Packings in Enneads

R.G. Stanton
Department of Computer Science
University of Manitoba
Winnipeg, Canada R3T 2N2

1. Introduction.

In earlier papers, we have illustrated how the packings of pairs into k-sets depend critically upon whether the total number of varieties v does or does not exceed the value $k^2 - k + 1$. The behaviour is also critically affected by the existence or non-existence of a projective geometry on $k^2 - k + 1$ points.

From these points of view, the packings of v elements into enneads (for values of v less than 73) are of particular interest since the geometry on 73 points is the first example of a geometry of even order that displays much structure (the geometries on 7 and on 21 points are too small to illustrate much general behaviour; cf.[8]).

First, we review the terminology that we shall be using. We start with v elements, and we wish to determine the packing number D(2,k,v) for v small. This is the cardinality of the largest family of k-sets chosen from the v elements in such a way that no pair occurs more than once. Saying that v is "small" means that $v \leq k^2 - k + 1$ (in this paper, we shall thus be considering the case that $k = 9$, $v \leq 73$).

The reason for this restriction is that, if a projective geometry with k points per line does exist, then it provides a perfect packing of all pairs selected from the $v = k^2 - k + 1$ points in exactly $b = k^2 - k + 1$ blocks. For v values that exceed $k^2 - k + 1$, the ordinary Fisher-Yates counting process gives the bound $bk \leq Rv$ (R being the maximum replication number for any element in the packing). This bound is, of course, equally valid for $v \leq k^2 - k + 1$, but it is usually far from providing an accurate answer.

In other papers, various results have been given for packings in sextuples, septuples, and octuples (cf. [3], [8], [10]). These numerical results have suggested general results (in turn, the general results have strengthened and simplified some of the numerical results). In this paper, we extend our results to the case $k = 9$.

We summarize the concept of the weight of a design, since the weight function will be one of our principal tools. Since we only consider packing designs in this paper, we restrict our definitions to that case, although they can be made more general (cf. [1], [4], [6], [7], [9]).

We define the weight of a block B to be

$$w(B) = (b - 1) - \Sigma(r_i - 1),$$

where the summation is over all the elements in the block B. It is easy to see that, in a packing design, w(B) is also equal to x_0, the number of blocks that are disjoint from B, and consequently w(B) is a non-negative quantity. The weight of the whole design is then found by summing w(B) over all blocks and so is

$$w(D) = b(b - 1) - \Sigma r_i(r_i - 1),$$

where the summation is now over all varieties in the design; w(D) is likewise non-negative. It is essential to note that, for a fixed b, the maximum value of w(D) occurs when the frequencies r_i are as nearly equal as possible (cf. [2]). Consequently, for a fixed b, we have bk elements in the packing array, and can compute the quantity

$$bk = av + t,$$

where $t < v$. Then the design of maximum weight in b blocks will occur when the design contains t elements of frequency (a + 1) and v - t elements of frequency a. Any alteration in the frequencies will increase the value of the quantity $\Sigma r_i(r_i - 1)$, and consequently will decrease the total weight of the design.

2. The Trivial Packings.

As for smaller values of k, the existence of the BIBD having parameters (10,45,9,2,1) shows that D(2,9,45) = 10, cf. [3].

Furthermore, we can write down the packing array for 45 elements as the dual of this pair-design and read off all smaller packing numbers by the peeling procedure indicated in [3]. The array follows.

1 2, 3, 4, 5, 6, 7, 8, 9
1, 10, 11, 12, 13, 14, 15, 16, 17,
2, 10, 18, 19, 20, 21, 22, 23, 24,
3, 11, 18, 25, 26, 27, 28, 29, 30,
4, 12, 19, 25, 31, 32, 33, 34, 35,
5, 13, 20, 26, 31, 36, 37, 38, 39,
6, 14, 21, 27, 32, 36, 40, 41, 42,
7, 15, 22, 28, 33, 37, 40, 43, 44,
8, 16, 23, 29, 34, 38, 41, 43, 45,
9, 17, 24, 30, 35, 39, 42, 44, 45.

We can then make the following array of packing numbers:

v	D	v	D	v	D	v	D
10	1	19	2	28	3	37	5
11	1	20	2	29	3	38	5
12	1	21	2	30	3	39	5
13	1	22	2	31	4	40	6
14	1	23	2	32	4	41	6
15	1	24	2	33	4	42	6
16	1	25	3	34	4	43	7
17	2	26	3	35	4	44	8
18	2	27	3	36	5	45	10

3. Packings for $46 \leq v \leq 57$.

The next natural dividing line comes at the value 57, since there is a triple system having parameters (19,57,9,3,1). From the general results in [3], we know that D(2,9,57) = 19, D(2,9,56) = 16, D(2,9,55) = 15.

The weight bound shows that D(2,9,46) < 11 and that D(2,9,47) < 11; hence the value of the packing number is 10 in each case.

A packing of 48 elements in 11 blocks would have weight 2 and would contain 3 elements of frequency 3 and 45 elements of frequency 2. Such a packing is easily found by dualizing the PBD on {1,2,3,4,5,6,7,8,9,a,b} that consists of blocks 123, 456, 789, and all missing pairs except the pair (a,b). Hence D(2,9,48) = 11.

Since a packing of 49 elements in 12 blocks would have negative weight, we also have that D(2,9,49) = 11.

A packing of 50 elements in 12 blocks has weight zero and is easily displayed as the dual of a PBD consisting of 8 triples and 42 pairs. One such PBD is given by (1,2,3), (1,4,5), (2,6,7), (3,8,9), (4,10,11), (5,6,12), (7,8,10), (9,11,12), together with all missing pairs. Since any packing of 51 elements in 13 blocks would have negative weight, we have established that D(2,9,51) = 12.

A packing of 52 elements in 13 blocks has weight zero, and so may be achieved as the dual of a PBD of 13 triples and 39 pairs. Such a PBD is easily displayed by cycling an initial block (1,2,4), modulo 13, and adding all missing pairs. Also, a packing of 53 elements in 14 blocks would have negative weight, and so we also have D(2,9,53) = 13.

Finally, the weight bound rules out a packing of 54 elements in 15 blocks. A packing in 14 blocks would have maximum weight 2. Such a packing would be the dual of a near-PBD on 14 elements consisting of 18 triples and 36 pairs (one pair being missing). Such a near-PBD can be displayed as consisting of the following blocks:

fourteen triples (1,2,4), (2,3,5), (3,4,6), (4,5,7), (5,6,8), (6,7,9), (7,8,10), (8,9,11), (9,10,12), (10,11,13), (11,12,14), (12,13,1), (13,14, 2), (14,1,3)

four triples (1,6,11), (2,7,12), (3,8,13), (4,9,14)

all missing pairs except for (5,10).

4. Packings with $58 \leq v \leq 62$.

First we prove a useful Lemma. Suppose that a design exists with parameters (v,b,r,k,1), as happens with the triple system on 57 elements. Then the packing number D(2,r,b) = v and we want to know whether D(2,r,b+1) can be v+1.

If we employ the weight procedure, we find that

$$r(v+1) = (r-k)(k+1) + (b+1-(r-k))k.$$

So the packing of maximal weight in v+1 blocks would have weight

$$(v+1)v - (r-k)(k+1)k - (b+1)k(k-1) + (r-k)k(k-1)$$

$$= (v+1)v - 2(r-k)k - rv(k-1) - k(k-1)$$

$= 2v - 2k(r-k) - k(k-1)$

$= 2 + k^2 + k - 2r.$

Thus we can state

Lemma 1. If (v,b,r,k,1) exists as a BIBD, then D(2,r,b+1) = v whenever the quantity $2+ k^2 + k$ is less than 2r. In particular, for k = 3, this always occurs for r > 7.

In virtue of this Lemma, D(2,9,58) = 19.

Now consider the packing on 59 elements. The weight bound is 20, and the packing of maximal weight has weight 8 (3 elements of frequency 4, 56 elements of frequency 3). The dual of this packing is a near-PBD with 3 quadruples, 56 triples, and 4 pairs that do not occur.

The 8 elements (call them set X) that appear in the missing pairs all appear in the triples, but not in the quadruples. The other 12 elements appear in the quadruples as (1,2,3,4), (5,6,7,8), (9,a,b,c).

Let us try to place the 8 elements in 8 triples. This can be done, since D(2,3,8) = 8, and the 4 missing pairs from the (2,3,8) packing can be taken as the 4 pairs not in the near-PBD. Then there are 48 triples that contain a single element from X, as well as a pair of elements from the 12 elements of A = {1,2,3,4,5,6,7,8,9,a,b,c} that appear in the quadruples.

Clearly, all that we need to do is to produce 8 one-factors on the elements of A which, together with the 3 quadruples, give a complete covering of all the pairs. This can be done by writing down the following 4 cycles (easily found in a couple of minutes by conducting a tree-search by hand):

(1,5,9,2,6,a,3,7,b,4,8,c) (1,6,b,2,7,c,3,8,9,4,5,a)
(1,8,a,2,5,b,3,6,c,4,7,9) (1,7,a,4,6,9,3,5,c,2,8,b)

Each cycle gives two 1-factors, and so we have 6 pairs from A to go with each element from X. This gives 48 triples; the (2,3,8) packing of the 8 elements of X contributes an additional 8 triples. So we have the required PBD with 4 quadruples and 56 triples, each element having frequency 9. The dual is then our (2,9,59) packing; hence we have established that D(2,9,59) = 20.

When we consider 60 elements, the weight bound is 21. The packing of maximal weight has weight 6 and contains 9 elements of frequency 4, 51 elements of frequency 3. Among the packings of weight zero is a packing consisting of 12 elements of frequency 4, 45 elements of frequency 3, and 3 elements of frequency 2.

This packing is the dual of a PBD on 21 elements containing 12 quadruples, 45 triples, 3 pairs. Suppose that the pairs are xy, xz, yz. Then each quadruple contains one of x, y, or z, and there are 9 triples containing x, y, and z (3 triples contain each letter). The PBD generated by deleting x, y, and z is a PBD on 18 elements and consists of 48 triples and 9 pairs.

Clearly, this PBD might be obtained by taking a suitable triple system on 19 elements and deleting element 19. In order to be able to add the elements x, y, and z, the triple system should have the following form.

9 triples of the form (19,1,2), (19,3,4), (19,5,6), etc.
4 triples on points 1,2,3,4,5,6,7,8,9,10,11,12, to go with x
4 triples on points 7,8,9,10,11,12,13,14,15,16,17,18, to go with y
4 triples on points 1,2,3,4,5,6,13,14,15,16,17,18, to go with z

Given the large number of triple systems on 19 points, it seemed likely that one of them could satisfy these rather mild requirements. I am grateful to Douglas Stinson for supplying one such triple system. He points out that Design #9 on page 64 of the paper "Steiner Triple Systems with Rotational Automorphisms" by K.T. Phelps and A. Rosa, that appeared in Discrete Mathematics **33** (1981), pages 57-66, serves our purpose. The point set of that triple system is $Z_9 \times \{1,2\}$, together with an invariant point P. The base blocks developed modulo 9 are given by (P,01,02), (01,31,61), (02,12,32), (61,02,42), (11,81,02), (41,51,02), (31,71,02), where the second base block gives the short orbit.

The three partial parallel classes that we require come from rearranging the 12 blocks obtained from developing the second and fourth base blocks in the following way.

11,41,71	21,51,81	01,31,61
31,62,12	41,72,22	21,52,02
61,02,42	71,12,52	51,82,32
01,32,72	11,42,82	81,22,62

Now employ the relabelling whereby 01 = 1, 02 = 2, 31 = 3, 32 = 4, 61 = 5, 62 = 6, 11 = 7, 12 = 8, 41 = 9, 42 = 10, 71 = 11, 72 = 12, 21 = 13, 22 = 14, 51 = 15, 52 = 16, 81 = 17, 82 = 18. With this relabelling, the first partial class uses points 1-12; the second partial class uses points 7-18, and the third partial class uses points 1-6, 13-18. Also, the base block (P,01,02), when developed modulo 9 and relabelled, becomes

(19,1,2), (19,3,4), ... , (19,17,18).

This triple system, with point 19 deleted, gives exactly the PBD on 18 points that we need; after adjoining x, y, and z, we have the PBD in 21 blocks with 12 quadruples, 45 triples, and 3 pairs. The dual of this PBD gives our packing, and so D(2,9,60) = 21.

5. Packings in the Conic Bound Region ($64 \le v \le 73$).

Of course D(2,9,73) = 73, since PG(2,8) exists. From Theorems A and B of **[3]**, we have D(2,9,72) = 64, D(2,9,71) = 56. Using the general results of **[3]**, we see that deletion of successive points from a no-3-collinear set produces the following additional bounds:

$$\begin{array}{ll} D(2,9,70) \ge 49, & D(2,9,69) \ge 43, \\ D(2,9,68) \ge 38, & D(2,9,67) \ge 34, \\ D(2,9,66) \ge 31, & D(2,9,65) \ge 29, \\ D(2,9,64) \ge 28, & D(2,9,63) \ge 28. \end{array}$$

Since the weight bound rules out the possibility of a packing of 70 elements in 50 blocks, we have D(2,9,70) = 49. On the other hand, D(2,9,69) would be 46 if the Wallis Design (46,69,9,6,1) should exist (cf. **[5]**), and so we can only state that $43 \le D(2,9,69) \le 46$.

Similarly, from Theorem B of **[3]**, we would have D(2,9,68) = 40 if the BIBD (46,69,9,6,1) should exist; on combining this fact with the conic bound, we have $38 \le D(2,9,68) \le 40$.

The weight bound for 67 is 37, and so $34 \le D(2,9,67) \le 37$. As in **[10]**, it is easy to show that the bound of 37 can only be achieved with a packing of weight 8 and frequency distribution $5^{65},4^2$ or with a packing of weight 0 and frequency distribution $6^4,5^{57},4^6$.

The weight bound for 66 is 33, and so $31 \le D(2,9,66) \le 33$.

The weight bound for 65 is 30, and so $30 \le D(2,9,65) \le 31$.

The weight bound for 64 is 29, and so $28 \le D(2,9,64) \le 29$.

The weight bound for 63 is 28, which is also the conic bound. Consequently, D(2,9,63) = 28, and the packing can be achieved by dualizing the BIBD with parameters (28,63,9,4,1).

The weight bound for 61 is 22, and a packing in 22 blocks would have maximal weight 6, with 15 elements of frequency 4 and 45 elements of frequency 3.

This frequency distribution also generates all of the possible frequency distributions for packings of weight 4, 2, and 0. Among the packings of weight zero is a packing with one element of frequency 6, 12 elements of frequency 4, and 48 elements of frequency 3; since this packing has weight zero, it is the dual of a PBD on 22 elements.

The required PBD would be achieved with a block B = (a,b,c,d,e,f); twelve blocks of length 4, none of which contains any element from B; 48 blocks of length 3, each containing an element from B. We would then get another PBD by deleting the elements of B. It would be a design on 16 elements and would contain 12 quadruples and 48 pairs. In order to reconstruct the original design, we would need to have the the 48 pairs arranged in 6 one-factors on the 16 elements involved.

To obtain such a design, all we need to do is take the 16 elements as the points of the affine geometry EG(2,4). Use 3 of its resolution classes for the 12 quadruples. Take the other two resolution classes and split each of them into three one-factors. This is easily done; for example, the resolution class (1,2,3,4), (5,6,7,8), (9,10,11,12), (13,14,15,16) generates three one-factors as follows.

(1,2), (3,4), (5,6), (7,8), (9,10), (11,12), (13,14), (15,16)
(1,4), (2,3), (5,8), (6,7), (9,12), (10,11), (13,16), (14,15)
(1,3), (2,4), (5,7), (6,8), (9,11), (10,12), (13,15), (14,16)

We then add back the 6 elements a,b,c,d,e,f, placing them all in the single block B and constructing 48 triples by adding a to the first 1-factor, b to the second 1-factor, c to the third one-factor, etc. The resulting Pairwise Balanced Design is the dual of the packing that we are seeking, and so D(2,9,61) = 22.

Since there is a BIBD with parameters (28,63,9,6,1), it follows from Theorem B of [3] that D(2,9,62) = 24.

6. Conclusion.

We summarize the results of the last few sections in the following table of the packing numbers D(2,9,v) for $46 \leq v \leq 73$.

v	D	v	D	v	D	v	D
46	10	54	14	62	24	70	49
47	10	55	15	63	28	71	56
48	11	56	16	64	28-29	72	64
49	11	57	19	65	29-30	73	73
50	12	58	19	66	31-33		
51	12	59	20	67	34-37		
52	13	60	21	68	38-40		
53	13	61	22	69	43-46		

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