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A sharp refinement of a result of Alon, Ben-Shimon and Krivelevich on bipartite graph vertex sequences

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Abstract

We give a sharp refinement of a result of Alon, Ben-Shimon and Krivelevich. This gives a sufficient condition for a finite sequence of positive integers to be the vertex degree list of both parts of a bipartite graph. The condition depends only on the length of the sequence and its largest and smallest elements.

1 Introduction

Recall that a finite sequence $\underline{d} = (d_1, \dots, d_n)$ of positive integers is *graphic* if there is a simple graph with n vertices having $\underline{d}$ as its list of vertex degrees. A pair $(\underline{d}_1, \underline{d}_2)$ of sequences (possibly of different length) is *bipartite graphic* if there is a simple, bipartite graph whose parts have $\underline{d}_1, \underline{d}_2$ as their respective lists of vertex degrees. We say that a sequence $\underline{d}$ is *bipartite graphic* if the pair $(\underline{d}, \underline{d})$ is bipartite graphic; that is, if there is a simple, bipartite graph whose two parts each have $\underline{d}$ as their list of vertex degrees. The classic Erdős–Gallai Theorem gives a necessary and sufficient condition for a sequence to be graphic. Similarly, the Gale–Ryser Theorem [5, 7] gives a necessary and sufficient condition for a pair of sequences to be bipartite graphic. In particular, the Gale–Ryser Theorem gives a necessary and sufficient condition for a single sequence to be bipartite graphic. Further results on bipartite graphic sequences are given in [3, 6].

In [8, Theorem 6], Zverovich and Zverovich gave a sufficient condition for a sequence to be graphic, depending only on the length of the sequence and its largest and smallest elements. A sharp refinement of this result is given in [4]. In [1, Corollary 2.2], Alon, Ben-Shimon and Krivelevich gave a result for bipartite graphic sequences, which is directly analogous to the theorem of Zverovich–Zverovich. The purpose of the present paper is to give a sharp refinement of the Alon–Ben-Shimon–Krivelevich result.

Here is the Alon–Ben-Shimon–Krivelevich result:

Theorem 1 ([1, Corollary 2.2]). *Suppose that $\underline{d}$ is a finite sequence of positive integers having length n, maximum element a and minimum element b. If for a real number $x \geq 1$, we have*

$$a \leq \min\left\{xb, \frac{4xn}{(x+1)^2}\right\}, \tag{1}$$

then $\underline{d}$ is bipartite graphic.

As we will explain at the end of this introduction, Theorem 1 can be rephrased in the following equivalent form:

Theorem 2. *Suppose that $\underline{d}$ is a finite sequence of positive integers having length n, maximum element a and minimum element b. Then $\underline{d}$ is bipartite graphic if*

$$nb \geq \frac{(a+b)^2}{4}. \tag{2}$$

The main aim of this paper is to prove the following result.

Theorem 3. *Suppose that $\underline{d}$ is a finite sequence of positive integers having length n, maximum element a and minimum element b. Then $\underline{d}$ is bipartite graphic if*

$$nb \geq \left\lfloor \frac{(a+b)^2}{4} \right\rfloor, \tag{3}$$

where $\lfloor . \rfloor$ denotes the integer part. Moreover, for any triple (a, b, n) of positive integers with $b < a \leq n$ that fails (3), there is a non-bipartite-graphic sequence of length n with maximal element a and minimal element b.

Let us contrast the above result with the sharp result for graphic sequences given in [4]. We will require this result later in Section 5.

Theorem 4 ([4]). *Suppose that $\underline{d}$ is a finite sequence of positive integers with even sum having length n, maximum element a and minimum element b. Then $\underline{d}$ is graphic if*

$$nb \geq \begin{cases} \left\lfloor \frac{(a+b+1)^2}{4} \right\rfloor - 1 & : \text{ if } b \text{ is odd, or } a+b \equiv 1 \pmod 4, \\ \left\lfloor \frac{(a+b+1)^2}{4} \right\rfloor & : \text{ otherwise.} \end{cases} \tag{4}$$

Moreover, for any triple (a, b, n) of positive integers with $b < a < n$ that fails (4), there is a non-graphic sequence of length n having even sum with maximal element a and minimal element b.

We give two proofs of Theorem 3. The first proof is in the spirit of the original paper of Zverovich and Zverovich, and uses the notion of *strong indices*. The preparatory results for this proof, notably Theorem 7 and Lemma 2, may be of independent interest. Our second proof is much shorter, and uses the sharp version of Zverovich–Zverovich from [4] and recent results relating bipartite graphic sequences to the degree sequences of graphs having at most one loop at each vertex [3].

The paper is organised as follows. Section 2 gives a necessary and sufficient condition for a sequence of the form (a^s, b^{n-s}) to be bipartite graphic. Here and throughout the paper, the superscripts indicate the number of repetitions of the element. So, for example, the sequence $(5, 5, 5, 4, 4)$ is denoted $(5^3, 4^2)$. In Section 2 we also prove Theorem 3 for sequences of the form (a^s, b^{n-s}), and we give examples showing that Theorem 3 is sharp. Section 3 presents results about bipartite graphic sequences, which are used in the first proof of Theorem 3 found in Section 4. Section 5 presents the second proof of Theorem 3.

To complete this introduction, let us establish the equivalence of Theorems 1 and 2. If $nb \geq \frac{(a+b)^2}{4}$, then setting $x = \frac{a}{b}$, we have that (1) holds. Thus Theorem 2 follows from Theorem 1. Conversely, fix a, b, n and note that the hypothesis of Theorem 1 is that $a \leq xb$ and $a \leq \frac{4xn}{(x+1)^2}$. Observe that $\frac{4xn}{(x+1)^2}$ is a monotonic decreasing function of x for $x \geq 1$. So if $a \leq \frac{4xn}{(x+1)^2}$ holds for some $x \geq \frac{a}{b}$, then $a \leq \frac{4xn}{(x+1)^2}$ holds for $x = \frac{a}{b}$, in which case (2) holds. Hence Theorem 1 follows from Theorem 2.

2 Two-element sequences

We consider two-element sequences; that is, sequences of the form (a^s, b^{n-s}) with $a, b \in \mathbb{N}$ where $\mathbb{N}$ is the set of positive integers.

Theorem 5. *Let $a, b, n, s \in \mathbb{N}$ with $b < a \leq n$ and $s \leq n$. Then the sequence (a^s, b^{n-s}) is bipartite graphic if and only if $s^2 - (a + b)s + nb \geq 0$.*

Proof. We will employ [8, Theorem 8], from which we have in particular: a two-element sequence $\underline{d} = (a^s, b^{n-s})$ is bipartite graphic if and only if

$$\sum_{i=1}^{s}(a + in_{s-i}) \leq sn \qquad \text{and} \qquad \sum_{i=1}^{s}(a + in_{n-i}) + \sum_{i=s+1}^{n}(b + in_{n-i}) \leq n^2, \tag{5}$$

where n_j is the number of elements of $\underline{d}$ equal to j; that is,

$$n_j = \begin{cases} s & : \text{ if } j = a \\ n - s & : \text{ if } j = b \\ 0 & : \text{ otherwise.} \end{cases}$$

Notice that the second inequality in (5) is always satisfied. Indeed,

$$\sum_{i=1}^{s}(a+in_{n-i})+\sum_{i=s+1}^{n}(b+in_{n-i}) = as+(n-s)b+\sum_{j=0}^{n-1}(n-j)n_j$$
$$= s(a-b)+nb+(n-a)s+(n-b)(n-s)=n^2.$$

So, rewriting the first inequality in (5), we have that $\underline{d}=(a^s,b^{n-s})$ is bipartite graphic if and only if

$$\sum_{j=0}^{s-1}(s-j)n_j \le s(n-a). \qquad (6)$$

If $b<s\le a$, then $\sum_{j=0}^{s-1}(s-j)n_j=(s-b)(n-s)$ and hence

$$\sum_{j=0}^{s-1}(s-j)n_j \le s(n-a) \iff s^2-(a+b)s+nb\ge 0,$$

as required. It remains to consider the cases $s\le b$ and $a<s$. If $s\le b$, then

$$\sum_{j=0}^{s-1}(s-j)n_j = 0 \le s(n-a).$$

If $a<s$, then

$$\sum_{j=0}^{s-1}(s-j)n_j=(s-a)s+(s-b)(n-s)=s(n-a)-b(n-s)\le s(n-a).$$

The inequality $s^2-(a+b)s+nb\ge 0$ holds in both these cases. Indeed, the minimum of the function $f(s)=s^2-(a+b)s+nb$ occurs at $s=\frac{a+b}{2}$ so $f(s)$ is decreasing for $s\le b$, and increasing for $a<s$, and $f(a)=f(b)=(n-a)b\ge 0$. □

Example 1. First assume $a\equiv b \pmod 2$ and $4nb<(a+b)^2$. Then the sequence

$$(a^{\frac{a+b}{2}}, b^{\frac{2n-a-b}{2}})$$

is not bipartite graphic by Theorem 5. Now assume $a\not\equiv b \pmod 2$ and $4nb<(a+b)^2-1$. Then

$$(a^{\frac{a+b+1}{2}}, b^{\frac{2n-a-b-1}{2}})$$

is not bipartite graphic, again by Theorem 5. These examples show that the bound given in Theorem 3 is sharp.

Remark 1. Note that for two-element sequences, we can deduce Theorem 3 from Theorem 5. Indeed, suppose that $\underline{d}=(a^s,b^{n-s})$ and that

$$nb\ge\left\lfloor\frac{(a+b)^2}{4}\right\rfloor.$$

As we observed in the proof of Theorem 5, the minimum of the function $f(s) = s^2 - (a+b)s + nb$ occurs at $\frac{a+b}{2}$. If $a+b$ is even, then

$$f(s) \geq f\left(\frac{a+b}{2}\right) = nb - \frac{(a+b)^2}{4} = nb - \left\lfloor \frac{(a+b)^2}{4} \right\rfloor \geq 0,$$

and so $\underline{d}$ is bipartite graphic by Theorem 5. So we may suppose that $a+b$ is odd. Then as s is an integer,

$$f(s) \geq f\left(\frac{a+b-1}{2}\right) = nb - \frac{(a+b)^2-1}{4} = nb - \left\lfloor \frac{(a+b)^2}{4} \right\rfloor \geq 0.$$

Hence $\underline{d}$ is bipartite graphic by Theorem 5.

3 Strong indices

In this section, $\underline{d} = (d_1, \ldots, d_n)$ is a (not necessarily strictly) decreasing sequence of nonnegative integers and for each integer j, the number of elements in $\underline{d}$ equal to j is denoted n_j. As a particular case of [8, Theorem 7], one has the following.

Theorem 6 ([8]). *The sequence $\underline{d}$ is bipartite graphic if and only if $\sum_{i=1}^{k}(d_i + in_{k-i}) \leq kn$, for all indices k.*

Recall the following standard definition.

Definition 1. In the sequence $\underline{d}$, an index is said to be *strong* if $d_k \geq k$.

The following result improves Theorem 6.

Theorem 7. *The sequence $\underline{d}$ is bipartite graphic if and only if $\sum_{i=1}^{k}(d_i + in_{k-i}) \leq kn$, for all strong indices k.*

Proof. Necessity follows from Theorem 7 in [8]. To prove sufficiency, define

$$F_k = kn - \sum_{i=1}^{k}(d_i + in_{k-i}) = kn - \sum_{i=1}^{k} d_i - \sum_{i=0}^{k}(k-i)n_i.$$

Suppose that $F_k \geq 0$ for all strong indices k. We will show that $F_k \geq 0$ for all indices k. To do this, we show that the minimum value of F_k, for $k = 1, 2, \ldots, n$, is nonnegative, and to do this we look at the smallest k for which F_k assumes the minimum value. Thus it suffices to show that F_1 and F_n are nonnegative and $F_k \geq 0$ for all $k = 2, \ldots, n-1$ such that $F_{k-1} > F_k$ and $F_{k+1} \geq F_k$. We will make use of the following lemma. Define the function $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ as follows: $f(k) = \max\{p : d_p \geq k+1\}$, with the convention that $\max \varnothing = 0$.

Lemma 1. *For the sequence $\underline{d}$, suppose that $n \geq d_1$. For a given $k = 0, 1, \ldots, n$, denote $p = f(k)$. Then, in the above notation,*

(a) *if $k, p > 0$, then at least one of them is a strong index,*

(b) $\sum_{s=k+1}^{n} n_s = p$ *and* $\sum_{s=0}^{n} n_s = n$,

(c) $\sum_{s=k+1}^{n} sn_s = \sum_{i=1}^{p} d_i$ *and* $\sum_{s=0}^{n} sn_s = \sum_{i=1}^{n} d_i$,

(d) $F_k = \sum_{i=1}^{n} d_i - \sum_{i=1}^{k} d_i - \sum_{i=1}^{p} d_i + kp$. *In particular, if $f(p) = k$, then $F_k = F_p$.*

Proof. (a) Suppose k is not a strong index, so that $k > d_k$. As $p = f(k)$ is assumed to be positive we have $p \in \{1, \dots, n\}$ and moreover, $d_p \geq k + 1 > d_k$. So, as $\underline{d}$ is decreasing, $p < k$. Thus $d_p \geq k + 1 > p$ and so p is a strong index, as required.

(b) The left-hand side of the first equality equals $\#\{s : d_s \geq k + 1\} = p$ by definition. The second equality is obvious.

(c) For an arbitrary $s \geq 0$ we have $sn_s = \sum_{i:d_i=s} d_i$. It follows that $\sum_{s=k+1}^{n} sn_s = \sum_{s=k+1}^{n} \sum_{i:d_i=s} d_i = \sum_{i:d_i \geq k+1} d_i = \sum_{i=1}^{p} d_i$. This proves the first equality; the second equality is obvious.

(d) We have by (b) and (c):

$$\begin{aligned} F_k &= kn - \sum_{i=1}^{k} d_i - k \sum_{i=0}^{k} n_i + \sum_{i=0}^{k} in_i \\ &= k\Big(n - \sum_{i=0}^{k} n_i\Big) - \sum_{i=1}^{k} d_i + \sum_{i=0}^{n} in_i - \sum_{i=k+1}^{n} in_i \\ &= kp - \sum_{i=1}^{k} d_i + \sum_{i=1}^{n} d_i - \sum_{i=1}^{p} d_i, \end{aligned}$$

as required. If not only $f(k) = p$, but also $f(p) = k$, then $F_k = F_p$, as the latter expression for F_k is symmetric with respect to k and p. □

Continuing with the proof of the theorem, by Lemma 1(b),

$$F_{k+1} - F_k = n - d_{k+1} - \sum_{i=0}^{k} n_i = \sum_{i=k+1}^{n} n_i - d_{k+1} = f(k) - d_{k+1}. \tag{7}$$

Moreover, $F_n = n^2 - \sum_{i=1}^{n} d_i - n \sum_{i=0}^{n} n_i + \sum_{i=0}^{n} in_i = 0$ by Lemma 1(b, c) and $F_1 \geq 0$ by assumption, as $d_1 \geq 1$. By (7) and Lemma 1(b), the inequalities $F_{k-1} > F_k$ and $F_{k+1} \geq F_k$ give

$$\begin{aligned} &F_{k+1} - F_k = f(k) - d_{k+1} \geq 0, \\ &F_k - F_{k-1} = f(k-1) - d_k = f(k) + n_k - d_k < 0. \end{aligned}$$

That is,

$$d_{k+1} \leq f(k) < d_k - n_k. \tag{8}$$

Let k be a non-strong index for which (8) holds. Denote $p = f(k)$. If $p > 0$, then p is a strong index by Lemma 1(a), hence $F_p \geq 0$ by assumption. Moreover, by (8) we have $d_{k+1} \leq p$ and $d_k > p + n_k$ so $d_k \geq p + 1$ and $d_{k+1} < p + 1$. It follows that $k = \max\{s : d_s \geq p + 1\}$, so $f(p) = k$ by definition. Then, by Lemma 1(d), we have $F_k = F_p \geq 0$. So we may assume that $p = 0$. Then $d_{k+1} = 0$, by (8), and hence $d_j = 0$

for all $j > k$. Furthermore, as $f(k) = p = 0$, we have $\{s : d_s \geq k+1\} = \varnothing$, and so $n_i = 0$ for all $i > k$. So by (7), for every $j > k$ we have $F_j - F_{j-1} = \sum_{i=j}^{n} n_i - d_j = 0$. Thus $F_k = F_n$. As $F_n = 0$ from the above, we get $F_k = 0$, as required. □

In the next section, we will also need the following lemma, which is a variation of [4, Lemma 1].

Lemma 2. *Suppose that $\underline{d}$ has maximum element $a = d_1 \leq n$ and minimum element $b = d_n$. For every strong index $k > b$, we have*

$$\sum_{i=1}^{k}(d_i + in_{k-i}) \leq n(k-b) + K(a+b) - K^2,$$

where K is the largest strong index, $K = \max\{k : d_k \geq k\}$.

Proof. Let $k > b$ be a strong index. We have $\sum_{i=1}^{k} d_i \leq ka$. Furthermore, since $n_j = 0$ for $j < b$, we have

$$\sum_{i=1}^{k} in_{k-i} = \sum_{j=0}^{k-1}(k-j)n_j \leq (k-b)\sum_{j=0}^{k-1} n_j.$$

The sum $\sum_{j=0}^{k-1} n_j$ counts the number of elements of $\underline{d}$ strictly less than k, hence $\sum_{j=0}^{k-1} n_j \leq n - K$ as $d_K \geq K \geq k$. Hence

$$\sum_{i=1}^{k}(d_i + in_{k-i}) \leq ka + (k-b)(n-K). \tag{9}$$

As $a \geq d_K \geq K$, we have $a + 1 - K \geq 1$. Thus, using $k \leq K$, inequality (9) gives

$$\begin{aligned}\sum_{i=1}^{k}(d_i + in_{k-i}) \leq ka + (k-b)(n-K) &= kn + k(a-K) + bK - bn \\ &\leq kn + K(a-K) + bK - bn \\ &= n(k-b) + K(a+b) - K^2,\end{aligned}$$

as required. □

4 First Proof of Theorem 3

Let $\underline{d}$ be a sequence satisfying hypothesis (3) of Theorem 3. If $a \equiv b \pmod 2$, then the result follows from Theorem 2. So we may assume that a, b have different parity. Let k be a strong index and suppose first that $k > b$. By Lemma 2,

$$\sum_{i=1}^{k}(d_i + in_{k-i}) \leq n(k-b) + K(a+b) - K^2, \tag{10}$$

where K denotes the largest strong index. As a quadratic in K, the maximal value of $n(k-b)+K(a+b)-K^2$ is attained at $K=\frac{a+b\pm 1}{2}$ and

$$n(k-b)+\frac{(a+b\pm 1)}{2}(a+b)-\left(\frac{a+b\pm 1}{2}\right)^2 = n(k-b)+\frac{1}{4}(a+b)^2-\frac{1}{4}.$$

Hence, since $nb \geq \left\lfloor \frac{(a+b)^2}{4} \right\rfloor = \frac{(a+b)^2}{4}-\frac{1}{4}$, we have

$$n(k-b)+K(a+b)-K^2 \leq n(k-b)+\frac{1}{4}(a+b)^2-\frac{1}{4} \leq kn.$$

So by (10), we have $\sum_{i=1}^{k}(d_i+in_{k-i}) \leq kn$. On the other hand, if $k \leq b$, then $\underline{d}$ contains no elements less than k and hence

$$\sum_{i=1}^{k}(d_i+in_{k-i}) = \sum_{i=1}^{k} d_i \leq ka. \tag{11}$$

Note that $n \geq a$, since otherwise by (3), we would have $ab > nb \geq \frac{(a+b)^2-1}{4}$, and hence $(a-b)^2 < 1$, giving $a=b$, which is impossible as a,b have different parity. So (11) gives $\sum_{i=1}^{k}(d_i+in_{k-i}) \leq kn$ once again. Hence $\underline{d}$ is bipartite graphic by Theorem 7.

5 Second Proof of Theorem 3

Suppose we have a decreasing sequence $\underline{d}=(a,\ldots,b)$ of length n, and suppose it satisfies hypothesis (3) of Theorem 3. By Remark 1, we may assume that $\underline{d}$ has at least 3 distinct elements. Suppose that $n_a=s$; that is, $\underline{d}$ has precisely s elements equal to a. Now consider the sequence $\underline{d}'$ obtained from $\underline{d}$ by reducing the first s elements of $\underline{d}$ by 1. So $\underline{d}'$ has maximal element $a'=a-1$. Note that $\underline{d}$ has at least 3 distinct elements, hence the minimum element of $\underline{d}'$ is still b. Suppose for the moment that $\underline{d}'$ has even sum. We will show that $\underline{d}'$ is graphic. From (3), we have

$$nb \geq \begin{cases} \frac{(a+b)^2}{4} & : \text{ if } a \equiv b \pmod 2, \\ \left\lfloor \frac{(a+b)^2}{4} \right\rfloor & : \text{ otherwise,} \end{cases}$$

We will show that

$$nb \geq \begin{cases} \left\lfloor \frac{(a'+b+1)^2}{4} \right\rfloor - 1 & : \text{ if } b \text{ is odd, or } a'+b \equiv 1 \pmod 4, \\ \left\lfloor \frac{(a'+b+1)^2}{4} \right\rfloor & : \text{ otherwise,} \end{cases} \tag{12}$$

from which we can conclude that $\underline{d}'$ is graphic by Theorem 4. Consider two cases according to whether or not $a \equiv b \pmod 2$. If $a \equiv b \pmod 2$, then our hypothesis is $nb \geq \frac{(a+b)^2}{4}$, and hence

$$nb \geq \frac{(a'+b+1)^2}{4} = \left\lfloor \frac{(a'+b+1)^2}{4} \right\rfloor,$$

and so (12) holds. Similarly, if $a \not\equiv b \pmod 2$, then our hypothesis is $nb \geq \left\lfloor \frac{(a+b)^2}{4} \right\rfloor$, and hence

$$nb \geq \left\lfloor \frac{(a'+b+1)^2}{4} \right\rfloor,$$

and again (12) holds. Thus in either case, $\underline{d}'$ is graphic.

We now use a result of [3]. By a *graph-with-loops* we mean a graph, without multiple edges, in which there is at most one loop at each vertex. For a graph-with-loops, the *reduced degree* of a vertex is taken to be the number of edges incident to the vertex, with *loops counted once*. This differs from the usual definition of degree in which each loop contributes two to the degree. By [3, Corollary 1], a sequence $\underline{d}$ of positive integers is the sequence of reduced degrees of the vertices of a graph-with-loops if and only if $\underline{d}$ is bipartite graphic. In our case, $\underline{d}'$ is graphic. Take a realization of $\underline{d}'$ as the degree sequence of some graph G', and label the vertices of G' in the same order as $\underline{d}'$. Now add a loop to each of the first s nodes of G' and call the resulting graph-with-loops G. So the sequence of reduced degrees of G is $\underline{d}$. Thus by [3, Corollary 1], $\underline{d}$ is bipartite graphic.

It remains to deal with the case where $\underline{d}'$ has odd sum. Since $\underline{d}$ has at least 3 distinct elements, we can modify the above construction as follows: we take the sequence $\underline{d}''$ obtained from $\underline{d}$ by reducing the first $(s+1)$ elements of $\underline{d}$ by 1. Then $\underline{d}''$ has even sum, maximum element $a-1$ and minimum element b, and we proceed exactly as above, only adding $s+1$ loops.

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