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Some generalizations of extension theorems for linear codes over finite fields

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Abstract

We give four new extension theorems for linear codes over $\mathbb{F}_q$: (a) For $q = 2^h$, $h \geq 3$, every $[n, k, d]_q$ code with d odd whose weights are congruent to 0 or $d \pmod{q/2}$ is extendable. (b) For $q = 2^h$, $h \geq 3$, every $[n, k, d]_q$ code with $\gcd(d, q) = 2$ whose weights are congruent to 0 or $d \pmod q$ is doubly extendable. (c) For integers h, m, t with $0 \leq m < t \leq h$ and prime p, every $[n, k, d]_q$ code with $\gcd(d, q) = p^m$ and $q = p^h$ is extendable if $\sum_{i \not\equiv d \pmod{p^t}} A_i < q^{k-1} + r(q)q^{k-3}(q-1)$, where $q+r(q)+1$ is the smallest size of a non-trivial blocking set in $\mathrm{PG}(2, q)$. (d) Every $[n, k, d]_q$ code with $\gcd(d, q) = 1$ whose diversity is $(\theta_{k-1} - 2q^{k-2}, q^{k-2})$ is extendable. These are generalizations of some known extension theorems by Hill and Lizak (1995), Simonis (2000) and Maruta (2005).

1 Introduction

Let $\mathbb{F}_q^n$ denote the vector space of n-tuples over $\mathbb{F}_q$, the field of q elements. A q-ary linear code of length n and dimension k or an $[n, k]_q$ code is a k-dimensional subspace of $\mathbb{F}_q^n$. An $[n, k, d]_q$ code is an $[n, k]_q$ code with minimum (Hamming) distance d. The *weight* of a vector $\boldsymbol{x} \in \mathbb{F}_q^n$, denoted by $wt(\boldsymbol{x})$, is the number of nonzero coordinate positions in $\boldsymbol{x}$. The weight distribution of $\mathcal{C}$ is the list of numbers $(A_0, A_1, \dots, A_n)$, where A_i denotes the number of codewords of $\mathcal{C}$ with weight i. A_i is usually omitted from the list if $A_i = 0$. The weight distribution $(A_0, A_d, \dots) = (1, \alpha, \dots)$ is expressed

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as $0^1 d^\alpha \dots$ in this paper. A q-ary linear code $\mathcal{C}$ is called *w-weight (mod q)* if $\mathcal{C}$ has exactly w distinct weights of codewords under modulo q reduction. We only consider linear codes over finite fields having no coordinate which is identically zero. For an $[n,k,d]_q$ code $\mathcal{C}$ with a generator matrix G, $\mathcal{C}$ is called *extendable* (to $\mathcal{C}'$) if there exists a vector $h \in \mathbb{F}_q^k$ such that the extended matrix $[G, h^{\mathrm{T}}]$ generates an $[n+1,k,d+1]_q$ code $\mathcal{C}'$. Then $\mathcal{C}'$ is called an *extension* of $\mathcal{C}$. $\mathcal{C}$ is *doubly extendable* if one of its extensions $\mathcal{C}'$ is also extendable. The most well-known extension theorem is the following by Hill and Lizak [6]; see also [5] and [10].

Theorem 1.1 ([6])**.** *Every $[n,k,d]_q$ code with gcd$(d,q) = 1$, whose weights (of codewords) are congruent to* 0 *or d* (mod q)*, is extendable.*

For even $q \geq 8$, we give a stronger result:

Theorem 1.2. *For $q = 2^h$, $h \geq 3$, every $[n,k,d]_q$ code with d odd whose weights are congruent to* 0 *or d* (mod $q/2$) *is extendable.*

Theorem 1.1 is an extension theorem for 2-weight (mod q) linear codes. As for the extension theorems for 3-weight (mod q) linear codes, see [14]. Theorem 1.2 is applicable to 4-weight (mod q) linear codes whose weights are $0, q/2, d, d+q/2$ (mod q), and is the first extension theorem for 4-weight (mod q) linear codes.

The extendability of $[n,k,d]_q$ codes with $\gcd(d,q) = 2$ was first investigated in [17]. The condition "$\gcd(d,q) = 1$" in Theorem 1.1 cannot be replaced by "$\gcd(d,q) = 2$" for $q = 4$ since there is a counterexample; see [17]. But for $q \geq 8$, we prove the following.

Theorem 1.3. *For $q = 2^h$, $h \geq 3$, every $[n,k,d]_q$ code with gcd$(d,q) = 2$ whose weights are congruent to* 0 *or d* (mod q) *is doubly extendable.*

Simonis [16] gave the following generalization of Theorem 1.1.

Theorem 1.4 ([16])**.** *Every $[n,k,d]_q$ code with gcd$(d,q) = 1$, $q = p^h$, p prime, is extendable if $\sum_{i \not\equiv d \pmod p} A_i = q^{k-1}$.*

We give a generalization of Theorem 1.4:

Theorem 1.5. *Let h, m, t be integers with $0 \leq m < t \leq h$. For $q = p^h$ with prime p, every $[n,k,d]_q$ code with gcd$(d,q) = p^m$ is extendable if*

$$\sum_{i \not\equiv d \pmod{p^t}} A_i = q^{k-1}. \tag{1.1}$$

Note that Theorem 1.4 is the case $m = 0$, $t = 1$ in Theorem 1.5. The condition (1.1) can be weakened to the following.

Theorem 1.6. *Let h, m, t be integers with $0 \leq m < t \leq h$. For $q = p^h$ with prime p, every $[n,k,d]_q$ code with gcd$(d,q) = p^m$ is extendable if*

$$\sum_{i \not\equiv d \pmod{p^t}} A_i < q^{k-1} + r(q) q^{k-3}(q-1), \tag{1.2}$$

where $q + r(q) + 1$ is the smallest size of a non-trivial blocking set in $PG(2,q)$.

A *non-trivial blocking set in* $PG(2, q)$ is a set of points in the projective plane over $\mathbb{F}_q$ meeting every line in at least one point but containing no line; see Chapter 13 of [7]. As for $r(q)$, it is known that $r(3) = r(4) = 2$, $r(5) = 3$, $r(7) = 4$. It can be shown that the inequality (1.2) implies the equality (1.1). The following result is known as another extension theorem making use of $r(q)$.

Theorem 1.7 ([9]). *Every* $[n, k, d]_q$ *code with* $gcd(d, q) = 1$ *is extendable if*

$$\sum_{i \not\equiv 0,\, d \pmod q} A_i \le q^{k-3} r(q). \tag{1.3}$$

Since the condition on weights of codewords in Theorem 1.1 can be written as $\sum_{i \not\equiv 0,d \pmod q} A_i = 0$, Theorem 1.7 is also a generalization of Theorem 1.1, and the inequality (1.3) was recently improved as follows.

Theorem 1.8 ([15]). *Every* $[n, k, d]_q$ *code with* $gcd(d, q) = 1$ *is extendable if*

$$\sum_{i \not\equiv 0,d \pmod q} A_i < q^{k-2}(q-1).$$

To give one more extension theorem, we introduce the diversity of a linear code. For an $[n, k, d]_q$ code $\mathcal{C}$ with $\gcd(d, q) < q$, let

$$\Phi_0 = \frac{1}{q-1} \sum_{q|i,\, i>0} A_i, \quad \Phi_1 = \frac{1}{q-1} \sum_{i \not\equiv 0,d \pmod q} A_i,$$

where the notation $q|i$ means that q is a divisor of i. The pair of integers (Φ_0, Φ_1) is called the *diversity* of $\mathcal{C}$ ([11], [12]). Theorem 1.8 shows that $\mathcal{C}$ is extendable if $\Phi_1 < q^{k-2}$ and $\gcd(d, q) = 1$. Next, we consider the case when $\Phi_1 = q^{k-2}$. We denote $\theta_j = (q^{j+1} - 1)/(q - 1)$ for $\mathbb{F}_q$. As for ternary linear codes ($q = 3$), it is known that an $[n, k, d]_3$ code with $\gcd(3, d) = 1$, $k \ge 3$, is extendable if

$$(\Phi_0, \Phi_1) \in \{(\theta_{k-2}, 0), (\theta_{k-3}, 2 \cdot 3^{k-2}), (\theta_{k-2}, 2 \cdot 3^{k-2}), (\theta_{k-2} + 3^{k-2}, 3^{k-2})\},$$

see [12]. For an $[n, k, d]_q$ code $\mathcal{C}$ with $\gcd(d, q) = 1$, $k \ge 3$, it follows from Theorem 1.1 that $\mathcal{C}$ is extendable if $(\Phi_0, \Phi_1) = (\theta_{k-2}, 0)$. We generalize the case $(\Phi_0, \Phi_1) = (\theta_{k-2} + 3^{k-2}, 3^{k-2})$ for ternary linear codes to q-ary linear codes.

Theorem 1.9. *Let* $\mathcal{C}$ *be an* $[n, k, d]_q$ *code with diversity* (Φ_0, Φ_1)*,* $gcd(d, q) = 1$*. Then* $\mathcal{C}$ *is extendable if* $(\Phi_0, \Phi_1) = (\theta_{k-1} - 2q^{k-2}, q^{k-2})$.

Example 1.1.

(a) Let $\mathcal{C}_1$ be a $[100, 3, 87]_8$ code. Considering the possible residual codes, it can be proved that all possible weights of $\mathcal{C}_1$ are $87, 88, 91, 92, 95, 96$. So, $A_i = 0$ for all $i \not\equiv 0, 3 \pmod 4$. Hence $\mathcal{C}_1$ is extendable by Theorem 1.2. Actually, the possible weight distributions for $\mathcal{C}_1$ are $0^1 87^{413} 88^{63} 95^{35}$, $0^1 87^{420} 88^{56} 95^{28} 96^7$, $0^1 87^{392} 88^{49} 91^{56} 92^{14}$ and $0^1 87^{378} 88^{63} 91^{70}$.

(b) There exists a $[73, 4,62]_8$ code $\mathcal{C}_2$ with weight distribution $0^1 62^{1764} 64^{1883} 70^{252} 72^{196}$, see [8]. Since the weights of $\mathcal{C}_2$ are congruent to 0 or 6 (mod 8), $\mathcal{C}_2$ is doubly extendable to a $[75, 4, 64]_8$ code by Theorem 1.3.

(c) There exists a $[30, 3, 22]_4$ code $\mathcal{C}_3$ with weight distribution $0^1 22^{45} 24^{15} 30^3$, see [3]. $\mathcal{C}_3$ is extendable by Theorem 1.6 with $m = 1$, $t = 2$, $p = 2$.

(d) Let $\mathcal{C}_4$ be a $[15, 3, 7]_4$ code with generator matrix

$$G_4 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & \bar{\omega} & \bar{\omega} & 1 & \omega & 1 & \bar{\omega} & \omega & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & \omega & 1 & 0 & 1 & 0 & 0 & \bar{\omega} & 1 \end{bmatrix},$$

where $\mathbb{F}_4 = \{0, 1, \omega, \bar{\omega}\}$. The weight distribution of $\mathcal{C}_4$ is $0^1 7^3 8^3 9^3 11^9 12^{36} 13^9$ with diversity $(13, 4)$. So, $\mathcal{C}_4$ is extendable by Theorem 1.9. Indeed, by adding the column $(1, 0, 1)^{\mathrm{T}}$ to G_4, one gets a $[16, 3, 8]_4$ code $\mathcal{C}_4'$ with weight distribution $0^1 8^3 9^6 12^{12} 13^{42}$. See also Example 2.1 in Section 2.

Problem. (i) Can the conditions "$q = 2^h$" and " (mod $q/2$)" in Theorem 1.2 be generalized to "$q = p^h$" and " (mod q/p)" for an odd prime p?
(ii) Is Theorem 1.9 valid for the case $\gcd(d, q) \geq 2$?
(iii) Find more diversities such that every code over $\mathbb{F}_q$ is extendable.

2 Proof of the main theorems

We first give the geometric method to investigate linear codes over $\mathbb{F}_q$ through projective geometry. A *j-flat* of $\mathrm{PG}(r, q)$ is a projective subspace of dimension j in $\mathrm{PG}(r, q)$. The 0-flats, 1-flats, 2-flats and $(r-1)$-flats are called *points, lines, planes* and *hyperplanes*, respectively. The number of points in a j-flat is $|\mathrm{PG}(j, q)| = \theta_j = (q^{j+1} - 1)/(q - 1)$, where $|T|$ denotes the number of elements in the set T. We refer to [7] for geometric terminologies.

We assume $k \geq 3$. Let $\mathcal{C}$ be an $[n, k, d]_q$ code with diversity (Φ_0, Φ_1) and a generator matrix G with no all-zero column. Let g_i be the i-th row of G for $1 \leq i \leq k$. We consider the mapping w_G from $\Sigma :=\mathrm{PG}(k - 1, q)$ to $\{i \mid A_i > 0\}$, the set of non-zero weights of $\mathcal{C}$. For $P = \mathbf{P}(p_1, \ldots, p_k) \in \Sigma$, *the weight of* P *with respect to* G, denoted by $w_G(P)$, is defined as $w_G(P) = wt(\sum_{i=1}^{k} p_i g_i)$, see [14].

Lemma 2.1 ([13]). *For a line* $L = \{P_0, P_1, \ldots, P_q\}$ *in* Σ, *the following holds:*

$$w_G(L) := \sum_{i=0}^{q} w_G(P_i) \equiv 0 \pmod{q}.$$

Let $F_d = \{P \in \Sigma \mid w_G(P) = d\}$. Recall that a hyperplane H of Σ is defined by a non-zero vector $h = (h_1, \ldots, h_k) \in \mathbb{F}_q^k$ as $H = \{\mathbf{P}(p_1, \ldots, p_k) \in \Sigma \mid h_1 p_1 + \cdots + h_k p_k = 0\}$. The vector h is called a *defining vector of* H.

Lemma 2.2 ([13]). *$\mathcal{C}$ is extendable if and only if there exists a hyperplane H of Σ such that $F_d \cap H = \emptyset$. Moreover, the extended matrix of G by adding a defining vector of H as a column generates an extension of $\mathcal{C}$.*

Now, let

$$\begin{aligned} F_0 &= \{P \in \Sigma \mid w_G(P) \equiv 0 \pmod q\}, \\ F_1 &= \{P \in \Sigma \mid w_G(P) \not\equiv 0, d \pmod q\}, \\ F_2 &= \{P \in \Sigma \mid w_G(P) \equiv d \pmod q\} \supset F_d. \end{aligned}$$

Note that $(\Phi_0, \Phi_1) = (|F_0|, |F_1|)$. Since $(F_0 \cup F_1) \cap F_d = \emptyset$ if $\gcd(d, q) < q$, we get the following.

Lemma 2.3. *$\mathcal{C}$ is extendable if $\gcd(d, q) < q$ and if there exists a hyperplane H of Σ such that $H \subset F_0 \cup F_1$.*

A set $\mathcal{B}$ of points in $\mathrm{PG}(r, q)$ is called a *blocking set with respect to s-flats* if every s-flat in $\mathrm{PG}(r, q)$ meets $\mathcal{B}$ in at least one point. A blocking set in $\mathrm{PG}(r, q)$ with respect to s-flats is called *non-trivial* if it contains no $(r - s)$-flat.

Lemma 2.4 ([1],[2],[4]). *Let $\mathcal{B}$ be a blocking set with respect to s-flats in $PG(r, q)$.*

(a) *$|\mathcal{B}| \geq \theta_{r-s}$, where the equality holds if and only if $\mathcal{B}$ is an $(r - s)$-flat.*

(b) *$|\mathcal{B}| \geq \theta_{r-s} + q^{r-s-1} r(q)$ if $\mathcal{B}$ is non-trivial, where $q + r(q) + 1$ is the smallest size of a non-trivial blocking set in $PG(2, q)$.*

The following result is essential in the proofs of Theorems 1.2 and 1.3.

Lemma 2.5 ([17]). *Let K be a set of points in $\Sigma = \mathrm{PG}(k - 1, q)$, $k \geq 3$, $q = 2^h$, $h \geq 3$, meeting every line in exactly 1, $q/2 + 1$, or $q + 1$ points. Then, K contains a hyperplane of Σ.*

Now, we are ready to prove our results.

Proof of Theorem 1.2. For $q = 2^h$, $h \geq 3$, let $\mathcal{C}$ be an $[n, k, d]_q$ code with d odd whose weights are congruent to 0 or $d \pmod{q/2}$. For a generator matrix G of $\mathcal{C}$ and a line L in $\Sigma = \mathrm{PG}(k - 1, q)$, we have $w_G(L) = \sum_{P \in L} w_G(P) \equiv 0 \pmod q$ by Lemma 2.1. Let $\tilde{F_0} := \{Q \in \Sigma \mid w_G(Q) \text{ is even}\}$. Then, $\tilde{F_0} \cap F_d = \emptyset$. Assume that the t points on L have odd weights and that the other have even weights. Then, from the condition, we have $td \equiv 0 \pmod{q/2}$, so, $t \equiv 0 \pmod{q/2}$, for d is odd. Hence $t = 0, q/2$ or q. Thus, $|\tilde{F_0} \cap L| = 1, q/2 + 1$ or $q + 1$, and $\tilde{F_0}$ contains a hyperplane of Σ by Lemma 2.5. Hence our assertion follows from Lemma 2.2. □

Proof of Theorem 1.3. For $q = 2^h$, $h \geq 3$, let $\mathcal{C}$ be an $[n, k, d]_q$ code with $\gcd(d, q) = 2$ whose weights are congruent to 0 or $d \pmod q$. For a generator matrix G of $\mathcal{C}$ and a line L in $\Sigma = \mathrm{PG}(k-1, q)$, we have $w_G(L) = \sum_{P \in L} w_G(P) \equiv 0 \pmod q$

by Lemma 2.1. Note that $\Sigma = F_0 \cup F_2$, $F_0 \cap F_2 = \emptyset$. Assume $|L \cap F_2| = t$. Then, from the condition, we have $td \equiv 0 \pmod q$, so, $t \equiv 0 \pmod{q/2}$, for $\gcd(d,q) = 2$. Hence $t = 0, q/2$ or q. Thus, $|F_0 \cap L| = 1, q/2+1$ or $q+1$, and F_0 contains a hyperplane of Σ, say H, by Lemma 2.5. Hence $\mathcal{C}$ is extendable by Lemma 2.3. Let $\mathcal{C}'$ be the extension with generator matrix $G' = [G, h^{\mathrm{T}}]$, where h is a defining vector of H. Let $F_{d'} = \{P \in \Sigma \mid w_{G'}(P) = d+1\}$. Note that $w_G(P) = w_{G'}(P) \equiv 0 \pmod q$ for any point P of H. Since $d+1$ is odd, we have $H \cap F_{d'} = \emptyset$. Hence, $\mathcal{C}'$ is also extendable by Lemma 2.2. □

Proof of Theorem 1.6. For integers h, m, t with $0 \le m < t \le h$ and for $q = p^h$ with prime p, let $\mathcal{C}$ be an $[n,k,d]_q$ code with $\gcd(d,q) = p^m$ and assume $\sum_{i \not\equiv d \pmod{p^t}} A_i < q^{k-1} + r(q)q^{k-3}(q-1)$. For a generator matrix G of $\mathcal{C}$ and a line L in $\Sigma = \mathrm{PG}(k-1,q)$, we have $w_G(L) = \sum_{P \in L} w_G(P) \equiv 0 \pmod q$ by Lemma 2.1. Let $\bar{F_0} = \{Q \in \Sigma \mid w_G(Q) \not\equiv d \pmod{p^t}\}$ and $\bar{F_2} = \{Q \in \Sigma \mid w_G(Q) \equiv d \pmod{p^t}\}$. Then, $\bar{F_0} \cap F_d = \emptyset$ and $|\bar{F_0}| < \theta_{k-2} + r(q)q^{k-3}$. Suppose $L \subset \bar{F_2}$. Then, we have $d \equiv 0 \pmod{p^t}$, a contradiction. Thus $\bar{F_0}$ forms a blocking set with respect to lines in Σ. Hence $\bar{F_0}$ contains a hyperplane of Σ by Lemma 2.4, and $\mathcal{C}$ is extendable by Lemma 2.2. □

Lemma 2.6. *Let K be a set of points in $\Sigma = \mathrm{PG}(r,q)$ with $K \ne \Sigma$. Then K is a hyperplane of Σ if and only if every line meets K in either one or $q+1$ points.*

A line ℓ is called an (i,j)-*line* if $|\ell \cap F_0| = i$ and $|\ell \cap F_1| = j$. Note that a $(1,1)$-line and a $(0,1)$-line do not exist by Lemma 2.1.

Proof of Theorem 1.9. Let $\mathcal{C}$ be an $[n,k,d]_q$ code with diversity $(\Phi_0, \Phi_1) = (\theta_{k-1} - 2q^{k-2}, q^{k-2})$, $\gcd(d,q) = 1$, $k \ge 3$. Then, we have $|F_1| = |F_2| = q^{k-2}$. For $R \in F_2$, there exist at least θ_{k-3} lines through R containing no point of F_1, for $|F_1| = q^{k-2}$. Such lines are $(1,0)$-lines, for $\gcd(d,q) = 1$. Let $l_1, \ldots, l_{\theta_{k-3}}$ be such lines and let $H = \bigcup_{i=1}^{\theta_{k-3}} l_i$. Since $|F_2 \cap H| = (q-1)\theta_{k-3} + 1 = |F_2|$, we have $F_2 \subset H$. Hence, every line through two points of F_2 is a $(1,0)$-line. For $R_i \in l_i$ and $R_j \in l_j$ with $i \ne j$ and $R_i, R_j \ne R$, the line $l = \langle R_i, R_j \rangle$ is a $(1,0)$-line. Let P be the point of F_0 on l. If there exists a point of F_1 on the line $l_P = \langle R, P \rangle$, then there exists a $(1,1)$-line or a $(0,1)$-line on the plane $\langle l_i, l_j \rangle$, a contradiction. Hence l_P is also a $(1,0)$-line, and l is contained in H. It follows that H forms a hyperplane of $\Sigma = \mathrm{PG}(k-1,q)$. Since H contains only $(1,0)$-lines or $(q+1,0)$-lines, $H_0 = F_0 \cap H$ is a hyperplane of H by Lemma 2.6. Now, take a hyperplane H_1 through H_0 with $H_1 \ne H$. Then, we have $H_1 \subset F_0 \cup F_1$ since $F_2 = H \setminus H_0$. Hence $\mathcal{C}$ is extendable by Lemma 2.3. □

Example 2.1. Let us investigate the $[15,3,7]_4$ code $\mathcal{C}_4$ in Example 1.1 (d). We denote by $[a,b,c]$ the line in $\mathrm{PG}(2,4)$ with defining vector (a,b,c). From the generator matrix G_4, we have $F_0 = \{(1,1,0), (1,\bar{\omega},0), (0,1,1), (1,\omega,1), (1,0,\omega), (0,1,\omega), (1,1,\omega), (1,\bar{\omega},\omega), (1,0,\bar{\omega}), (0,1,\bar{\omega}), (1,1,\bar{\omega}), (1,\bar{\omega},\bar{\omega}), (1,0,0)\}$ and $F_1 = \{(1,0,1), (0,1,0), (1,1,1), (1,\bar{\omega},1)\}$, where (x,y,z) stands for the point $\mathbf{P}(x,y,z)$ of $\mathrm{PG}(2,4)$.

Hence, $F_0 \cup F_1$ contains a $(1,4)$-line $[1,0,1]$, which gives a $[16,3,8]_4$ code $\mathcal{C}_4'$ in Example 1.1 (d). On the other hand, F_0 contains a $(5,0)$-line $[0,1,\omega]$, giving a $[16,3,8]_4$ code with weight distribution $0^1 8^3 9^6 12^{12} 13^{42}$.

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