radius $\frac{1}{\sqrt{2}} + \varepsilon$ could be properly 4-colored—after all, it has a hole in the middle—but the present authors have been unable to construct such a coloring. The nature of the difficulty will be illuminated in the next section.

3 Radial Colorings

Definition 3. Let $0 < r < \frac{1}{2}$. A coloring of A_r is called *radial* if there exists a sequence of radii $r_1, r_2, \dots, r_n = r_1$ such that the sector strictly between r_i and r_{i+1} is colored with a single color. Let $\chi^{\text{radial}}(A_r)$ denote the least number of colors needed to give a proper coloring of A_r using radial colorings only.

The radial colorings are the “nice” colorings mentioned at the end of Section 1. We emphasize that the boundaries between sectors can be colored arbitrarily, while the sectors between radial boundaries must be monochromatic. See Figure 6.

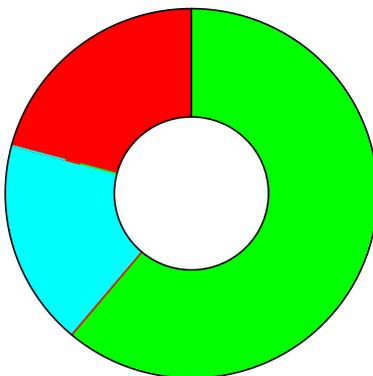


Figure 6: A radial coloring

Definition 4. A *unit chord* in A_r is a chord of unit length. A *unit sector* is a sector of minimal size that contains a unit chord. See Figure 7.

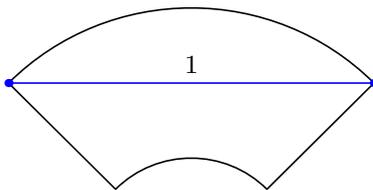


Figure 7: A unit chord contained within a unit sector

The following lemma will be powerful enough to allow us to determine $\chi^{\text{radial}}(A_r)$ for all $0 < r < \frac{1}{2}$.

Lemma 5. Let $0 < r < \frac{1}{2}$ and let A_r be given a proper radial coloring. Then the interior of any color class is included in some unit sector.

Proof. Call the intersection of a radial ray with A_r a *radial segment*. See Figure 8.

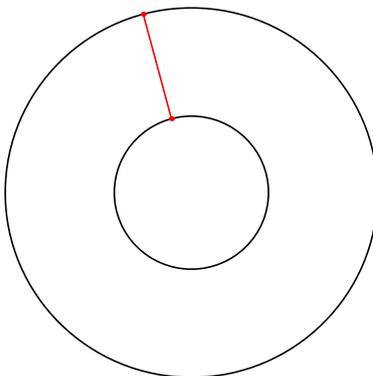


Figure 8: A radial segment

By way of contradiction, suppose there exist two radial segments of the same color (red, say) such that the chord joining their farthest-apart points has length greater than 1 (Figure 9).

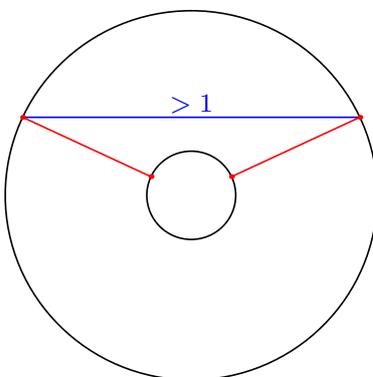


Figure 9: Two radial segments

Call one of the segments S_1 (Figure 10), and call the point at which it intersects the outer circle P . Call the other segment S_2 . Let A and B be the two points that are a unit chord away from P (Figure 11).

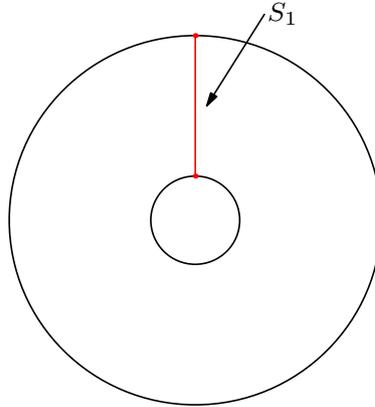


Figure 10: The segment S_1

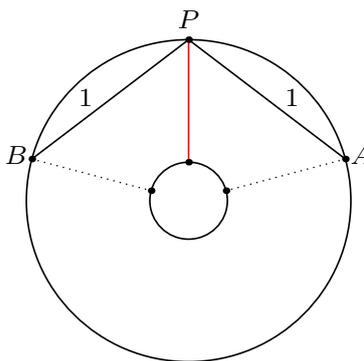


Figure 11: Points A , B , and P

Let P' be antipodal to P , and draw a segment from A to P' (Figure 12).

Now take the unit-length rod that goes from P to A , and slide it continuously so that one endpoint stays on S_1 and the other stays on AP' (Figure 13).

Thus every radial segment between A and P' has a point on it that is unit distance from some point of S_1 . (Similarly for those between P' and B .) But this is a contradiction, since some point on S_2 must be on AP' or BP' . \square

Theorem 6. Let $0 < r < \frac{1}{2}$. Then $\chi^{\text{radial}}(A_r)$ is the least integer N such that one can “walk around” the outside circumference using N unit-chord steps, i.e., $N = \lceil \frac{2\pi}{\theta} \rceil$, where $\theta = \arccos \left(1 - \frac{1}{2(\frac{1}{2} + r)^2} \right)$.

Proof. That $\chi^{\text{radial}}(A_r) \geq N$ follows immediately from the previous lemma. It is easy to color A_r in N colors using $N - 1$ unit sectors as different colors and one “leftover” sector in the N^{th} color. \square

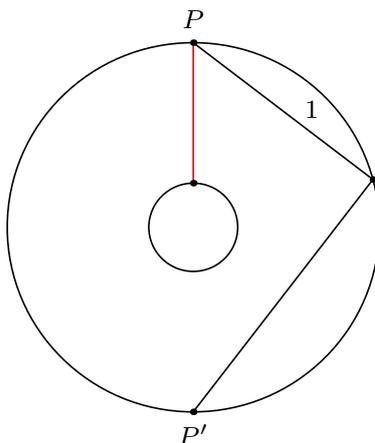


Figure 12: The segment from A to P'

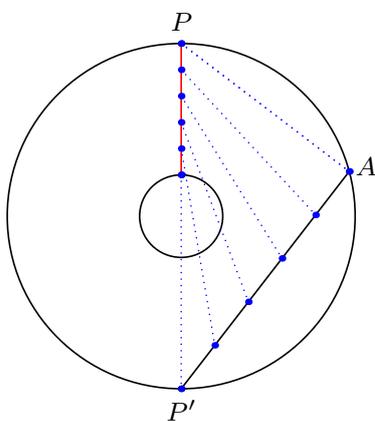


Figure 13: Sliding the segment AP

Below is a table of values for $\chi^{\text{radial}}(A_r)$.

We note that if we let $r = 1/2$, then we get a disk of radius 1, which can be given a proper radial coloring by coloring the 6 unit sectors in distinct colors and the center of the disk in the 7th color.

4 Further Directions

It may be interesting to consider other connected but not simply connected regions of the plane. For instance, consider the well-known periodic hexagonal coloring of the plane in 7 colors. Take one color class, enlarge it slightly and delete it, leaving a “holey plane.” Certainly, this holey plane can be colored in 6 colors. Can it be colored in 5? Townsend’s result on colorings using tiles would seem not to apply.

Bounded and not simply connected subsets may also be interesting to consider, although it would seem that their analysis promises little additional insight into the larger problem of understanding plane colorings and determining the chromatic

$0 < r \leq \frac{2-\sqrt{3}}{2\sqrt{3}}$	$\chi^{\text{radial}} = 3$
$\frac{2-\sqrt{3}}{2\sqrt{3}} < r \leq \frac{2-\sqrt{2}}{2\sqrt{2}}$	$\chi^{\text{radial}} = 4$
$\frac{2-\sqrt{2}}{2\sqrt{2}} < r \leq \frac{-1}{2} + \sqrt{\frac{2}{5-\sqrt{5}}}$	$\chi^{\text{radial}} = 5$
$\frac{-1}{2} + \sqrt{\frac{2}{5-\sqrt{5}}} < r < \frac{1}{2}$	$\chi^{\text{radial}} = 6$

Table 1: A table of values for $\chi^{\text{radial}}(A_r)$

number of the plane. After all, the authors were not able to find colorings of A_r that improve Kruskal’s corresponding bounds for circular regions; the “donut hole” did not seem to be useful for arbitrary colorings.

A more promising approach might be to try to prove (or find a counterexample for) a Moser-spindle version of Kruskal’s tri-rod lemma. If $r > \frac{3}{\sqrt{11}} - \frac{1}{2}$, then a spindle can be embedded into the interior of A_r (see Figure 14). If a tri-rod can be embedded, then A_r is not 3-colorable. Perhaps if a spindle can be embedded, A_r is not 4-colorable.

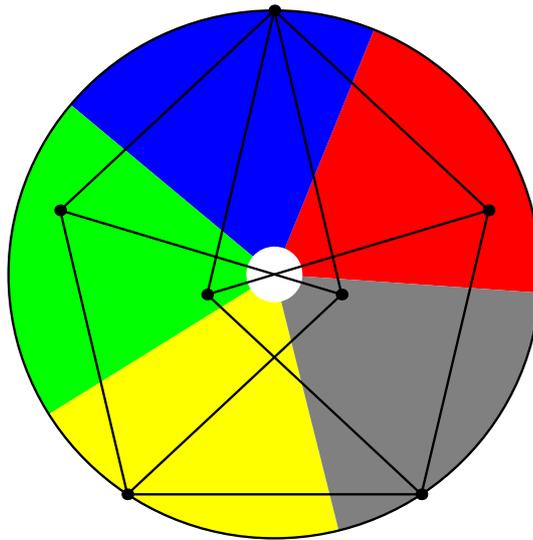


Figure 14: The Moser spindle can be embedded into A_r if $r > \frac{3}{\sqrt{11}} - \frac{1}{2}$. A radial coloring of A_r requires 5 colors in this case.

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(Received 14 Oct 2013; revised 28 Feb 2014, 12 Aug 2014)