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On the edge-reconstruction number of a tree

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Abstract

The edge-reconstruction number $\mathrm{ern}(G)$ of a graph G is equal to the minimum number of edge-deleted subgraphs $G-e$ of G which are sufficient to determine G up to isomorphism. Building upon the work of Molina and using results from computer searches by Rivshin and more recent ones which we carried out, we show that, apart from three known exceptions, all bicentroidal trees have edge-reconstruction number equal to 2. We also exhibit the known trees having edge-reconstruction number equal to 3 and we conjecture that the three infinite families of unicentroidal trees which we have found to have edge-reconstruction number equal to 3 are the only ones.

1 Introduction

Trees have often been the test-bed for various graph theoretic conjectures, not least being the Reconstruction Conjecture. Kelly's proof that trees are reconstructible

[8] was the first substantial reconstructibility proof. This result was later improved by various authors who showed that trees can be reconstructed using only their endvertex- or peripheral-vertex- or cutvertex-deleted subgraphs [7, 3, 11].

All graphs considered will be finite, simple and undirected. A vertex-deleted subgraph $G - v$ of G is called a *card* of G; the collection of cards of G is called the *deck* of G, denoted by $\mathcal{D}(G)$. Our main focus in this paper will be on the analogously defined *edge-cards* of G which are the edge-deleted subgraphs $G - e$ of G; the collection of edge-cards of G is called the *edge-deck* of G and is denoted by $\mathcal{ED}(G)$.

In [6], Harary and Plantholt introduced the notion of reconstruction numbers. The *reconstruction number* $\mathrm{rn}(G)$ of a graph G is defined to be the least number of vertex-deleted subgraphs of G which alone reconstruct G uniquely, up to isomorphism. The *class reconstruction number* $\mathcal{C}\mathrm{rn}(G)$ is defined as follows. Let $\mathcal{C}$ be a class of graphs closed under isomorphism. Then the class reconstruction number of a graph G in $\mathcal{C}$ is the minimum number of vertex-deleted subgraphs of G which, together with the information that G is in $\mathcal{C}$, reconstruct G uniquely. It is clear that the reconstruction number of a graph is always at least 3 and that $\mathcal{C}\mathrm{rn}(G) \leq \mathrm{rn}(G)$. In fact, the class reconstruction number can even be 1, for example, when $\mathcal{C}$ is the class of regular graphs. The *edge-reconstruction number* $\mathrm{ern}(G)$ of a graph G and the *class edge-reconstruction number* $\mathcal{C}\mathrm{ern}(G)$ for a graph G in $\mathcal{C}$ are analogously defined.

In [4], Harary and Lauri tackled the reconstruction number of a tree. Let $\mathcal{T}$ be the class of trees. In their paper, Harary and Lauri tried to show that $\mathcal{T}\mathrm{rn}(T) \leq 2$. Although they managed to achieve this in many of the cases they considered, in some cases they had to settle for the upper bound of 3. So, what was accomplished in [4] was to show that $\mathcal{T}\mathrm{rn}(T) \leq 3$ and to make plausible their conjecture that, in fact, $\mathcal{T}\mathrm{rn}(T) \leq 2$ for all trees T. Myrvold [15] soon improved the first result by showing that $rn(T) \leq 3$. The conjecture $\mathcal{T}\mathrm{rn}(T) \leq 2$, however, still stood. A significant step forward was recently taken by Welhan [17] who proved that the class reconstruction number of trees is at most 2 for trees without vertices of degree 2.

The situation for the edge-reconstruction numbers of trees is less clear, somewhat surprisingly when compared with what happens in the Reconstruction Problem where edge-reconstruction is easier than vertex-reconstruction. Although Harary and Lauri conjectured that $\mathcal{T}\mathrm{rn}(T) \leq 2$ for all trees T, they presented in [4] a few trees with class edge-reconstruction number $\mathcal{T}\mathrm{ern}$ equal to 3 even though their class (vertex) reconstruction number was equal to 2. In [14], Molina started to tackle the edge-reconstruction number of trees. In summary, these are Molina's main results.

1. Let T be a unicentroidal tree with at least four edges, then $\mathrm{ern}(T) \leq 3$.
2. Let T be bicentroidal with centroidal vertices a and b, and let G and H be the two components of $T - ab$ with a in G and b in H. Then
   (a) If one of the centroidal vertices has degree equal to two, then $\mathrm{ern}(T) \leq 3$.

(b) If both centroidal vertices have degree at least three and if G or H has an irreplaceable endvertex (defined below), then $\mathrm{ern}(T) = 2$.

(c) If both centroidal vertices have degree at least three and if either G or H has no irreplaceable endvertex, then $\mathrm{ern}(T) \leq 3$.

In this paper we shall improve the above results on bicentroidal trees by showing that $\mathrm{ern}(T) = 2$ when the degrees of the centroidal vertices are 2 and even when both G and H have no irreplaceable vertices, giving our main result that all bicentroidal trees, with only three exceptions, have ern equal to 2. We shall also prove some results on unicentroidal trees and, based on these results and empirical evidence which we shall present, we give a conjecture stating which infinite classes of unicentroidal trees have ern equal to 3.

2 Main techniques

We shall here present the main techniques and supporting results used in this paper. Many of these were first used or proved in [4]. While all work on the reconstruction of trees prior to [4] depended on the centre of a tree, in [4] the centroid was used instead. Since then, all investigations of reconstruction numbers of trees except [17] depended heavily on centroids. Non-pseudosimilarity and irreplaceabilty of endvertices were also very important techniques first used in the proofs in [4]. These ideas will be explained below. We shall also present, in Section 2.2 a new idea, that of conjugate pairs of trees and a result (Theorem 2.4) characterising such pairs, which will be used for the first time in this paper. One technique employed at the end of Case 1 of Theorem 2.4 is the existence of a unique isomorphism between trees preserving the longest path. The application of this isomorphism is the mechanism by which all of the exceptional trees occurring in [17] are discovered.

2.1 The centre and the centroid of a tree

The *diameter* $\mathrm{diam}(G)$ of a connected graph G is the length of a longest path in G. The *eccentricity* of a vertex v in G is the longest distance from v to any other vertex in the graph. The *centre* of G is the set of vertices with minimum eccentricity. It is well-known that if G is a tree then the centre consists of either one vertex or two adjacent vertices.

We now turn our attention to the centroid. Define the *weight* of a vertex v of a tree T, denoted by $wt(v)$, to be the number of vertices in a largest component of $T - v$. For example all endvertices in an n-vertex tree have weight $n - 1$. The *centroid* of a tree T is the set of all vertices with minimum weight denoted by $wt(T)$. A *centroidal vertex* is a vertex in the centroid. It is well-known that the centroid of a tree consists of either one vertex or two adjacent vertices. A tree with one centroidal vertex is called *unicentroidal* while a tree with two centroidal vertices is called *bicentroidal*. In the latter case, the edge joining the centroidal vertices is called the *centroidal edge*.

When T is bicentroidal with centroidal edge e, the two components of $T - e$ are also said to be *centroidal components.*

The following simple observation will be very useful. The second part, especially, tells us that for a graph T which we know to be a tree, if it is bicentroidal, then one can determine from an edge-deleted subgraph $T - e$ of T alone, whether or not e is the centroidal edge of T and also, if e is the centroidal edge, the isomorphism types of the two centroidal components.

Observation 2.1 *Let T be a tree of order n and let v be a vertex of T. Then $wt(v) \leq \frac{n}{2}$ if and only if v is in the centroid of T. Also, T is bicentroidal with centroidal vertices a and b if and only if $T - ab$ has two components G, H each of order $\frac{n}{2}$.*

Notation. In the rest of the paper, a and b will denote the centroidal vertices of a bicentroidal tree with centroidal components G and H such that a is in G and b is in H.

A vertex of degree 1 is said to be an *endvertex.* A cutvertex in a tree which is adjacent to only one vertex of degree greater than 1 is said to be an *end-cutvertex.* An edge incident to an endvertex is called an *end-edge.*

2.2 Pseudosimilar vertices, irreplaceable edges and conjugate pairs of trees

Most of the works which we mentioned concerning reconstruction numbers of trees make heavy use of the impossibility of endvertices being pseudosimilar in a tree and of the fact that only one very special type of tree, called quasipaths, have the property that any end-edge can be exchanged with another giving us a tree isomorphic to the one which we started with. Since we shall be using these results even in this paper we shall explain them and their general use in this section. We shall also prove another result in this vein which we shall be needing, namely a result about a pair of trees such that each one can be obtained from the other by exchanging some end-edges in a particular way.

Let u and v be two vertices in a graph K such that an automorphism of K maps u into v. Then u and v are said to be *similar* in K. Now suppose that u and v are such that $K - u$ is isomorphic to $K - v$; we call such a pair of vertices *removal-similar.* If u and v are removal-similar in K but not similar, then u and v are said to be *pseudosimilar vertices* in K. The following results say that endvertices and end-cutvertices in a tree cannot be pseudosimilar.

Theorem 2.1 ***(Harary and Palmer)*** *[5] (i) Any two removal-similar endvertices in a tree are similar.*
(Kirkpatrick, Klawe and Corneil) *[9] (ii) Any two removal-similar end-cutvertices in a tree are similar.*

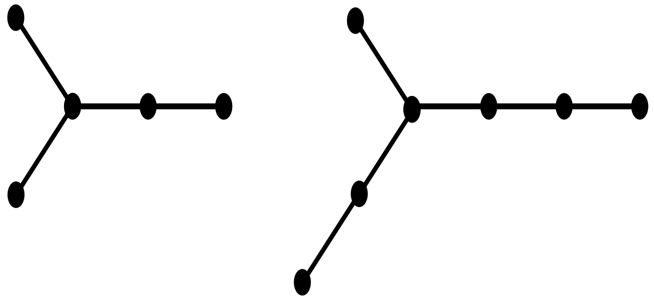

Figure 1: The trees: (a) S_1; and (b) S_2

Since we shall be expanding on this and the subsequent result in this paper it is interesting to see one way in which these two results have been extended by Krasikov in [10]. Let T be a tree and $a, b \in V(T)$, and let A, B be two rooted trees (a *rooted tree* is a tree which has one vertex designated as the root). Then $T_{a,b}(A, B)$ denotes the tree obtained by identifying the root of A with a and the root of B with b. Krasikov proved the following.

Theorem 2.2 *[10] If A and B are two non-isomorphic rooted trees and*

$$T_{a,b}(A, B) \simeq T_{a,b}(B, A)$$

then a and b are similar in T.

Clearly, if we take A to be the tree on two vertices and B a single vertex, then this result gives that endvertices cannot be pseudosimilar in a tree.

Now let $e = xv$ be an end-edge of T with $\deg(v) = 1$. Let $y \neq x$ be another vertex of T and let $T' = T - e + e'$, where $e' = yv$. If T' is isomorphic to T, then e is called a *replaceable end-edge* and v a *replaceable endvertex.* If there is no such vertex y then e and v are called, respectively, an *irreplaceable end-edge* and an *irreplaceable endvertex.* Let S_1 and S_2 be the graphs shown in Figure 1. A tree which is isomorphic either to a path P_k on k vertices or to one of S_1 or S_2 is said to be a *quasipath.*

The following theorem was proved in [4] and was also profitably used in [14].

Theorem 2.3 *Any tree which is not a quasipath has an irreplaceable end-edge.*

The use of non-pseudosimilarity of endvertices and irreplaceable edges are important techniques which are used in these two broad scenarios in this paper. First of all, suppose that we have two trees G, H and we know that a tree T is to be reconstructed by joining together with a new edge an endvertex a of G to another endvertex b of H (we do not know which vertices a and b are). Suppose, however, that we know the isomorphism types of both $G' = G - a$ and $H' = H - b$. Then, since endvertices in a tree cannot be pseudosimilar, we can pick any endvertex x in G such that $G - x \simeq G'$ and similarly any endvertex y in H such that $H - y \simeq H'$, and join the two vertices x and y giving the reconstruction of T which is unique up to isomorphism.

The second scenario is basically this. Suppose that we know again that the tree T to be reconstructed is obtained by joining vertex a in H to vertex b in G (a, b need not be endvertices now). We are also given the tree T' which is composed of H joined correctly to G', where G' is G less an endvertex and we can identify the edge ab in T'. We therefore know from T' the components H and G' and how they are connected. We just need to be able to put back the missing endvertex in G'. In order to have unique reconstruction up to isomorphism, non-pseudosimilarity of the missing endvertex is not enough here. We now require that the missing edge be irreplaceable in G.

In this paper, we shall also need a notion which is in some way an extension of the idea of replaceable endvertices. Instead of asking that exchanging an end-edge in a tree gives us the same tree, we ask that a pair of trees are related by a particular exchange of end-edges. This is quite a natural occurrence when considering reconstruction of trees. First we need a technical definition which, however, will find its natural place in our reconstruction result later in Theorem 3.2. One little issue of notation first: we shall often encounter a situation where we add a new end-edge e to a tree T. By this we mean that we are also adding a new endvertex w incident to e, that is, the new end-edge e is the edge $\{v, w\}$ where v is a vertex already in T. Also, when we delete an end-edge from a tree T we mean that we are also deleting the endvertex incident to the end-edge.

Suppose G and H are two *non-isomorphic trees*. Let a, b be endvertices of G and H, respectively. Suppose also that:

1. $G - a + e_1 \simeq H$ for some new end-edge e_1 added to $G - a$;
2. $G + aa' - e_2 \simeq H$ for some new endvertex a' added to G and some end-edge e_2 of G;
3. $H - b + e_3 \simeq G$ for some new end-edge e_3 added to $H - b$;
4. $H + bb' - e_4 \simeq G$ for some new endvertex b' added to H and some end-edge e_4 of H.

Then G and H are said to be a *conjugate pair* of trees.

The theorem we shall need is the following. We give two more definitions first. A *caterpillar* is a tree such that the removal of all of its endvertices results in a path. This path is called the *spine* of the caterpillar. A caterpillar whose spine is the path $v_1 v_2 \dots v_s$ and such that the vertex v_i is adjacent to a_i endvertices will be denoted by $C(a_1, \dots, a_s)$. Two examples are shown in Figure 2. A path on n vertices is denoted by P_n.

Finally, let P be a path in a tree and let v be an internal vertex on P. The *branch* at v relative to P is the component of $T - E(P)$ containing v and rooted at v.

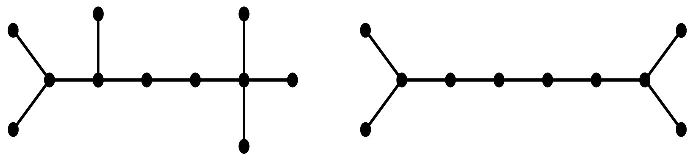

Figure 2: Caterpillars: (a) $C(2, 1, 0^2, 3)$; (b) $C(2, 0^4, 2)$

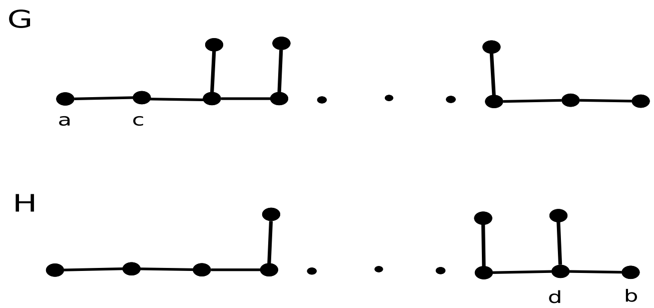

Figure 3: A conjugate pair of trees

Theorem 2.4 *Let G and H be a conjugate pair of trees as in the definition. Then G and H must be the caterpillars $C(1^n)$ and $C(1, 0, 1^{n-3}, 2)$ for some $n \geq 3$ as shown in Figure 3.*

Proof. Let c be the neighbour of a in G and d the neighbour of b in H, as in Figure 3. We shall first show that $\text{diam}(G) = \text{diam}(H)$. For, suppose not. By the symmetry of the four equations defining conjugate pairs of trees, we may assume, without loss of generality, that $\text{diam}(H) > \text{diam}(G)$. Therefore, from Equation 3, removing the endvertex b from H and adding the end-edge e_3 must reduce the diameter of H. Therefore b must be on a longest path of H. But then consider Equation 4. Adding the edge bb' increases the diameter of H, making it at least $2 + \text{diam}(G)$. But then, deleting the end-edge e_4 can, at most, bring down the diameter of $H + bb' - e_4$ to $1 + \text{diam}(G)$, contradicting Equation 4.

We shall now consider two cases:

Case 1: At least one of the trees on the left-hand-side of equations (1)–(4) in the definition of conjugate pairs (resulting from adding and deleting end-edges) has a different centre from the corresponding original tree, G or H.

Suppose that the graph $G - a + e_1$ (from Equation 1 of the definition) has a different centre from G. The arguments for Equation 2 are similar, and those for Equations 3 and 4 follow by symmetry. The bicentral case is similar. We shall first show that a is on the unique longest path of G. First of all, if a is not on a longest path then, in order for the centre of $G - a + e_1$ to be different from that of G, the end-edge e_1 must be appended to a longest path of G. But then, the graph $H \simeq G - a + e_1$ would have diameter larger than that of G, a contradiction. Therefore a must be on a longest path of G. Now, suppose that G has another longest path Q. If Q remains a longest path in $G - a + e_1$ than the centre of G has not moved. But for Q to be no

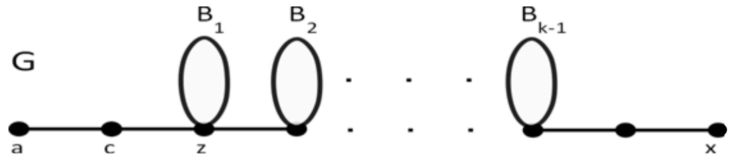

Figure 4: The tree G and the branches relative to P

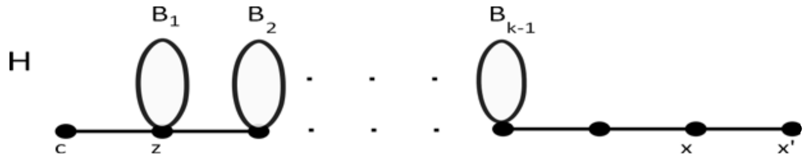

Figure 5: The tree H shown as $G - a + xx'$

longer a longest path in $G - a + e_1$, the end-edge e_1 must be appended to Q, again giving that the diameter of H is larger than diam(G). Therefore the longest path of G on which a lies is the only longest path of G.

Let us call this path P. Let x be the vertex at the other end of P. It also follows that $\deg(c) = 2$. If z is the other neighbour of c on P, then the branch at z relative to P must be at most a star; the branch at the next vertex z' cannot have a vertex distant more than 2 from z', and so on. Let the branches relative to P at the internal vertices of P be $B_1, B_2, \ldots, B_k$ in that order starting from the branch at z as shown in Figure 4. Since, by definition, G and H are not isomorphic, the branches cannot all be trivial (consisting of only the root vertex). However, note that B_k is trivial due to the unique longest path ending at x.

Now, consider H given as $G-a+xx'$ as depicted in Figure 5. Which would be the vertex b in H which satisfies Equation 4? Recall that $H + bb' - e_4$, being isomorphic to G, would need to have a unique path of maximum length and the branches of the internal vertices of this path must be $B_1, B_2, \ldots, B_k$ in that order (B_k is the branch meeting the vertex of P at distance 2 from x).

Therefore b cannot be x' nor any endvertex in any of $B_1, B_2, \ldots, B_k$ (except possibly the endvertex of B_1, but this vertex is anyway similar to c in H). Therefore b must be the vertex c and d must be the vertex z in Figure 5.

But, in order to satisfy Equation 3, the branch B_1 must be a single edge and the end-edge e_3 must be as shown in Figure 6. But then, comparing $H - b + e_3 \simeq G$ as in Figure 6 with G in Figure 4, the fact that the only possible isomorphisms must preserve the longest path shows that all the branches are single edges and G is as in Figure 3. This finishes Case 1.

Case 2: Every tree on the left-hand-side of equations (1)–(4) in the definition of conjugate pairs (resulting from adding and deleting end-edges) has the same centre as that in the original tree.

We shall prove that this leads to a contradiction, therefore only Case 1 can hold. We shall only consider the unicentral case. The bicentral case can be treated similarly.

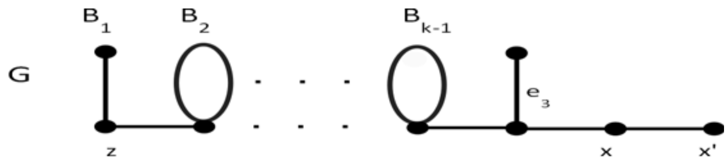

Figure 6: The graph G as $H - b + e_3 \simeq H - c + e_3$

First we need to define exactly what we mean by a central branch of a unicentral tree T with central vertex v. Let A be a component of $T - v$ and let u be the neighbour of v in A. Let $B = A + uv$. Then B will be called a *central branch* of T. Clearly, the number of central branches of T is equal to $\deg(v)$.

Consider first Equation 1: $G - a + e_1 \simeq H$. Let B_1 be the central branch of G containing the edge ac. We now have two sub-cases.

Case 2.1: The edge e_1 is incident to a vertex in B_1.

Therefore G and H have exactly the same collection of branches except that H has the branch $B_1' \simeq B_1 - a + e_1$ instead of B_1. So, the only way of obtaining G back from H in the way stipulated by Equations 3 and 4 is by changing B_1' to B_1 and, similarly, the only way of going from G to H, according to Equations 1 and 2, is by changing B_1 to B_1'. Therefore Equations 1 to 4 hold for the trees B_1 and B_1', that is, they form a conjugate pair of trees. Applying induction on the number of vertices gives us that B_1 and B_1' are as specified by the theorem, that is, as in Figure 3. Therefore G is the tree B_1 with extra branches joined to v (which is an endvertex in B_1). But then, G cannot satisfy Equations 1 to 4, that is, it cannot be a member of a conjugate pair of trees.

Case 2.2: The edge e_1 is not incident to a vertex in B_1.

Let B_2 be the central branch of G containing the vertex incident to of e_1. Therefore the branches of G and H are identical except that H has $B_1' = B_1 - a$ instead of B_1 and $B_2' = B_2 + e_1$ instead of B_2; G and H are as shown in Figure 7.

Now, the endvertex a of G which is in B_1 is also involved in Equation 2: $G + aa' - e_2 \simeq H$. Let us consider where the edge e_2 can come from so that H is isomorphic to both $G - a + e_1$ and $G + aa' - e_2$. We point out that we need to obtain the same collection of branches for H (with B_1' and B_2' instead of B_1 and B_2, respectively) and that we cannot do this by moving the centre. That is, we can only make modifications to the existing central branches.

The only way this can happen is if e_2 comes from some third central branch B_3. Let the orders of B_1, B_2 and B_3 be r, s, t, respectively. Then, a moment's consideration shows that we must have that $r = p + 1$, $s = p$ and $t = p + 2$, for some p.

Therefore, H is isomorphic to the tree shown in Figure 8. So we get, for example by considering orders, that $B_3 \simeq B_1 + aa'$ and $B_2 \simeq B_1 - a$. Switching over from

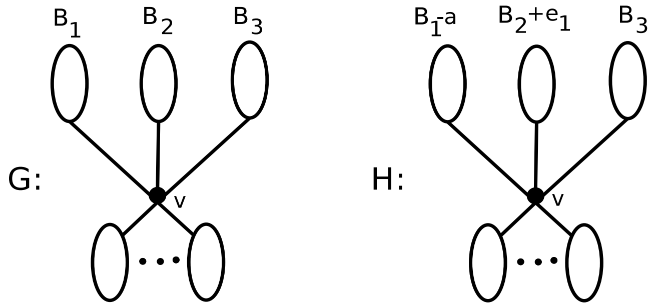

Figure 7: The graphs G and $H = G - a + e_1$

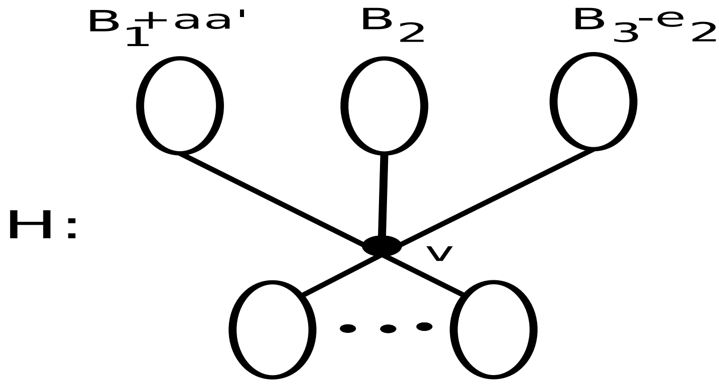

Figure 8: The graph $H = G + aa' - e_2$

G to H and from H to G using Equations 1 to 4 involves exchanging endvertices between these three branches (or three branches in G or H isomorphic to them).

So, when we are considering G in Equations 1 and 2, the vertex equivalent to a would be in that branch which has order $p+1$, the new edge e_1 would be attached to the branch of order p, and e_2 would be removed from the branch of order $p+2$. The argument is similar for the case where we are considering H with b, e_3 and e_4 as indicated in Equations 3 and 4. But we are always permuting between the same (up to isomorphism) three branches which become isomorphic to B_1, B_2, B_3 in G and B_1', B_2', B_3' in H. But this would force the two trees in Figure 9 to be isomorphic, therefore G and H would be isomorphic, a contradiction which completes our proof. □

Note that it is the fact that G and H are not isomorphic which forces conjugate pairs to be as described in the theorem and which gives us our final contradiction. If G and H are allowed to be isomorphic then, for example, two trees both isomorphic to the one shown in Figure 10 do satisfy Equations 1 to 4. Note, in this example, that the three central branches are as described in Case 2.2 of the above proof.

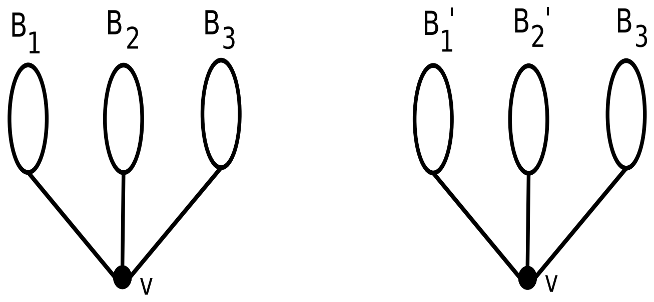

Figure 9: These two subtrees must be isomorphic

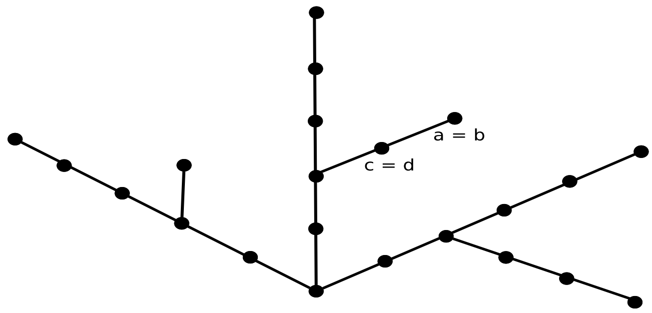

Figure 10: $G \simeq H$ would be a conjugate pair if allowed to be isomorphic

2.3 Recognising trees and Molina's Lemma

There is a simple but very useful result proved by Molina in [14] which often allows us to identify a graph as a tree from two given edge-cards. We reproduce its short proof for completeness' sake.

Lemma 2.1 *Let G be a graph with edges e_1 and e_2. Suppose that the edge-card $G-e_1$ has two components which are trees of orders p_1 and p_2 while the edge-card $G-e_2$ has another two components which are trees of orders q_1 and q_2. If $\{p_1,p_2\} \neq \{q_1,q_2\}$, then G is a tree.*

Proof. Suppose G is not a tree. Because of this, e_1 must join two vertices in the same component of $G-e_1$; call this component H. Note that $H+e_1$ contains a cycle. Therefore, to obtain the second edge-card with two trees as components an edge must be removed from the cycle in $H+e_1$. But this contradicts that $\{p_1,p_2\} \neq \{q_1,q_2\}$. □

3 Bicentroidal trees

We now come to our principal theorems on the edge-reconstruction number of bicentroidal trees. We consider two main cases.

3.1 The centroidal component G is not a quasipath and $\deg(b)$ is at least 3

Theorem 3.1 *Let T be a bicentroidal tree with bicentroidal edge ab, bicentroidal components G, H, $a \in V(G)$, $b \in V(H)$. Suppose $\deg(b) \geq 3$ and G is not a quasipath. Then $ern(T) = 2$.*

Proof. Since T is bicentroidal and ab is the centroidal edge then the two components G and H of the card $T - ab$ have the same number of vertices, namely $\frac{|V(T)|}{2}$. Let f be an irreplaceable end-edge of G (such an f exists since G is not a quasipath). We claim that T is reconstructible from $T - ab$ and $T - f$.

By Lemma 2.1 we can recognise from $T - ab$ and $T - f$ that the graph to be reconstructed is a tree. By Observation 2.1 one can therefore recognise from the edge-card $T - ab$ that the missing edge (which must join G to H) is the centroidal edge and also that G and H are the centroidal components of T.

Now, we would like to show that the centroidal edge is recognisable in the edge-card $T-f$. There is surely an edge e such that $(T-f)-e$ has non-trivial components $G - f$ and H, but we can uniquely identify e as the edge ab only if:

(i) there is only one edge e such that the non-trivial components of $(T - f) - e$ are isomorphic to H and some $G - f'$ for some $f' \in E(G)$;

and

(ii) there is no edge e' such that the non-trivial components of $(T - f) - e'$ is isomorphic to G and some $H - f'$ for some $f' \in E(H)$.

If both (i) and (ii) hold then we can distinguish the centroidal edge in $T - f$ and $G - f$ as well, and we can reconstruct uniquely by putting f back into $G - f$, since f is an irreplaceable end-edge (note that this proof also works if the end-edge f happens to be adjacent to the centroidal edge).

But conditions (i) and (ii) can fail to occur only if the degree of the centroidal vertex b is two. Since $\deg(b) > 2$ it follows that T is reconstructible from $T - ab$ and $T - f$. □

We shall come back to what happens when $\deg(b) = 2$ but G is still not a quasipath in Lemma 5.2 after having obtained some more results and discussed some special cases.

3.2 Both $\deg(a)$ and $\deg(b)$ equal 2 and neither G nor H is a quasipath

Theorem 3.2 *Let T be a bicentroidal tree with bicentroidal edge ab. Let $\deg(a) = \deg(b) = 2$ and suppose that neither of the two centroidal components G, H is a quasipath. Then $ern(T) = 2$.*

Proof. Without loss of generality, suppose that $b \in V(H)$; let $d \in V(H)$ be the other neighbour of b. We shall first try to show that T is reconstructible from $T - ab$ and $T - bd$ and we shall see where this can go wrong.

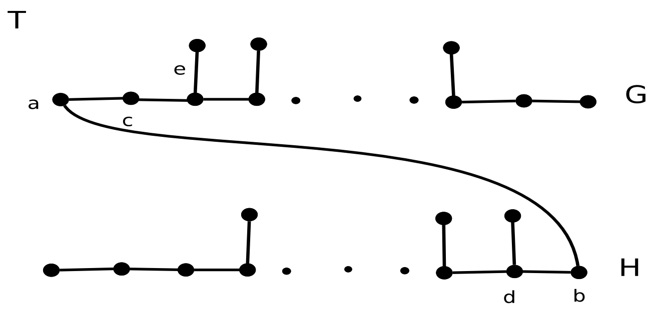

Figure 11: The tree T when the two centroidal components are a conjugate pair

As before, from the two given edge-cards we can recognise that T is a tree and that G, H are its centroidal components. Consider $T - bd$. If we can definitely tell that the larger component of $T - bd$ is G plus some edge as opposed to H plus some edge, then we would only need to decide which is the extra end-edge in the larger component. But, since endvertices cannot be pseudosimilar, we can choose any endvertex whose deletion gives G. We therefore know, up to isomorphism, which of the vertices of G is incident to the centroidal edge. Now we would need to do the same with H.

Let H' be the smaller component of $T - bd$. Having identified the larger component, we know that H' is H minus some edge. We look for any endvertex b' such that $H - b' \simeq H'$. Again, by non-pseudosimilarity of endvertices, any such choice is equivalent to b up to isomorphism. So we also know the vertex of H which is incident to the centroidal edge, hence T can be uniquely reconstructed.

This proof fails if we cannot tell whether the larger component is G plus an end-edge or H plus an end-edge. This ambiguity can only happen if $G \not\simeq H$ and $G + ab - \alpha \simeq H$ for some end-edge α of G and $H - b + \beta \simeq G$ for some new end-edge β.

Therefore let us assume that this is the case and let us proceed to reconstruct, this time from $T - ab$ and $T - ac$, where c is the other neighbour of a in G.

Reconstruction will proceed as above unless we cannot tell whether the larger component of $T - ac$ is G plus an end-edge or H plus an end-edge. But this ambiguity can only happen if $H + ab - \gamma \simeq G$ for some end-edge γ of H and $G - a + \delta \simeq H$ for some new end-edge δ.

But this means that G and H are conjugate pairs and, by Theorem 2.4, T is therefore as shown in Figure 11. But then it can be checked that T is reconstructible from $T - ab$ and $T - e$, where e is as shown in Figure 11. Therefore $ern(T) = 2$. □

4 Edge-reconstruction number 3: three infinite families

Molina, in [14] had stated that $\text{ern}(P_n) = 3$ if T is a path with four or more edges. We shall show that his statement is correct provided that n, the number of vertices, is odd, that is, P_n is unicentroidal. In the following theorem we shall show that

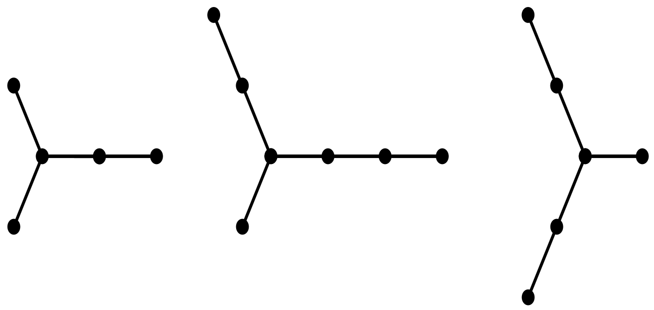

Figure 12: The trees: (a) $S_{1,1,2}(=S_1)$; (b) $S_{1,2,3}(=S_2)$; and (c) $S_{1,2,2}$

$\text{ern}(P_n) = 2$ when n is even. To show that $\text{ern}(P_n) = 3$, when n is odd, we need to show that, for each pair of cards in the edge-deck $\mathcal{ED}(P_n)$, there exists a graph $H \not\simeq P_n$ which has the same pair of edge-cards in its edge-deck, that is, H is a blocker for that particular pair of edge-cards. By the term "blocker" we mean the following: suppose we are considering $\text{rn}(G)$ or $\text{ern}(G)$ and suppose that a graph $H \not\simeq G$ has in its deck (edge-deck) the cards (edge-cards) $G - v_1, \ldots, G - v_k$ $(G - e_1, \ldots, G - e_k)$; we then say that H is a *blocker* for these cards (edge-cards) or that H *blocks* these cards (edge-cards).

We shall need the definition of one more special type of tree. A tree denoted by $S_{p,q,r}$ is similar to a star (that is, the tree on n vertices, $n-1$ of which are endvertices) and it consists of three paths on p, q and r edges, respectively, emerging from the central vertex. Some examples are shown in Figure 12. Note that the quasipaths S_1 and S_2 (shown in Figure 1) are $S_{1,1,2}$ and $S_{1,2,3}$, respectively.

Theorem 4.1 *If n is even, then $ern(P_n) = 2$ while if n is odd, then $ern(P_n) = 3$.*

Proof. Consider the graph P_n, n even. Let e_1 be the central edge of P_n and e_2 either of the two edges adjacent to e_1. We claim that the two edge-cards $C_1 = P_n - e_1 = P_{\frac{n}{2}} \cup P_{\frac{n}{2}}$ and $C_2 = P_n - e_2 = P_{\frac{n}{2}+1} \cup P_{\frac{n}{2}-1}$ reconstruct P_n.

By Molina's Lemma the graph to be reconstructed must be a tree. Consider the missing edge of $P_n - e_1$. This edge can be made incident to (i) two endvertices of $P_n - e_1$; or (ii) two vertices of degree two in $P_n - e_1$; or (iii) one endvertex and one vertex of degree two. Case (i) gives P_n, and Case (ii) is impossible because no other edge-card of the resulting tree can be equal to the union of two paths. Therefore we need only consider Case (iii).

Let w be the vertex of degree three incident to e_1 after this edge is put back into $P_n - e_1$ to get H. Then the second edge-card C_2 must be obtained by removing one of the other two edges incident to w. But this will always give a component P_k with $k > \frac{n}{2} + 1$, which is a contradiction. This proves our claim.

We now consider the odd path P_n for $n = 2s+1$. When two edge-cards are obtained by deleting the two edges incident to the central vertex, then, for all $n \geq 5$, a blocker would consist of the cycle C_s union the path P_{s+1}. We therefore consider any other pair of deleted edges. Let the edges of P_n be ordered as

$$e_1, e_2, \ldots, e_{n-1}.$$

Suppose we are given the two cards $P_n - e_i = P_i \cup P_{2s-i+1}$ and $P_n - e_j$, where $i \leq j$. (We can assume, by symmetry, that $j \leq s$. Also, we may assume that we do not have $i = j = s$, corresponding to $i = (n-1)/2$ and $j = (n+1)/2$ since we have already observed that the blocker then is $C_{s+1} \cup P_s$.) The blocker will then consist of $S_{p,q,r}$ where $p = i$, $q = j$ and $r = 2s - i - j$ and where the edge-cards come from the edges incident to the vertex of degree 3. Therefore $\mathrm{ern}(P_n) > 2$. But Molina has shown that for any tree T on at least four edges $\mathrm{ern}(T) \leq 3$, therefore $\mathrm{ern}(P_n) = 3$ when n is odd. □

We now show that a class of caterpillars also has ern = 3.

Theorem 4.2 *The caterpillars $C(2, 0, \ldots, 0, 2)$ of even diameter greater than 3 have edge-reconstruction number equal to 3.*

Proof. Let $C = C(2, 0, \ldots, 0, 2)$ have even diameter $d > 3$. Let $(v_0, \ldots, v_d)$ be a longest path of C. By the first result of Molina presented in the introduction, $\mathrm{ern}(C) \leq 3$, so we only have to prove that $\mathrm{ern}(C) > 2$. Thus, we have to prove that for every pair of edge-cards A and B of C (A and B might be isomorphic), there is a blocker, that is, a graph X, non-isomorphic to C, having two edge-cards isomorphic to edge-cards A and B, respectively.

Let F_i be the forest obtained by deleting edge $v_{i-1}v_i$, $i = 1, \ldots, d$. Note that, because of symmetry, we need only consider $F_1, \ldots, F_{d/2}$. For $d > 5$ we argue as follows:

- If the pair F_1, F_1 is chosen, we construct the graph X by adding to F_1 the edge v_0v_{d-1}. From X, we obtain F_1, by deleting v_0v_{d-1}, and also by deleting $v_{d-1}v_d$. Note that X is a tree.

- If the pair F_1, F_i is chosen, $2 \leq i \leq d/2$, we construct graph X by adding to F_1 the edge v_0v_{d-i-1}. From X, we obtain F_i by deleting $v_{d-i}v_{d-i+1}$. Also in this cases, X is a tree.

- If the pair F_j, F_i is chosen, $j = 2, \ldots, (d/2) - 1$, $j \leq i \leq d/2$, we construct X by adding to F_j the edge $v_{j-1}v_{d-i}$. From X we obtain F_i by deleting $v_{d-i}v_{d-i+1}$. Again, X is a tree.

- If the pair $F_{d/2}$, $F_{d/2}$ is chosen, construct X by adding to $F_{d/2}$ the edge $v_{(d/2)+1}v_d$. From X we obtain $F_{d/2}$ by deleting the edge $v_{d-1}v_d$. In this last case, X is not a tree, and it can be seen that there is no tree, non-isomorphic to C, having $F_{d/2}$ as two of its edge-cards.

For $d = 4$, we have the caterpillar $C(2, 0, 2)$ which, as we now show, has ern = 3. We give the same analysis as for $d > 5$ above.

- If the pair F_1, F_1 is chosen, we construct X by adding to F_1 the edge v_0v_3. From X, we obtain F_1, by deleting, of course, v_0v_3, and also by deleting v_3v_4. In this case X is a tree.

- If the pair F_1, F_2 is chosen, we construct X by adding to F_1 again the edge v_0v_3. From X, we obtain F_2 by deleting v_2v_3. The graph X is a tree in this case too.

- If the pair F_2, F_2 is chosen, we construct X by adding to F_2 the edge v_0x, where x is the other vertex of degree 1 in the same connected component as v_0. From X, we obtain F_2 both by deleting v_0x , and by deleting v_1x. In this case X is a not a forest.

□

Comment. The caterpillars $C(2, 0, \ldots, 0, 2)$ of odd diameter d all have ern $=$ 2. Indeed, it can be directly verified that (with the same notation as before) the pair $F_{(d-1)/2}$, $F_{(d+1)/2}$ is a pair of edge-cards which are not in the edge-deck of any other graph not isomorphic to $C(2, 0, \ldots, 0, 2)$. This observation and also Rivshin's computer search (see Section 5) show that $C(2, 0, 0, 0, 0, 2)$, which is not covered by our previous results, does indeed have ern $= 2$.

Finally, we note that the trees T_k ($k \geq 2$) in the infinite family shown in Figure 14 also has ern $= 3$. (Note that, when $k = 2$, T_k is the caterpillar $C(2, 0, 2)$ and, when $k = 3$, T_k is the graph G_1 shown in Figure 13(a).) Since there are only two types of edges up to isomorphism in T_k, it is easy to verify that $\mathrm{ern}(T_k) = 3$. For example, if e_1 and e_2 are two edges of T_k incident to the central vertex then $T_k - e_1$ and $T_k - e_2$ are isomorphic. The blocker having two copies of these graphs in its edge-deck is $T_{k-1} \cup R$, where R is a triangle. Therefore these two subgraphs do not reconstruct T_k.

5 Empirical Evidence

Empirical evidence, which was provided to us by David Rivshin [16], showed that out of more than a billion graphs on at most eleven vertices and at least four edges, only seventeen trees have edge-reconstruction number equal to 3. Four of these trees are paths of odd order which we have already considered in the previous section. Other trees are the graphs $S_{2,2,2}$ and $S_{3,3,3}$ which were already noticed by Harary and Lauri [4]. Nine other trees are the caterpillars $C(2, 2)$, $C(2, 0, 2)$, $C(1, 0, 1, 0, 1)$, $C(2, 1, 2)$, $C(2, 0^3, 2)$, $C(2, 3, 2)$, $C(2, 1, 1, 2)$, $C(1, 0, 1, 0, 1, 0, 1)$ and $C(2, 0^5, 2)$, while the remaining two trees are G_1 and G_2 shown in Figure 13.

One can notice that only three out of the seventeen trees are bicentroidal namely the two caterpillars $C(2, 2)$ and $C(2, 1, 1, 2)$, and the graph named G_2 in Figure 13. These trees do not contradict Theorems 3.1 and 3.2 since, in all three cases, the centroidal components are both quasipaths. These small examples show that the

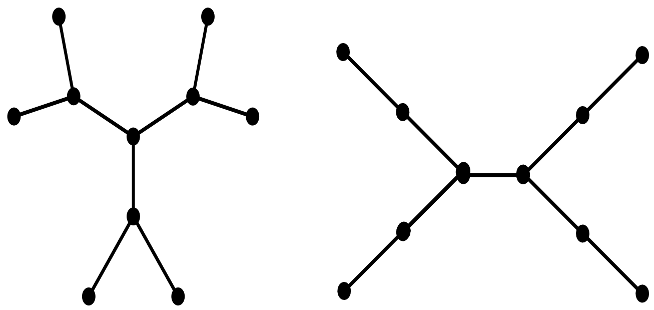

Figure 13: The graphs: (a) G_1; and (b) G_2

condition that at least one centroidal component is not a quasipath is required for ern to be equal to 2. Therefore the only bicentroidal trees T for which we have not determined their ern because they are not covered by Theorem 3.1 or Theorem 3.2 are: (i) those with both centroidal components equal to quasipaths; or (ii) those with one centroidal component being a quasipath and the centroidal vertex in the other component having degree 2. In the next section, we shall return to the arbitrarily large instances of these two cases, that is, when the quasipaths involved are paths. But for the smaller cases we now present our computer search which not only covers these cases but also gives empirical evidence for our later conjecture on unicentroidal trees.

Rivshin's computer analysis considered all graphs and went up to order 11. Here, by considering only trees, we extend the analysis up to order 25. Further computer search is feasible, but given that no new interesting trees were detected on more than 15 vertices, further search is not likely to turn up any more interesting examples.

5.1 The computer search

The program *geng* distributed with Brendan McKay's program *nauty* [13] was used to generate the trees on up to $n = 20$ vertices and nauty was used for isomorphism testing. For trees on 21–24 vertices, Li and Ruskey's program [12] was used because it generates the trees much faster. The approach applied to determine the edge reconstruction number of each tree T works as follows: For each way to select two different edges e_1 and e_2 of T, first create the set Y_1 of all graphs having a card $T - e_1$ by adding one edge back to $T - e_1$ in all possible ways. The number of ways to add back an edge is $n(n-1)/2-(n-2)$. Similarly, determine the set Y_2 of graphs having a card isomorphic to $T - e_2$. These graphs are put into their canonical forms using nauty (two isomorphic graphs have the same canonical form). Next find the intersection Y of Y_1 and Y_2 which is equal to the set of all graphs having both cards. If the two cards are not isomorphic to each other and Y only contains one graph then return the message that $\mathrm{ern}(T)$ is equal to two. If the two cards are isomorphic to each other, then remove from Y any graphs having only one card isomorphic to $T - e_1$. If $|Y|$ is equal to one after removing these graphs then return the message that $\mathrm{ern}(T)$ is equal to two. If all pairs of edges are tested without determining that

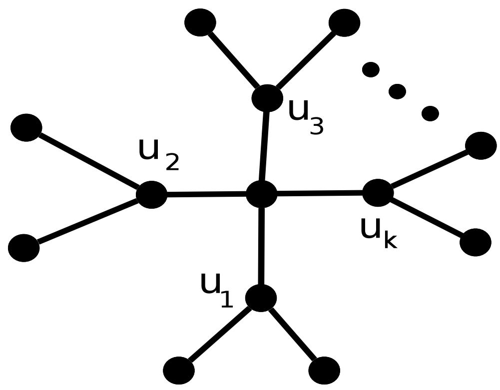

Figure 14: An infinite family of trees T_k with ern = 3

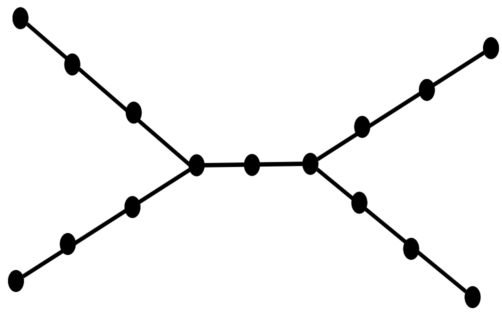

Figure 15: The tree G_{15} on fifteen vertices with ern = 3

ern(T) is equal to two, return the message that ern(T) is equal to three.

The results of this computer search match the results that came from David Rivshin's data for up to 11 vertices. This search also enabled us to discover the infinite family of trees T_k with ern = 3 which we described at the end of Section 4. We also found the graph G_{15} on fifteen vertices (shown in Figure 15) which does not fall within any known infinite class but which also has ern = 3.

5.2 Bicentroidal trees: the remaining cases

Let us now take stock of the situation for bicentroidal trees in the light of the results we have presented. We can summarise the situation as follows. If both centroidal components are not quasipaths, then $\mathrm{ern}(T) = 2$. If one component is $S_1 = S_{1,1,2}$ then $\mathrm{ern}(T) = 2$ except when the other component is also S_1 and T is the caterpillar $C(2, 1, 1, 2)$, in which case $\mathrm{ern}(T) = 3$. If both components are $S_2 = S_{1,2,3}$ then $\mathrm{ern}(T) = 2$. The case when only one centroidal component is S_2 is covered by our computer search which confirms that, in this case too, all these trees have ern = 2.

Now, if one of the two centroidal components is P_k, for $k \leq 5$, then $\mathrm{ern}(T) = 2$ except when both components are P_3 and therefore T is the caterpillar $C(2, 2)$, and when the two components are P_5 and therefore T is the graph G_2 of Figure 13. We shall therefore consider next the case when both components are P_k for $k > 5$.

First a bit of notation: Let T be a bicentroidal tree with centroidal edge ab and such that the two components of $T - ab$ are both isomorphic to P_k, the path on k vertices. Let the two maximal paths starting from a in $T - ab$ have $p + 1$ and $q + 1$ vertices, and similarly for b, let the lengths be $r+1$ and $s+1$. Then we say that T is of *type* $T(p, q; r, s)$. We shall only give a sketch of the proof of the following lemma.

Lemma 5.1 *Let T be a bicentroidal tree with both centroidal components equal to $P_k, k > 5$. Then $ern(T) = 2$.*

Proof. As usual, let the centroidal edge be ab and let T be of type $T(p, q; r, s)$, as defined above. Suppose first that $p \neq q$. Let P be the P_k path containing a. Then P contains an edge e such that the component of $P - e$ containing a has a as its central vertex. Then, T is reconstructible from the two edge-cards $T - ab$ and $T - e$, since, from $T - ab$, we know that the two centroidal components are P_k and, from $T - e$ we have four ways of adding the missing edge to give the second component P_k, but all these ways give isomorphic trees. Similarly, if $r \neq s$, then T can be reconstructed from $T - ab$ and one other edge-card. Therefore we may assume that T is of type $T(p, p; p, p)$.

Let x be a neighbour of a different from the other centroidal vertex b, and let x' be the other neighbour of x; x' exists since $k \geq 5$. We shall, in this case, use the edge cards $T - ax$ and $T - xx'$to reconstruct T. Checking all the possibilities one finds that, again since T is not the graph G_2, the only way to join the two components of $T - xx'$ in such a way that the resulting tree has an edge-card isomorphic to $T - ax$ is by joining the vertex x to either one of the two endvertices in the other component (isomorphic to a path) of $T - xx'$. □

Therefore the only remaining case of an infinite class of bicentroidal tree whose ern is not known is when one of the centroidal components is the path P_k and the other component is not a quasipath. We can now easily deal with this case in our final result which therefore neatly complements our first reconstruction result, Theorem 3.1.

Lemma 5.2 *Let T be a bicentroidal tree one of whose centroidal components is the path P_k ($k > 5$) while the other component is not a quasipath. Then $ern(T) = 2$.*

Proof. As usual, let the two centroidal components of T be G and H containing, respectively, the centroidal vertices a and b. Suppose H is the path P_k. Therefore, since G is not a quasipath, we may assume that $\deg(b) = 2$, otherwise we know that $\text{ern}(T) = 2$, by Theorem 3.1. Let d be the other neighbour of b, therefore $d \in V(H)$. We shall first try to reconstruct T from $T - ab$ and $T - bd$.

As usual, we can determine from these two edge-cards that T must be a tree, by Lemma 2.1, and we therefore know the two centroidal components of T. We now consider $T - bd$. The two usual situations can arise: (1) the large component of $T - bd$ is isomorphic to G plus an edge and the smaller component is isomorphic to P_k less an edge; or (2) vice-versa, with the roles of G and P_k reversed. Suppose the first but not the second case holds. Therefore we need to find, in the large component K of $T - bd$, an end-edge e such that $K - e$ is isomorphic to G. Reconstruction would then proceed by joining the endvertex incident to e with an endvertex of the other component of $T - bd$. The edge ab is surely one such edge e. But even if there is another edge vw with $\deg(w) = 1$ such that $K - vw \simeq G$, then w and b would be similar in K. Therefore, joining the vertex w or the vertex b to an endvertex of the other component would give isomorphic reconstructions. This shows that T is reconstructible from $T - ab$ and $T - bd$.

We can now suppose that case (2) holds. This means that $G + ab - \alpha \simeq P_k$ for some end-edge α of G. So, since G is not P_k, it must be the path P_{k-1} with an extra end-edge incident to one of its interior vertices. Therefore T is the path P_{2k-1} plus an end-edge incident to some interior vertex.

But in this case it is easy to show that $\text{ern}(T) = 2$. Let the vertices of the path P_{2k-1} be, in order, $v_1, v_2, \ldots, v_{2k-1}$. Let the extra end-edge be $v_i x$ where i is not equal to 1 or $2k - 1$. Since T is bicentroidal, v_i is not the central vertex v_k of P_{2k-1}. Also, we may assume, without loss of generality, that $i < k$. But then it is easily checked that T is reconstructible from the edge-cards $T - v_i x$ and $T - v_{2i-1}v_{2i}$. □

This final result and the previous comments gives the main result of this paper.

Theorem 5.1 *Every bicentroidal tree except $C(2, 2)$, $C(2, 1, 1, 2)$ and the graph G_2 shown in Figure 13(b) has edge-reconstruction number equal to 2.*

Note that, from this theorem, it also follows that $\mathcal{T}\text{ern} = 2$ for any tree except possibly $C(2, 2)$ and $C(2, 1, 1, 2)$ and G_2. One can easily check that, if T is one of $C(2, 2)$, or G_2 then, for any two edges e_1, e_2 of T, there is a *tree* which is a blocker for the two edge-cards $T - e_1, T - e_2$. Therefore $\mathcal{T}\text{ern}$ for these two trees is also equal to 3. However, if $T = C(2, 1, 1, 2)$ as shown in Figure 16, then it can be checked that the only blocker of $T - e_1$ and $T - e_2$ is not a tree, as is also the only blocker of $T - f_1$ and $T - f_2$. Therefore $\mathcal{T}\text{ern}(C(2, 1, 1, 2)) = 2$. It might be interesting to note here that, from [2], there is evidence to believe that $C(2, 2)$ and G_2 are also the only trees with "degree associated" reconstruction number equal to 3 rather than 2 (for the degree associated reconstruction number, the reconstructor is given $\deg(v)$ together with each $G - v$).

6 Final comments

We have managed to fill in the gaps in our knowledge of the edge-reconstruction number of bicentroidal trees. The computer search described above also leads us to make this conjecture for unicentroidal trees.

Conjecture 6.1 *With only finitely many exceptions, the only trees which have ern = 3 are the paths on an odd number of vertices, the caterpillars $C(2, 0, \ldots, 0, 2)$ of even diameter, and the family of trees T_k depicted in Figure 14.*

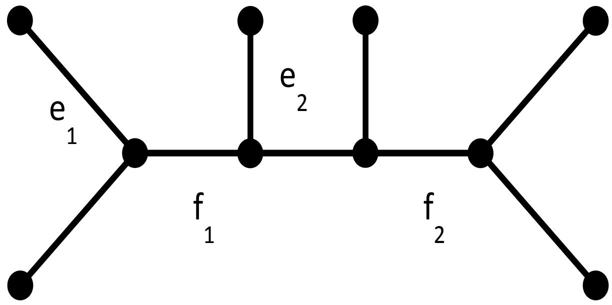

Figure 16: Blockers of the edge-cards of $C(2,1,1,2)$ obtained from e_1, e_2 and from f_1, f_2 are not trees

The known sporadic trees with ern = 3 are the following, all mentioned in Section 5: $S_{2,2,2}$, $S_{3,3,3}$, $C(1,0,1,0,1)$, $C(2,1,2)$, $C(2,3,2)$, $C(1,0,1,0,1,0,1)$, G_1 of Figure 13, and G_{15} of Figure 15.

Proving this conjecture might not be easy. The difficulty of determining which unicentroidal trees have ern equal to 2 or 3 when the (vertex) reconstruction number of trees is known is again evidence for the phenomenon, commented upon in [1], when determining the edge-reconstruction number of a class of graphs is sometimes more difficult than determining the (vertex) reconstruction number.

Acknowledgements

We are again grateful to David Rivshin whose data and computer programs helped us when we could not see the trees for the wood when considering the exceptional cases amongst smaller trees. We are also grateful to two anonymous referees whose careful reading of the manuscript and detailed comments and suggestions helped us improve the paper considerably.

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