AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **60(1)** (2014), Pages 11–26

A theory of 2-dipath colourings

Gary MacGillivray   Kailyn M. Sherk

Mathematics and Statistics
University of Victoria
P.O. Box 1700 STN CSC
Victoria, BC V8W 2Y2
Canada

Abstract

We study colourings of oriented graphs in which vertices joined by a directed path of length two are assigned different colours. There are two models, depending on whether adjacent vertices must also be assigned different colours. In each case we describe a homomorphism model, a dichotomy theorem for the complexity of the problem of deciding whether there exists such a colouring with a fixed number of colours, and a polynomial time algorithm for determining the minimum number of colours needed to colour a given multipartite tournament.

1 Introduction

In an *oriented colouring* of an oriented graph, colours are assigned to the vertices so that two conditions are satisfied. First, adjacent vertices get different colours. Second, the orientation of the arcs is respected in the sense that if there is an arc from a vertex of colour r to a vertex of colour g, then there is no arc from a vertex of colour g to a vertex of colour r. It follows that vertices joined by a directed path of length two are assigned different colours.

Chen and Wang [6] were the first to explicitly define and study *proper 2-dipath colourings*: proper colourings of oriented graphs in which vertices joined by a directed path of length two are assigned different colours. They proved that any orientation of a Halin graph admits such a colouring with at most seven colours, and this bound is best possible. Our goal in this paper is to develop some basic theory of these colourings, and the related type of colouring in which adjacent vertices may be assigned the same colour, including a homomorphism model and complexity results.

Proper 2-dipath colourings are a special case of the more general concept of $L(p, q)$-labellings. For integers $p \geq q \geq 0$, a k-$L(p, q)$-labelling of an oriented graph G is an assignment of the colours $0, 1, \ldots, k$ to the vertices of G so that adjacent vertices are assigned colours that differ in absolute value by at least p, and vertices joined by a directed path of length two are assigned colours that differ in absolute value by at least q. Hence, a *proper 2-dipath k-colouring* of an oriented graph G is a $(k-1)$-$L(1,1)$-labelling of G. Adjacent vertices are assigned different colours, and so are vertices joined by a directed path of length two.

Chang and Liaw were the first to study k-$L(p,q)$-labellings of oriented graphs [4]. Their focus was on k-$L(2,1)$-labellings, as was that of other authors [3, 5, 7, 13]. The corresponding problem for undirected graphs was introduced by Griggs and Yeh in 1992 [8] and has subsequently become an active area of research (see [2] for a survey).

Figure 1 gives an example showing that a 2-$L(1,1)$-labelling – a proper 2-dipath colouring – of an oriented graph G may not be an oriented colouring of G: the orientation of the arcs need not be respected by the colour assignment. Goncalves, Raspaud and Shalu define and study *oriented k-$L(p,q)$-labellings* [7]. These are k-$L(p,q)$-labellings which are also oriented colourings.

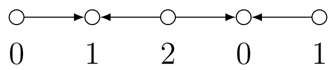


Figure 1: A 2-$L(1,1)$-labelling which is not an oriented colouring.

We also study colourings which are not proper: vertices joined by a directed path of length two are assigned different colours, but adjacent vertices may be assigned the same colour. These can also be seen as $L(p,q)$-labellings if the condition $p \geq q \geq 0$ is dropped from the definition. With that relaxation in mind, we define a *2-dipath k-colouring* of an oriented graph G to be a $(k-1)$-$L(0,1)$-labelling of G.

For the sake of simplicity, in (proper) 2-dipath k-colourings we typically use the colours $1,2,\ldots,k$ rather than $0,1,\ldots,k-1$.

An oriented graph G is *2-dipath k-colourable* if it admits a 2-dipath k-colouring, and *proper 2-dipath k-colourable* if it admits a proper 2-dipath k-colouring. Since every oriented graph G is (proper) 2-dipath $|V(G)|$-colourable, we can define the *2-dipath chromatic number* of G, $\chi_2(G)$, to be the smallest k such that G is 2-dipath k-colourable and the *proper 2-dipath chromatic number* of G, $\chi'_2(G)$, to be the smallest k such that G is proper 2-dipath k-colourable.

The parameters χ_2 and χ'_2 are related.

Proposition 1.1 *For an oriented graph G, $\chi_2(G) \leq \chi'_2(G) \leq 2\chi_2(G)$.*

Proof. The first inequality is clear. To show $\chi'_2(G) \leq 2\chi_2(G)$, consider a 2-dipath k-colouring of G with $k = \chi_2(G)$. For $i = 1,2,\ldots,k$, the subgraph of G induced by the set of vertices assigned colour i may contain arcs, but not a directed path of length two. For each i, recolour every vertex which is the head of an arc having both ends of colour i with the new colour $i+k$. The result of doing so is a proper 2-dipath $(2k)$-colouring of G. □

The remainder of this paper is organized as follows. Necessary definitions are introduced in the next section, as are some preliminary results. In the subsequent section, a homomorphism model and complexity results are given for 2-dipath colourings. The homomorphism model is used to obtain an upper bound on the oriented chromatic number, χ_o. A polynomial time algorithm for finding the 2-dipath chromatic number of a multipartite tournament is also described. The final section contains similar results for proper 2-dipath colourings.

2 Preliminaries

The purpose of this section is to review some necessary concepts.

An *oriented graph* is a directed graph such that, for any two different vertices x and y, at most one xy and yx is an arc. That is, it is a directed graph in which there are no directed cycles of length two. An oriented graph can be viewed as being obtained by assigning a direction to each edge of a simple undirected graph.

Let $D = (V, E)$ be a directed graph. The *out-neighbourhood* of a vertex x is $N^+(x) = \{y : xy \in E\}$, and the *in-neighbourhood of* x is $N^-(x) = \{y : yx \in E\}$. The *out-degree* of the vertex x is $d^+(x) = |N^+(x)|$, and the *in-degree* of x is $d^-(x) = |N^-(x)|$. We use $\delta^+(G)$ to denote $\min_{x\in V} d^+(x)$, and $\delta^-(G)$ to denote $\min_{x\in V} d^-(x)$.

The *underlying undirected graph* of a directed graph D is the graph $U[D]$ with vertex set $V(U[D]) = V(D)$ and edge set $E(U[D]) = \{xy : xy \in E(D) \text{ or } yx \in E(D)\}$. When graph terminology like 'connected' or 'bipartite' is used in reference to a directed graph D, it should be understood to mean that $U[D]$ has the given property.

A *homomorphism of a digraph* G *to a digraph* H is a function $f : V(G) \to V(H)$ such that $f(x)f(y) \in E(H)$ whenever $xy \in E(G)$. For ease of notation, we shall talk about a homomorphism $f : G \to H$, or a homomorphism $G \to H$ when the specific function f is unimportant.

Homomorphisms and colourings are closely related. A k-colouring of a graph G is a homomorphism $G \to K_k$. An *oriented* k*-colouring* of an oriented graph G is a homomorphism of G to some oriented graph on k vertices. Many other colouring parameters admit a *homomorphism model*: a theorem stating that a (directed) graph has a colouring of a given type if and only if it admits a homomorphism to a (directed) graph in a certain family. A wealth of information about colourings and homomorphisms is contained in the book by Hell and Nešetřil [10].

Let G be an oriented graph. Define the *auxiliary graph*, $Aux(G)$ to be the undirected graph with vertex set $V(Aux(G)) = V(G)$ and $xy \in E(Aux(G))$ if and only if the vertices x and y are joined by a directed path of length two in G. Further, let G^2 be the digraph obtained from G by adding the arc uv whenever u and v are joined by a directed path of length two in G.

Proposition 2.1 *Let G be an oriented graph. There is a 1–1 correspondence between the set of 1-dipath k-colourings of G and the set of k-colourings of $Aux(G)$. There is a $1-1$ correspondence between the set of proper 2-dipath k-colourings of G and the set of k-colourings $U[G^2]$.*

Corollary 2.2 *Let G be an oriented graph. Then $\chi_2(G) = \chi(Aux(G))$ and $\chi'_2(G) = \chi(U[G^2])$.*

Because of Corollary 2.2, bounds on χ_2 and χ'_2 can be obtained from bounds on $\chi(Aux(G))$ and $\chi(U[G^2])$, respectively, for example using Brooks' Theorem.

3 2-dipath colourings

The colourings considered in this section assign different colours to vertices joined by a directed path of length two, but may assign the same colour to adjacent vertices.

3.1 Homomorphism model

We shall define a set of oriented graphs G_k, $k \geq 1$, such that an oriented graph G is 2-dipath k-colourable if and only if there is a homomorphism $G \to G_k$. Similar, but not identical, oriented graphs have been used by Sopena [14] in work on the oriented chromatic number. The graph G_k has arisen in the study of injective oriented colourings (e.g. see [12, 15]).

For an integer $k \geq 1$, we define G_k to be the directed graph with vertex set

$$V(G_k) = \{(u_0; u_1, u_2, \dots, u_k) : u_0 \in \{1, 2, \dots, k\}, u_i \in \{+, -\}, 1 \leq i \leq k\}.$$

and edge set

$$E(G_k) = \{(u_0; u_1, u_2, \dots, u_k)(x_0; x_1, x_2, \dots, x_k) : u_{x_0} = +, x_{u_0} = -\}.$$

Note that, in the definition of $E(G_k)$, the integers u_0 and x_0 may be equal. The oriented graphs G_1 and G_2 are shown in Figure 2.

The integer u_0 is the *index* of the vertex $(u_0; u_1, u_2, \dots, u_k) \in V(G_k)$. For $i = 1, 2, \dots, k$, let S_i be the set of vertices of index i. The underlying idea is that, for $j \geq 1$, the entry u_j of $(u_0; u_1, u_2, \dots, u_k)$ indicates the sort of adjacencies between the vertex $(i; u_1, u_2, \dots, u_k)$ and vertices in S_j. When $u_j = +$ it is adjacent *to* some vertices in S_j (namely those with i-th entry $-$), and when $u_j = -$ it is adjacent *from* some vertices in S_j (namely those with i-th entry $+$). By definition, an arc has its origin at vertex $(i; u_1, u_2, \dots, u_k) \in S_i$ and terminus at vertex $(j; v_1, v_2, \dots, v_k) \in S_j$ if and only if $u_j = +$ and $v_i = -$. It follows that G_k is an oriented graph.

The oriented graph G_k has the following properties:

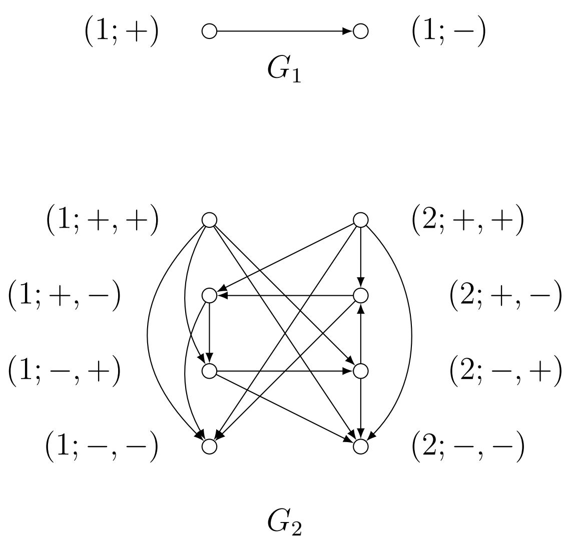


Figure 2: The oriented graphs G_1 and G_2.

- $|V(G_k)| = k \cdot 2^k$,
- for all vertices v, $deg^+(v) + deg^-(v) = k \cdot \frac{2^k}{2} = k \cdot 2^{k-1}$,
- $|E(G)| = \frac{1}{2} \cdot (k \cdot 2^k) \cdot (k \cdot 2^{k-1}) = k^2 \cdot 2^{2k-2}$, and
- for $1 \leq i \leq k$, the subdigraph induced by S_i is a bipartite tournament for which there is bipartition (A_i, B_i) such that all arcs have their origin in A_i.

Lemma 3.1 *There is a directed path of length two in G_k joining the vertices $(i; u_1, u_2, \ldots, u_k)$ and $(j; v_1, v_2, \ldots, v_k)$ if and only if $k \geq 2$, $i \neq j$, and there exists $\ell \geq 1$ such that $u_\ell \neq v_\ell$.*

Proof. ($\Rightarrow$) Without loss of generality, suppose that there is a directed path of length two in G_k from $(i; u_1, u_2, \ldots, u_k)$ to $(j; v_1, v_2, \ldots, v_k)$. Then there is a vertex $(\ell; w_1, w_2, \ldots, w_k)$ such that $u_\ell = +$, $w_i = -$, $w_j = +$ and $v_\ell = -$. In particular, $i \neq j$, so that $k \geq 2$, and $u_\ell \neq v_\ell$.

($\Leftarrow$) Suppose that $i \neq j$ and there exists ℓ such that $u_\ell \neq v_\ell$. Without loss of generality, $u_\ell = +$, and $v_\ell = -$. Then, any vertex $(\ell; w_1, w_2, \ldots, w_k)$ such that $w_i = -$ and $w_j = +$ is the midpoint of a directed path of length two from $(i; u_1, u_2, \ldots, u_k)$ to $(j; v_1, v_2, \ldots, v_k)$. □

Proposition 3.2 $\chi_2(G_k) = k$.

Proof. Assigning colour i to all vertices in S_i, the set of vertices of index i, is a 2-dipath k-colouring of G_k. To see that k is the minimum possible number of colours, suppose there is a 2-dipath colouring of G_k with fewer than k colours. Since $|V(G_k)| = k \cdot 2^k$, there exists i such that more than 2^k vertices have colour i. Hence there exist vertices u, v and w such that $u, v \in S_i$ and $w \in S_j$, $i \neq j$ and there is no directed path of length two joining any two of u, v and w. But Lemma 3.1 implies that there is a directed path of length two joining either u and w, or v and w, a contradiction. This completes the proof. □

Theorem 3.3 *An oriented graph G has a 2-dipath colouring with k colours if and only if there exists a homomorphism $G \to G_k$.*

Proof. (⇒) Suppose G has a 2-dipath colouring with colours, $1, 2, \ldots, k$. We shall describe the desired homomorphism. The image $(v_0; v_1, v_2, \ldots, v_k)$ of vertex $v \in V(G)$ with colour i has $v_0 = i$. For $j = 1, 2, \ldots, k$, set $v_j = -$ if v has an in-neighbour of colour j, and set $v_j = +$ otherwise. By definition of a 2-dipath colouring, v can not have both an in-neighbour u and an out-neighbour w of the same colour. Hence the image of v is well-defined. Suppose $uw \in E(G)$, with u having colour a and v having colour b. Then, by construction, the image $(u_0; u_1, u_2, \ldots, u_k)$ of u has entry $u_b = +$ and the image $(w_0; w_1, w_2, \ldots, w_k)$ of w has entry $w_a = -$. These two vertices are adjacent in G_k.

(⇐) Suppose there is a homomorphism $f : G \to G_k$. Assign vertex $v \in V(G)$ colour i if and only if its image under f has index i. Since vertices joined by a directed path of length two in G map to vertices joined by a directed path of length two in G_k, and any two such vertices of G_k have different indices, this is a 2-dipath colouring of G. □

Corollary 3.4 *If there exists a homomorphism $G \to H$, then $\chi_2(G) \leq \chi_2(H)$.*

Recall that the *wreath product* (or lexicographic product) of digraphs G and H is the digraph $G\,wr\,H$ with vertex set $V(G\,wr\,H) = V(G) \times V(H)$, and edge set

$$E(G\,wr\,H) = \{(g, h)(g', h') : gg' \in E(G), \text{ or } g = g' \text{ and } hh' \in E(H)\}.$$

Informally, $G\,wr\,H$ is the digraph obtained by replacing each vertex of G with a copy of H and, for each arc $uv \in E(G)$ adding all possible arcs from the copy of H that replaced u to the one that replaced v.

We use I_n to denote the directed graph with n vertices and no arcs. The following assertions are clear.

Proposition 3.5 *For $1 \leq i \leq k$, the oriented graph $G_{k+1} - S_i \cong G_k\,wr\,I_2$.*

Corollary 3.6 *For $k \geq 1$, G_k is an induced subgraph of G_{k+1}.*

Corollary 3.7 *For $k \geq 1$, there exists a homomorphism $G_k \to G_{k+1}$.*

3.2 Complexity

In this section we determine, for each fixed integer $k \geq 1$, the complexity of 2-*dipath k-colouring*, the problem of deciding whether a given oriented graph G is 2-dipath k-colourable. The problem turns out to be NP-complete when $k \geq 3$, and Polynomial when $k \leq 2$. We also give several descriptions of the oriented graphs that can be 2-dipath coloured with one or two colours.

The key to the NP-completeness proof is a construction that relates k-colourings of a given undirected graph G and 2-dipath k-colourings of an associated bipartite digraph, $Bip(G)$. Given an undirected graph G, let V_1 and V_2 be two disjoint copies of $V(G)$ so that to each $x \in V(G)$ there corresponds a vertex $x_1 \in V_1$ and a vertex $x_2 \in V_2$. Define $Bip(G)$ to be the oriented graph with vertex set $V(Bip(G)) = V_1 \cup V_2$, and edge set $E(Bip(G)) = \{x_1x_2, y_1y_2, y_2x_1, x_2y_1 : xy \in E(G)\}$. That is, the edge xy of G corresponds to the directed 4-cycle x_1, x_2, y_1, y_2, x_1 in $Bip(G)$.

Lemma 3.8 *A graph G is k-colourable if and only if $Bip(G)$ is 2-dipath k-colourable.*

Proof. ($\Rightarrow$) Let c be a k-colouring of the undirected graph G. We will show that the colouring of $Bip(G)$ obtained by assigning colour $c(x)$ to the vertices x_1 and x_2 corresponding to x is a 2-dipath colouring of $Bip(G)$. Suppose there is a directed path of length two from u to v in $Bip(G)$. Then either $u, v \in V_1$ or $u, v \in V_2$. Without loss of generality, assume $u, v \in V_1$, so that there are vertices x and y of G for which $u = x_1$ and $v = y_1$. Let z_2 be the midpoint of a directed path of length two from x_1 to y_1. Since $x_1z_2 \in E(Bip(G))$, we have by construction that $z_2 = x_2$, as x_2 is the only out-neighbour of x_1. Hence $z_2y_1 = x_2y_1$. Therefore $xy \in E(G)$, so that $u = x_1$ and $v = y_1$ are assigned different colours. This proves the implication.

($\Leftarrow$) Suppose $Bip(G)$ is 2-dipath k-colourable. By construction, if $xy \in E(G)$ then x_1 and y_1 are joined by a directed path of length two in $Bip(G)$. Hence, assigning $x \in V(G)$ the same colour as $x_1 \in V(Bip(G))$ gives a k-colouring of G. □

Theorem 3.9 *Let $k \geq 1$ be a fixed integer. If $k \leq 2$, then 2-dipath k-colouring is Polynomial. If $k \geq 3$, then 2-dipath k-colouring is NP-complete.*

Proof. Suppose $k \leq 2$. Let G be a given oriented graph. Since the undirected graph $Aux(G)$ can be constructed in polynomial time, the statement follows from Proposition 2.1.

Suppose $k \geq 3$. The transformation is from k-colouring. Given an instance H of k-colouring, the transformed instance of 2-dipath k-colouring is $Bip(H)$. The transformation can clearly be accomplished in polynomial time. The result now follows from Lemma 3.8. □

We now give several descriptions of the oriented graphs that are 2-dipath colourable with one or two colours.

It is clear that an oriented graph has a 2-dipath colouring with one colour if and only if it has no subgraph isomorphic to P_3. Equivalently, if and only if there is no homomorphism of P_3, the directed path on three vertices, to G, if and only if none of P_3, C_3 and T_3, the transitive tournament on three vertices, is an induced subgraph of G. The following proposition gives a different description of these oriented graphs.

Proposition 3.10 *Let G be an oriented graph. Then $\chi_2(G) = 1$ if and only if there is a partition (V_1, V_2) of $V(G)$ such that every arc is from a vertex in V_1 to a vertex in V_2.*

Proof. ($\Rightarrow$) Let G be an oriented graph with $\chi_2(G) = 1$. Then G has no directed path of length two. Therefore G cannot contain an oriented odd cycle, and the underlying undirected graph of G is bipartite. Further, every vertex must have in-degree zero or out-degree zero. It follows that the desired partition exists.

($\Leftarrow$) If $V(G)$ can be partitioned (V_1, V_2) such that every arc is from a vertex in V_1 to a vertex in V_2, then G has no directed path of length two and, consequently, $\chi_2(G) = 1$. □

We now characterize the oriented graphs G with $\chi_2(G) = 2$. Let $\mathcal{F}_1$ be the set of oriented graphs constructed from an undirected odd cycle by replacing each edge xy by either the directed path x, m_{xy}, y, or the directed path y, m_{xy}, x, where m_{xy} is a new vertex. Let $\mathcal{F}_2$ be the set of directed odd cycles.

Proposition 3.11 *If $F \in \mathcal{F}_1 \cup \mathcal{F}_2$, then $\chi_2(F) = 3$.*

Proof. If $F \in \mathcal{F}_1 \cup \mathcal{F}_2$, then the undirected graph $Aux(F)$ contains an odd cycle. □

Theorem 3.12 *Let G be an oriented graph. Then $\chi_2(G) \leq 2$ if and only if there is no $F \in \mathcal{F}_1 \cup \mathcal{F}_2$ for which there exists a homomorphism $F \to G$.*

Proof. ($\Rightarrow$) Since no $F \in \mathcal{F}_1 \cup \mathcal{F}_2$ admits a homomorphism to G_2, and $G \to G_2$, no such F admits a homomorphism to G.

($\Leftarrow$) Suppose $\chi_2(G) > 2$. Then $Aux(G)$ is not bipartite, so it has an odd cycle, C. By the construction of $Aux(G)$, every pair of adjacent vertices are joined by a directed path of length two in G. Hence C arises from a closed walk of length $4\ell + 2$ in G comprised of $2\ell + 1$ directed paths of length two. Thus, there exists a homomorphism $F \to G$ for some $F \in \mathcal{F}_1 \cup \mathcal{F}_2$. □

We now describe a forbidden subgraph characterization of the oriented graphs with $\chi_2 \leq 2$.

Suppose $f : G \to H$ is a homomorphism. Then the set of images of the vertices of G is a subset of $V(H)$. The function f induces a mapping of $E(G)$ to $E(H)$, and the set of images of the edges of G under this mapping is a subset of $E(H)$. In the case that both of these mappings are surjective — every vertex of H is the image of a vertex of G and every edge of H is the image of some edge of G — then H is called a *homomorphic image* of G. If H is a homomorphic image of G, then the pre-image of every vertex of H is a non-empty independent set in G. Thus, a homomorphic image of G is a directed graph that can be constructed from G by a sequence of identifications of independent vertices.

Notice that a homomorphic image of an oriented graph may contain directed cycles of length two. In a forbidden subgraph characterization of the oriented graphs with $\chi_2(G) \leq 2$, the forbidden subgraphs should be oriented graphs.

One way to derive the desired forbidden subgraph characterization from Theorem 3.12 is to use the set of oriented graphs which are homomorphic images of the oriented graphs in $\mathcal{F}_1 \cup \mathcal{F}_2$ as the forbidden subgraphs. It is easy to see that a homomorphic image of a directed odd cycle contains a directed odd cycle. However, it is not true that a homomorphic image of an oriented graph in $\mathcal{F}_1$ necessarily contains an element of $\mathcal{F}_1 \cup \mathcal{F}_2$ as a subgraph. Let $\mathcal{F}_1'$ be the intersection of the set of homomorphic images of the oriented graphs in $\mathcal{F}_1$ and the set of oriented graphs.

Corollary 3.13 *Let G be an oriented graph. Then $\chi_2(G) \leq 2$ if and only if no oriented graph in $\mathcal{F}_1' \cup \mathcal{F}_2$ is a subgraph of G.*

3.3 Multipartite tournaments

In this section we consider the 2-dipath chromatic number of multipartite tournaments. Tight bounds on this quantity are given. In addition, it is shown that the 2-dipath chromatic number of a multipartite tournament can be computed in polynomial time.

Lemma 3.14 *Let T be an m-partite tournament with m-partition $(V_1, V_2, \ldots, V_m)$, and let c be a 2-dipath k-colouring of T. For each ℓ, $1 \leq \ell \leq k$, there exist i and j such that $\{x \in V(T) : c(x) = \ell\} \subseteq V_i \cup V_j$.*

Proof. Observe that the subgraph of T induced by any three vertices that belong to different sets of the m-partition is a tournament, and hence contains a directed path of length two. Since the set of vertices assigned a colour ℓ contains no directed path of length two, it can not contain vertices from three different sets in the m-partition. The result follows. □

Corollary 3.15 *Let T be an m-partite tournament. Then $\frac{m}{2} \leq \chi_2(T) \leq |V(T)|$.*

Proof. The lower bound follows from Lemma 3.14. The upper bound is clear. □

Infinitely many m-partite tournaments that achieve equality in the lower bound can be constructed. Recall that the definition of wreath product and digraph I_n were given in Sub-section 3.1. If T_m denotes the transitive tournament on m vertices, then for $n \geq 1$, the wreath product T_m wr I_n is an m-partite tournament with $\chi_2 = m/2$.

Infinitely many m-partite tournaments that achieve equality in the upper bound can also be constructed. Since it is impossible for adjacent vertices of a bipartite tournament to be joined by a directed path of length two, equality can only occur when $m \geq 3$. For $m = 3$ consider the 3-partite tournament $S(3, n)$ obtained from the bipartite tournament B_n, $n \geq 1$, with vertex set $\{a_1, a_2, \ldots, a_n\} \cup \{b_1, b_2, \ldots, b_n\}$ and arc set $\{a_ib_j : 1 \leq i \leq j \leq n\} \cup \{b_ja_i : 1 \leq j < i \leq n\}$ by adding a new vertex x, arcs joining x to each of $a_1, a_2, \ldots, a_n$, and arcs joining each of $b_1, b_2, \ldots, b_n$ to x. Any two vertices of $S(3, n)$ are joined by a directed path of length two. Notice that $S(3, t)$ is strongly connected. For $m = 3t$, $t \geq 2$, consider the wreath product T_t wr $S(3, t)$, where T_t is the transitive tournament on t vertices. Again, every two vertices are joined by a directed path of length two.

Theorem 3.16 *There exists a polynomial time algorithm to compute the 2-dipath chromatic number of a given multipartite tournament.*

Proof. Let T be an m-partite tournament with m-partition $(V_1, V_2, \ldots, V_m)$. For $i = 1, 2, \ldots, m$, define an equivalence relation θ_i on V_i by $(x, y) \in \theta_i$ if and only if $N^+(x) = N^+(y)$ and $N^-(x) = N^-(y)$. Let T' be the subgraph of T induced by choosing a representative of each equivalence class of $\theta_i, 1 \leq i \leq m$. Then T' is an m-partite tournament. Suppose it has m-partition $(V_1', V_2', \ldots, V_m')$ where $V_i' \subseteq V_i$, $1 \leq i \leq m$. By definition of the relation θ_i, any two vertices in V_i' are joined by a directed path of length two.

By construction, $\chi_2(T') = \chi_2(T)$. Every 2-dipath colouring of T' gives a 2-dipath colouring of T: for $i = 1, 2, \ldots, m$, assign all vertices in each equivalence class of θ_i the same colour as its representative in T'.

Let H be the complement of $Aux(T')$. Then every edge of H has one end in a set V_i' and the other end in a set V_j', where $i \neq j$. The ends of an edge of H can be regarded as sets of vertices – that is, equivalence classes under θ_i and θ_j – that can be assigned the same colour in a 2-dipath colouring. By Lemma 3.14, a 2-dipath colouring of T' corresponds to a partition of $V(H)$ into sets that induce K_1 or K_2. It follows that $\chi_2(T) = \chi_2(T') = |V(T')| - |M|$, where M is a maximum matching in H.

The m-partite tournament and undirected graph H can be found in polynomial time, as can a maximum matching in H. The result follows. □

4 Proper 2-dipath colourings

The colourings considered in this section assign different colours to adjacent vertices and to vertices joined by a directed path of length two. Many of the results that follow mirror those in the previous section. Proofs that are substantially similar to those already given are omitted.

4.1 Homomorphism model

We shall define a set of oriented graphs G'_k, $k \geq 1$, such that an oriented graph G has a proper 2-dipath k-colouring if and only if there is a homomorphism of $G \to G'_k$. For completeness, we note once again that a similar, but not identical, oriented graph has been used by Sopena [14] in work on the oriented chromatic number. The graph G'_k has arisen in the study of injective oriented colourings (e.g. see [11, 15]).

For an integer $k \geq 1$, we define G'_k to be the directed graph with vertex set

$$V(G'_k) = \{(u_0; u_1, u_2, \ldots, u_k) : u_0 \in \{1, 2, \ldots, k\}, u_i \in \{+, -\} \text{ if } i \neq u_0,\ u_{u_0} = \cdot\}$$

and edge set

$$E(G'_k) = \{(u_0; u_1, u_2, \ldots, u_k)(x_0; x_1, x_2, \ldots, x_k) : u_{x_0} = +, x_{u_0} = -\}.$$

By definition of $E(G'_k)$, the integers u_0 and x_0 can not be equal.

The integer u_0 is the *index* of the vertex $(u_0; u_1, u_2, \ldots, u_k) \in V(G'_k)$. For $i = 1, 2, \ldots, k$, let S'_i be the set of vertices of index i. Then S'_i is an independent set. For $j \geq 1$, the idea underlying the entry u_j of $(u_0; u_1, u_2, \ldots, u_k)$ is the same as in the previous section – to indicate the sort of adjacencies between $(i; u_1, u_2, \ldots, u_k)$ and vertices in $S_j, j \neq i$. As before, it follows from the definition that G'_k is an oriented graph.

The oriented graph G'_k also has the following properties:

- $|V(G'_k)| = k \cdot 2^{k-1}$,
- for all vertices v, $\deg^+(v) + \deg^-(v) = (k-1) \cdot \frac{2^{k-1}}{2} = (k-1) \cdot 2^{k-2}$,
- $|E(G'_k)| = \frac{1}{2}(k \cdot 2^{k-1}) \cdot ((k-1) \cdot 2^{k-2}) = \binom{k}{2} \cdot 2^{2k-3}$.

The next four results are similar to Lemma 3.1, Proposition 3.2, Theorem 3.3 and Corollary 3.4, respectively, and have essentially the same proofs.

Lemma 4.1 *There is a directed path of length two in G'_k joining the vertices $(i; u_1, u_2, \ldots, u_k)$ and $(j; v_1, v_2, \ldots, v_k)$ if and only if $k \geq 3$, $i \neq j$ and there exists $\ell \notin \{i, j\}$ such that $u_\ell \neq v_\ell$.*

Proposition 4.2 $\chi'_2(G'_k) = k$.

Theorem 4.3 *An oriented graph G has a proper 2-dipath colouring with k colours if and only if there exists a homomorphism $G \to G'_k$.*

Corollary 4.4 *If there exists a homomorphism $G \to H$, then $\chi'_2(G) \leq \chi'_2(H)$.*

The *oriented chromatic number* of an oriented graph G is the least integer k for which there is a homomorphism of G to an oriented graph on k vertices, and is denoted by $\chi_o(G)$. The homomorphism model for proper 2-dipath colourings makes it possible to obtain bounds on the oriented chromatic number in terms of the proper 2-dipath chromatic number.

Theorem 4.5 [11] $\chi_o(G_k) \leq 2^k - 1$.

Corollary 4.6 *Let G be an oriented graph with $\chi'_2 = k$. Then $k \leq \chi_o \leq 2^k - 1$.*

The next three results are similar to Proposition 3.5, Corollary 3.6, and Corollary 3.7, respectively. Recall that I_2 denotes the directed graph with two vertices and no arcs.

Proposition 4.7 *For $1 \leq i \leq k$, the oriented graph $G'_{k+1} - S'_i \cong G'_k$ wr I_2.*

Corollary 4.8 *For $k \geq 1$, G'_k is an induced subgraph of G'_{k+1}.*

Corollary 4.9 *For $k \geq 1$, there exists a homomorphism $G'_k \to G'_{k+1}$.*

4.2 Complexity of proper 2-dipath colourings

We shall determine, for each fixed integer $k \geq 1$, the complexity of *proper 2-dipath k-colouring*, the problem of deciding whether a given oriented graph G has a proper 2-dipath k-colouring. The problem turns out to be NP-complete when $k \geq 3$, and Polynomial when $k \leq 2$. We also describe the oriented graphs that can be 2-dipath coloured with one or two colours.

The NP-completeness proof uses two main ingredients. The first is a theorem due to Barto, Kozik and Niven [1]. For a fixed directed graph H, the symbol Hom_H denotes the problem of deciding whether a given directed graph G has a homomorphism to H. Observe that, if H is an oriented graph, then the input graph G can be assumed be an oriented graph.

Theorem 4.10 [1] *If an oriented graph F has a subgraph H with $\delta^+(H) > 0$, $\delta^-(H) > 0$, and two directed cycles each having the property that its length does not divide the length of the other, then Hom_F is $\mathcal{NP}$-complete.*

The second main ingredient in the NP-completeness proof is a reduction method that is well established in the study of graph homomorphisms, and which was first formally stated in the classic paper by Hell and Nešetřil [9] (also see [10]).

Let I be a fixed directed graph with special vertices u and v. Given a directed graph H, the *indicator construction with respect to* (I, u, v) produces the directed graph H^* with vertex set $V(H^*) = V(H)$ and $xy \in E(H^*)$ if and only if there is a homomorphism $I \to H$ that maps u to x and v to y.

Lemma 4.11 [9] *Let H^* denote the result of applying the indicator construction with respect to (I, u, v) to the directed graph H. Then Hom_{H^*} polynomially transforms to Hom_H.*

Theorem 4.12 *Let $k \geq 1$ be a fixed integer. If $k \leq 2$, then proper 2-dipath k-colouring is Polynomial. If $k \geq 3$, then proper 2-dipath k-colouring is NP-complete.*

Proof. Suppose $k \leq 2$. Let G be a given oriented graph. Since the undirected graph $U[G^2]$ can be constructed in polynomial time, the statement follows from Proposition 2.1.

Now suppose $k \geq 4$. The subgraph H of G'_k induced by the vertices that contain both a $+$ and a $-$ has $\delta^+(H) > 0$ and $\delta^-(H) > 0$. Since it also contains directed cycles of length three and length four, the statement follows from Theorem 4.10.

Finally, suppose $k = 3$. Referring to Figure 3, let $D = G'_3$ and let D^* denote the result of applying the indicator construction with respect to (I, u, v) to G'_3. Clearly D^* has a subgraph that satisfies the hypotheses of Theorem 4.10. Hence Hom_{D^*} is NP complete. The result now follows from Lemma 4.11. □

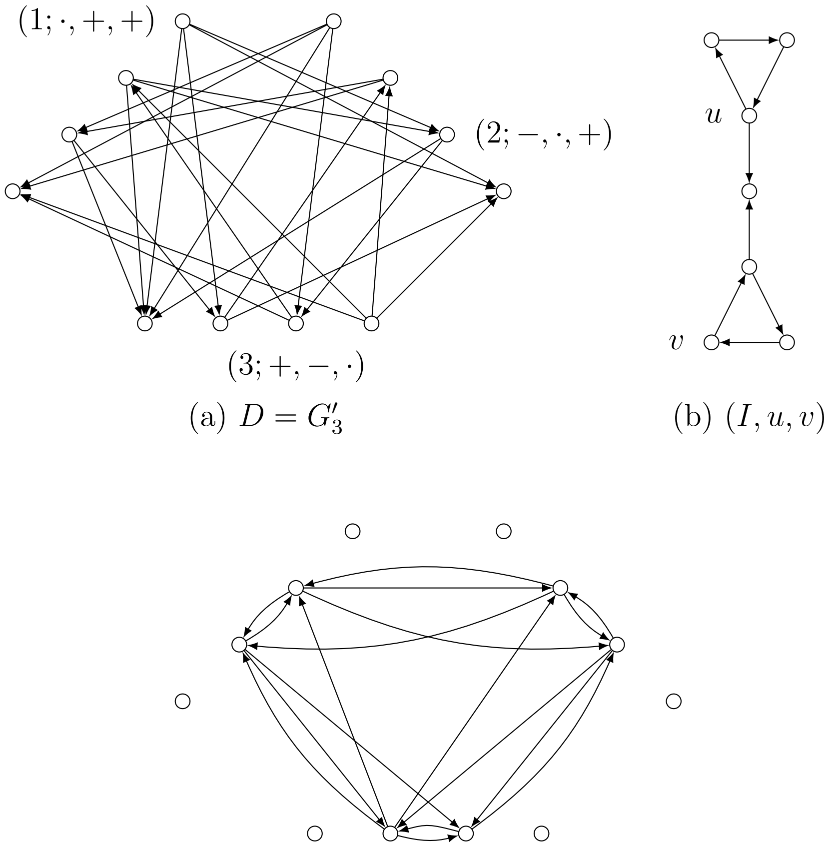
By Proposition 2.1, an oriented graph has a proper 2-dipath 1-colouring if and only if it has no arcs, and a proper 2-dipath 2-colouring if and only it does not contain a directed path of length two (which requires three colours). Thus the oriented graphs that have a proper 2-dipath 2-colouring are precisely those that have a 2-dipath 1-colouring. That is, an oriented graph G has a proper 2-dipath 2-colouring if and only if there is no homomorphism of P_3 to G, if and only if none of P_3, C_3 and T_3 is an induced subgraph of G, if and only if G is bipartite and $V(G)$ can be partitioned (V_1, V_2) such that every arc is from a vertex in V_1 to a vertex in V_2.

4.3 Multipartite tournaments

In this section we consider the proper 2-dipath chromatic number of multipartite tournaments. Tight bounds on this quantity are given. In addition, it is shown that the proper 2-dipath chromatic number of a multipartite tournament can be computed in polynomial time.

Proposition 4.13 *Let T be an m-partite tournament. Then $m \leq \chi'_2(T) \leq |V(T)|$.*

The multipartite tournament T_m *wr* I_n (see Sub-section 3.3) achieves equality in the lower bound. Equality in the upper bound is achieved by every tournament, and every m-partite tournament obtained using the construction in Sub-section 3.3.



(a) $D = G_3'$ (b) (I, u, v)

(c) D^*

Figure 3: Applying the indicator construction to G_3'

Theorem 4.14 *There exists a polynomial time algorithm to compute the proper 2-dipath chromatic number of a given multipartite tournament.*

Proof. Let T be an m-partite tournament with m-partition $(V_1, V_2, \ldots, V_m)$. Two vertices $x, y \in V(T)$ can be assigned the same colour in a proper 2-dipath colouring if and only if $N^+(x) = N^+(y)$ and $N^-(x) = N^-(y)$. Further, in a colouring with χ_2' colours, any two such vertices must receive the same colour.

For $i = 1, 2, \ldots, m$, define an equivalence relation θ_i on V_i by $(x, y) \in \theta_i$ if and only if $N^+(x) = N^+(y)$ and $N^-(x) = N^-(y)$. Suppose that θ_i has t_i equivalence classes. Then $\chi_2'(T) = \sum_{i=1}^{m} t_i$.

Since the equivalence classes of the relations θ_i can be found in polynomial time, the result follows. □

References

[1] L. Barto, M. Kozik and T. Niven, The CSP dichotomy holds for digraphs with no sources and no sinks (a positive answer to a conjecture of Bang-Jensen and Hell), *SIAM J. Computing* **38** (2008), 1782–1802.

[2] T. Calamoneri, The $L(h,k)$-Labelling Problem: An Updated Survey and Annotated Bibliography, *The Computer Journal* **54** (2011), 1344–1371.

[3] T. Calamoneri and B. Sinaimeri, $L(2,1)$-Labeling of Oriented Planar Graphs, *Discrete Applied Math.* **161** (2013), 1719–1225.

[4] G.J. Chang and S-C. Liaw, The $L(2,1)$-labeling problems on ditrees, *Ars Combin.* **66** (2003), 23–31.

[5] G.J. Chang, J-J. Chen, D. Kuo and S-C Liaw, Distance-two labelings of digraphs, *Discrete Applied Math.* **155** (2007), 1007–1013.

[6] M. Chen and W. Wang, The 2-dipath chromatic number of Halin graphs, *Inf. Proc. Letters* **99** (2006), 47–53.

[7] D. Goncalves, A. Raspaud and M. Shalu, On Oriented Labelling Parameters, *Formal Models, Languages and Applic.* **66** (2006), 34–45.

[8] J.R. Griggs and R.K. Yeh, Labelling graphs with a condition at distance 2, *SIAM J. Discrete Math.* **5** (1992), 586–595.

[9] P. Hell and J. Nešetřil, On the complexity of H-colouring, *J. Combin. Theory Ser. B* **48** (1990), 92–110.

[10] P. Hell and J. Nešetřil, *Graphs and Homomorphisms*, Oxford University Press, Oxford, UK, 2004.

[11] G. MacGillivray, A. Raspaud and J. Swarts, Injective Oriented Colourings, In: C. Paul and M. Habib (Eds.), WG 2009, *Lec. Notes Comp. Sci.* **5911**, Springer, Berlin, 2010, 262–272.

[12] G. MacGillivray, A. Raspaud and J. Swarts, Obstructions to Injective Oriented Colourings, *Electr. Notes Discrete Math.* **38** (2011), 597–605.

[13] S. Sen, 2-dipath and oriented $L(2,1)$-labelings of some families of oriented planar graphs, *Discussiones Mathematicae Graph Theory* **34** (2014), 31–48.

[14] E. Sopena, The chromatic number of oriented graphs, *J. Graph Theory* **25** (1997), 191–205.

[15] J. Swarts, *The complexity of digraph homomorphisms: Local tournaments, injective homomorphisms and polymorphisms*, Ph.D. Thesis, University of Victoria, Victoria, BC, Canada, 2009.
URL: `https://dspace.library.uvic.ca//handle/1828/1304`.

[16] K.M. Young [Sherk], 2-*dipath and Proper* 2-*dipath Colourings*, M.Sc. Thesis, University of Victoria, Victoria, BC, Canada, 2011.
URL: `https://dspace.library.uvic.ca/handle/1828/3277`.

(Received 16 Nov 2012; revised 1 Apr 2014, 30 May 2014)