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On graphs and codes associated to the sporadic simple groups HS and M_{22}

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Abstract

We provide a new construction of the strongly regular graphs associated with the two sporadic simple groups M_{22} and HS. Further, we give some new constructions of other known strongly regular graphs by taking the orbits of a certain subgroup of M_{22} on the planes of the Hermitian variety $\mathcal{H}(5,4)$. These geometric constructions can be used to produce cap codes with large parameters and automorphism groups containing M_{22} as a subgroup.

1 Introduction

The Higman-Sims group HS is a sporadic simple group of order 44,352,000 arising from the automorphism group of the so-called Higman-Sims graph, which is an undirected triangle-free graph with 100 vertices and 1100 edges where each vertex has valency 22, no neighbouring pair of vertices share a common neighbour and each non-neighbouring pair of vertices share six common neighbours; see [12, 13]. In other words, the Higman-Sims graph is a triangle-free strongly regular graph $\mathrm{srg}(100, 22, 0, 6)$. The uniqueness of a strongly regular graph with these parameters was proved by Gewirtz; see [11]. It should be remarked, however, that such a graph had been constructed earlier—and uniqueness was shown—by Mesner in his unpublished 1956 doctoral thesis; see [17, 19, 21, 22]. Further, an alternative new construction of the Highman-Sims graph can be found in [20].

The full automorphism group of the the Higman-Sims graph has order 88,704,000, and it turns out that HS is isomorphic to a subgroup of this automorphism group with index 2. Actually, the full automorphism group of the Higman-Sims graph is HS : 2.

The Higman-Sims graph can be constructed starting off with a Steiner system $S(3, 6, 22)$. Since every triplet of distinct points of an $S(3, 6, 22)$ determines exactly one block, a simple counting argument shows that there are exactly 77 blocks in an $S(3,6,22)$. Adjacent vertices are defined to be disjoint blocks. This graph is strongly regular; any vertex has 16 neighbors, any two adjacent vertices have no common neighbors, and any two non-adjacent vertices have four common neighbors. This graph has $\mathrm{M}_{22} : 2$ as its automorphism group. This graph is uniquely determined by its parameters; see [6, 7]. The Higman-Sims graph is then formed by appending the 22 points of $S(3,6,22)$ and a 100-th vertex ∞. The neighbors of ∞ are defined to be those 22 points. A point adjacent to a block is defined to be one that is included. Note that in the Higman-Sims graph the vertices at distance 2 from a vertex may be identified with the $\mathrm{srg}(77,16,0,4)$.

In this paper we provide a new description of both the Higman-Sims graph and the graph associated to M_{22}, exploring the geometry of the action of the absolutely irreducible representations of the groups $\mathrm{PSL}_2(11)$ and M_{22} as subgroups of $\mathrm{PSL}_{10}(2)$; see [1]. Our notation and terminology are standard; see for instance [23]. For a general account on design theory, Steiner systems and related topics see also [2, 3, 15].

2 The action of the groups $\mathrm{PSL}_2(11)$ and M_{22}

The group $\mathrm{PSL}_2(11)$ has an absolutely irreducible representation as a subgroup of $\mathrm{PSL}_{10}(2)$; see [1]. In this representation it fixes an elliptic quadric $Q^-(9,2)$. We assume that $Q^-(9,2)$ has equation

$$X_1X_2 + X_3X_4 + X_5X_6 + X_7X_8 + X_9X_{10} + X_1^2 + X_2^2 = 0,$$

therefore $\mathrm{PSL}_2(11)$ lies inside the group $\mathrm{P}\Omega_{10}^-(2)$. With the aid of MAGMA [4] we checked that the group $\mathrm{PSL}_2(11)$ has 11 point orbits of sizes 11, 11, 55, 55, 55, 55, 66, 110, 110, 165, 330 in $\mathrm{PG}(9,2)$. Two of the orbits of size 55, two of those of size 110 and the orbit of size 165 partition the point set of $Q^-(9,2)$. The two orbits of size 11 and one of the orbits of size 55 among those on $Q^-(9,2)$ are caps and their union gives rise to a 77–cap $\mathcal{O}$ which turns out to be complete in the projective space $\mathrm{PG}(9,2)$. Recall that a k-cap in a finite projective space is a set consisting of k points no three of which are collinear, and that a k-cap is said to be complete if it is not contained in a $(k+1)$-cap. The stabilizer of $\mathcal{O}$ in $\mathrm{PSL}_{10}(2)$ is isomorphic to $\mathrm{M}_{22} : 2$. From the ATLAS [1] we found out that the Mathieu group M_{22} can be generated by the following matrices:

$$\begin{pmatrix}
0&1&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0\\
0&0&1&1&1&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&1
\end{pmatrix};\quad
\begin{pmatrix}
0&0&1&0&0&0&0&0&0&0\\
0&1&1&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0\\
1&0&1&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&1\\
0&0&1&0&0&0&1&0&0&0\\
1&0&1&1&1&0&1&0&0&1\\
1&0&0&1&0&1&0&0&0&1
\end{pmatrix}.$$

This group turns out to have three orbits of sizes 77, 330 and 616 in its action on the points of $\mathrm{PG}(9,2)$.

3 Graphs associated to the group M_{22}

Denote by $\mathcal{O}$, O_1 and O_2 the three orbits of M_{22} of sizes 77, 330 and 616 on $\mathrm{PG}(9,2)$, respectively, as seen at the end of the previous section.

3.1 The unique $\mathrm{srg}(77,16,0,4)$

Define a graph $\mathcal{G}$ as follows. Vertices of the graph are the points of $\mathcal{O}$, with two vertices adjacent whenever the line joining them meets the longest orbit O_2. With the aid of MAGMA we found out that $\mathcal{G}$ has valence 16, it is triangle–free, and the number of vertices adjacent to two adjacent vertices is 4. It turns out that $\mathcal{G}$ is the unique $\mathrm{srg}(77,16,0,4)$ admitting $\mathrm{M}_{22} : 2$ as an automorphism group.

3.2 The Higman-Sims graph

The stabiliser of a point P of $\mathcal{O}$ in M_{22} has three orbits on $\mathcal{O}$ of sizes 1, 16 and 60. Points in the 16–orbit are adjacent to P. Notice that the crucial fact here is that $\mathcal{O}$ is a cap. As we already observed before, the group $\mathrm{PSL}_2(11)$ has two orbits of points of size 11 in $\mathrm{PG}(9,2)$, say L_1 and L_2, consisting of non–singular points with respect to the orthogonal polarity $\perp$ induced by $Q^-(9,2)$. It follows that for any point P in L_i, $i=1,2$, $P^\perp$ is a hyperplane of $\mathrm{PG}(9,2)$ intersecting $Q^-(9,2)$ in a parabolic quadric $Q(8,2)$. Hence a set W of 22 hyperplanes of $\mathrm{PG}(9,2)$ arises as the union of two orbits of the group $\mathrm{PSL}_2(11)$ of size 11, say X_1 and X_2. The set W turns out to be an orbit under the action of M_{22}. With the aid of MAGMA we checked that for any point P of $\mathcal{O}$ there are exactly 6 hyperplanes of W on P.

Define a graph $\mathcal{H}$ with three types of vertices as follows:

(i) a special vertex denoted by the symbol ∞;

(ii) the points of $\mathcal{O}$;

(iii) the hyperplanes of W.

Adjacency is defined over $\mathcal{H}$ as follows:

- the vertex ∞ is adjacent to all vertices of type (iii) and none of type (ii);
- a vertex P of type (ii) is adjacent to a vertex H of type (iii) if and only if $P \in H$;
- adjacency of vertices of type (ii) is inherited from that of the graph associated to the group M_{22} as seen in 3.1.

Theorem 1. *The graph $\mathcal{H}$ is a strongly regular graph* $\mathrm{srg}(100,22,0,6)$ *isomorphic to the Higman-Sims graph.*

Proof. We remark that in the Higman-Sims graph the vertices at distance 2 from a vertex may be identified with the srg$(77, 16, 0, 4)$ associated to M_{22}. All other parameters have been verified with the aid of MAGMA. □

3.3 The graph srg$(77, 60, 47, 45)$

Define a graph $\mathcal{G}_1$ as follows. Vertices of the graph are the points of $\mathcal{O}$, where two vertices are adjacent whenever the line joining them meets the orbit $\mathcal{O}_1$. The graph $\mathcal{G}_1$ is again a strongly regular graph with valence 60 and the other parameters 47 and 45, that is, $\mathcal{G}_1$ is a srg$(77, 60, 47, 45)$. In other words, $\mathcal{G}_1$ is the complement of G. Looking at the table of strongly regular graphs by Brouwer [5], it turns out that $\mathcal{G}_1$ is the strongly regular graph associated to the unique $3-(22, 6, 1)$ block design; see for instance [2].

3.4 The Hadamard design $\mathcal{H}_{11}$

Consider again the two orbits of hyperplanes X_1 and X_2 defined above. Every hyperplane of X_1 meets $Q^-(9, 2)$ in a parabolic quadric. It can be showed that a parabolic quadric arising from a hyperplane of X_i meets the parabolic quadrics arising from hyperplanes of X_j, with $i \neq j$, in either a hyperbolic quadric $Q^+(7, 2)$ or in a cone over a $Q(6, 2)$. More precisely, a parabolic quadric of X_i meets exactly five parabolic quadrics of X_2 in a cone. Also each pair of parabolic quadrics of X_i meets exactly two parabolic quadrics of X_j in a cone. In the end, we have constructed the $2-(11, 5, 2)$ biplane. This is the famous Hadamard design $\mathcal{H}_{11}$. The complementary design is a $2-(11, 6, 3)$ balanced incomplete block design.

3.5 More about the graph srg$(77, 16, 0, 4)$

As was already observed by Brouwer [7], the graph srg$(77, 16, 0, 4)$ associated to the group M_{22} is an object with Buekenhout-Tits diagram as in Figure 1; see also [8]. The vertices of type 1 represent the 77 points of the cap $\mathcal{O}$; those of type 2 represent the 2310 chords of $\mathcal{O}$ meeting the orbit $\mathcal{O}_1$; those of type 3 represent the 2310 4–sets $\{ z \mid p \sim z \sim q \}$, where z represents a vertex of the graph srg$(77, 16, 0, 4)$, for nonadjacent pairs (p, q), where $x \sim y$ indicates the existence of an edge between the vertices x and y; those of type 4 represent the 77 16–sets $\{ z \mid p \sim z \}$, where again z represents a vertex on the graph srg$(77, 16, 0, 4)$. Incidence is inclusion.

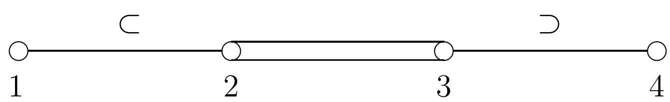

Figure 1.

3.6 Extending scalars

Embed PG$(9,2)$ in PG$(9,4)$ as a Baer subgeometry, and consider the action of the group M_{22} on points of PG$(9,4) \setminus$ PG$(9,2)$. It turns out that M_{22} has an orbit of size 1,232 that forms a cap. Such a cap is covered by extended lines of PG$(9,2)$ forming an orbit of size 616 that are secant to $\mathcal{O}$ and unisecant to $\mathcal{O}_2$. The 1,232-cap will be used in Section 5 to construct a very large even code.

4 Other constructions

From the ATLAS one sees that M_{22} has a 6-dimensional projective representation over $K = $ GF(4) in which M_{22} fixes a non–degenerate Hermitian form on the underlying vector space V. Let $\mathcal{H}(5,4)$ be the associated Hermitian variety of PG$(5,4)$ fixed by M_{22} with equation

$$X_0X_5^2 + X_1X_4^2 + X_2X_3^2 + X_3X_2^2 + X_4X_1^2 + X_5X_0^2 = 0.$$

It turns out that M_{22} is a subgroup of $PSU_6(4)$. With the following construction we intend to exploit this embedding.

Using MAGMA, we first construct the unique subgroup G of order 443,520 in $PSU_6(4)$, where necessarily $G \cong M_{22}$ as above. Our computations show that G has 4 orbits on the generators (totally singular planes) of $\mathcal{H}(5,4)$, of sizes 22, 77, 330 and 462. Planes in the orbit of size 77 are either disjoint or meet in a point. On the other hand, any two planes from the orbit of 22 generators meet in exactly one point. It is easy to check that the planes in this orbit form three 2-dimensional dual hyperovals embedded in $\mathcal{H}(5,4)$; see [16]. We recall that a family F of 2-dimensional subspaces of the finite 5-dimensional projective space PG$(5,4)$ is called a dual hyperoval if:

- every point of PG$(5,4)$ belongs to either 0 or 2 members in F;
- any two members of F have exactly one point in common;
- the set of points belonging to the members of F spans PG$(5,4)$.

In this setting we can define again both graphs associated to M_{22} and HS. Define a graph $\mathcal{G}$ as follows. Vertices of the graph are the planes in the 77-orbit with two vertices adjacent whenever two planes are disjoint. With the aid of MAGMA we checked that $\mathcal{G}$ is the graph srg$(77,16,0,4)$ associated to M_{22}. Joining the two orbits of size 77 and 22, and adding a new symbol ∞, we can construct the Higman-Sims graph $\mathcal{H}$ as follows. Define three types of vertices in $\mathcal{H}$:

(i) one special vertex denoted by the symbol ∞;

(ii) the planes of the 77–orbit;

(iii) the planes of the 22–orbit.

Define adjacency in $\mathcal{H}$ as follows:

- the vertex denoted by ∞ is adjacent to all vertices of type (iii);
- a vertex of type (ii) corresponding to a plane P is adjacent to a vertex of type (iii) corresponding to a plane Q if and only if P and Q meet in a line;
- adjacency of vertices of type (ii) is inherited from that of the graph associated to M_{22}.

4.1 Strongly regular graphs related to $\mathcal{H}(5,4)$

Note that among the subgroups of the group M_{22} fixing the Hermitian variety $\mathcal{H}(5,4)$ there is a group G isomorphic to $2:2:3:\mathrm{PSL}_3(4)$ which has an orbit $\mathcal{P}$ of length 105 and an orbit $\mathcal{P}'$ of length 120 in its action on the planes contained in $\mathcal{H}(5,4)$. Further, with the aid of MAGMA we checked that every two planes of $\mathcal{P}$ either meet at one point or are disjoint, and the same holds for the planes of $\mathcal{P}'$. This enables us to obtain an alternative simple geometric construction of four strongly regular graphs with a fairly large automorphism group; see [5].

Define a graph $\mathcal{G}_1$ whose vertices are the planes of $\mathcal{P}$ and two vertices $\pi, \pi' \in \mathcal{P}$ are adjacent whenever $|\pi \cap \pi'| = 1$. Then $\mathcal{G}_1$ turns out to be a strongly regular graph $\mathrm{srg}(105, 32, 4, 12)$.

Define a graph $\mathcal{G}_0$ whose vertices are the planes of $\mathcal{P}$ and two vertices $\pi, \pi' \in \mathcal{P}$ are adjacent whenever $\pi \cap \pi' = \emptyset$. Then $\mathcal{G}_0$ turns out to be a strongly regular graph $\mathrm{srg}(105, 72, 51, 45)$ which is the complement of $\mathcal{G}_1$.

Define a graph $\mathcal{G}_1'$ whose vertices are the planes of $\mathcal{P}'$ and two vertices $\pi, \pi' \in \mathcal{P}'$ are adjacent whenever $|\pi \cap \pi'| = 1$. Then $\mathcal{G}_1$ turns out to be a strongly regular graph $\mathrm{srg}(120, 77, 52, 44)$.

Define a graph $\mathcal{G}_0'$ whose vertices are the planes of $\mathcal{P}'$ and two vertices $\pi, \pi' \in \mathcal{P}'$ are adjacent whenever $\pi \cap \pi' = \emptyset$. Then $\mathcal{G}_0'$ turns out to be a strongly regular graph $\mathrm{srg}(120, 42, 8, 18)$ which is the complement of G_1'.

Uniqueness of the graph $\mathcal{G}_1$ was proved in [9], while uniqueness of the graph $\mathcal{G}_0'$ was proved in [10].

5 Related error correcting codes

Caps in projective spaces are closely related to a broad class of linear codes. If $\mathscr{K}$ is an n-cap in a projective space $\mathrm{PG}(r-1, q)$, then the coordinate vectors of the points of $\mathscr{K}$ are the columns of the parity check matrix H of an $[n, n-r, d]_q$ linear code C with minimum distance $d > 3$, that is, H is the generating matrix of an $[n, r, d']_q$ code $C^\perp$ which is the dual code of C; see [14, Chapter 14] for instance.

The cap $\mathcal{O} \subset \mathrm{PG}(9,2)$ arising from the action of the group $\mathrm{PSL}_2(11)$ on the elliptic quadric $Q^-(9,2)$, as described in Section 2, generates an even $[77, 10, 32]_2$ linear code with weight distribution

$$(0;1), (32;231), (40;770), (56;22),$$

whose dual is a $[77, 67, 4]_2$ linear code.

As it was pointed out in Section 2, at least one of the orbits of length 55 under the action of the group $PSL_2(11)$ on the elliptic quadric $Q^-(9,2)$ is a 55-cap in $PG(9,2)$. This generates a $[55,10,20]_2$ linear code with weight distribution

$$(0;1),(20;66),(24;220),(28;550),(32;165),(40;22),$$

whose dual is a $[55,45,4]_2$ linear code.

Taking one of the two 11-caps seen in Section 2 it is possible to generate an even $[11,10,2]_2$ linear code with weight distribution

$$(0;1),(2;55),(4;330),(6;462),(8;165),(10;11),$$

which is MDS, that is, with minimum distance d such that $d = n - k + 1$ (Singleton bound).

Joining the two orbits of length 11 seen in Section 2 we obtain a 22-cap in $PG(9,2)$ generating an even $[22,10,8]_2$ linear code with weight distribution

$$(0;1),(8;330),(12;616),(16;77),$$

whose dual is a $[22,12,6]_2$ linear code.

The 66-cap in $PG(9,2)$ obtained by joining the 55-cap seen in Section 2 with one of the two 11-caps generates a $[66,10,26]_2$ linear code whose weight distribution we omit due to the great number of different weights it has. Its dual is a $[66,56,4]_2$ linear code.

Finally, the 1,232-cap described in Setion 3.6 can be used to generate an even $[1{,}232,10,816]_4$ linear code with weight distribution

$$(0;1),(816;1{,}386),(832;693),(864;6{,}930),(904;36{,}960),(912;242{,}550),$$
$$(920;443{,}520),(936;110{,}880),(944;168{,}630),(960;36{,}960),(1{,}232;66),$$

whose dual is a $[1{,}232,1{,}222,4]_4$ linear code. It admits an automorphism group isomorphic to the group $2 : M_{22} : 3$.

The codes described in this section seem to be new, and admit an automoprhism group containing M_{22} as a subgroup. We remark that linear codes with large automorphism groups are considered interesting objects in their own right.

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